Triangle path transit functions, betweenness and pseudo-modular graphs

Manoj Changat\textsuperscript{a,}\textsuperscript{*}, G.N. Prasanth\textsuperscript{b}, Joseph Mathews\textsuperscript{c}

\textsuperscript{a}Department of Futures Studies, University of Kerala, Trivandrum - 695 034, India
\textsuperscript{b}Department of Mathematics, Govt. College Chittur, Palakkad - 678 104, India
\textsuperscript{c}Department of Mathematics, S.B.College, Changanassery - 686 101, India

Received 22 July 2006; received in revised form 1 December 2007; accepted 29 February 2008
Available online 14 April 2008

Abstract

The geodesic and induced path transit functions are the two well-studied interval functions in graphs. Two important transit functions related to the geodesic and induced path functions are the triangle path transit functions which consist of all vertices on all \(u,v\)-shortest (induced) paths or all vertices adjacent to two adjacent vertices on all \(u,v\)-shortest (induced) paths, for any two vertices \(u\) and \(v\) in a connected graph \(G\). In this paper we study the two triangle path transit functions, namely the \(I^\Delta\) and \(J^\Delta\) on \(G\). We discuss the betweenness axioms, for both triangle path transit functions. Also we present a characterization of pseudo-modular graphs using the transit function \(I^\Delta\) by forbidden subgraphs.

© 2008 Elsevier B.V. All rights reserved.

Keywords: Transit function; \(I^\Delta\) path; \(J^\Delta\) path; Betweenness axioms; Pseudo-modular graphs

1. Introduction

The geodesic interval function \(I(u,v)\) of a connected graph \(G\) is the set of vertices lying on all shortest-paths (geodesics) between \(u\) and \(v\), is an important tool in metric graph theory. This function is first systematically studied by Mulder in [17]. The interval function \(I\) has been studied from different perspectives, to name a few from the literature, convexity, see e.g. [13,17,20], medians and betweenness, see e.g. [15,17], monotonicity, [17]. Another well-studied function in graphs is the induced path function \(J\), where \(J(u,v)\) is the set of vertices lying on all induced paths between \(u\) and \(v\). Similar problems have been studied for \(J\) also, for example, convexity [7,12,14,16], median-type properties and betweenness [8,16] and monotonicity [5,7,8]. The all-paths function \(A(u,v)\) consists of the set of all vertices lying on all paths between \(u\) and \(v\), is also studied in similar lines, see [9].

The idea of transit function comes basically from these three functions in graphs. The transit function generalizes these functions in graphs. The term ‘transit function’ was coined by Mulder about ten years ago and finally written up in [18]. The purpose is to introduce a tool to study how to move around in discrete structures. Therefore transit functions have a role in discrete structures like graphs or partially ordered sets (posets) because of the interplay

\textsuperscript{*} Corresponding author.

E-mail addresses: mchangat@gmail.com (M. Changat), gnprasanth@gmail.com (G.N. Prasanth), jose.chingam@yahoo.co.in (J. Mathews).
between the additional properties defined by the structure (for example, the set of edges in a graph or the partial order in a poset). For instance, transit functions can be defined in terms of paths in a graph \( G \), such functions are called path transit functions on \( G \). Several well-studied transit functions using various types of paths in graphs are discussed in [10] using what is known as \( \phi \)-path transit functions. Here a \( \phi \)-path is a subset of all paths in \( G \). If \( u, v \in V(G) \), then \( \Phi(u, v) \) denotes the subset of all \( u, v \)-paths in \( \Phi \). The \( \phi \)-path transit function \( R_\phi \) on \( G \) is defined by 
\[
R_\phi(u, v) = \{ x \in V \mid x \text{ is in some } \phi \text{-path in } G \}.
\]
Prime examples of path transit functions on a graph are geodesic (I), induced (J) and all-paths (A) transit functions.

This exemplifies the basic idea for studying the concept of transit function: transfer ideas, various questions and problems from one transit function to another and see whether interesting situations arise. We follow this approach in this paper to study the problems of betweenness in triangle path transit functions.

A formal definition of the triangle path transit function as discussed in [10] is as follows. Let \( P = u_1 \rightarrow u_2 \rightarrow \cdots \rightarrow u_k \) be a path in \( G \). Let \( z_i \) be a vertex not on \( P \) but adjacent to two consecutive vertices \( u_i, u_{i+1} \) of \( P \). Then we say that the path \( Q = u_1 \rightarrow u_2 \rightarrow \cdots \rightarrow u_i \rightarrow z_i \rightarrow u_{i+1} \rightarrow \cdots \rightarrow u_k \) is obtained from \( P \) by replacing the edge \( u_i \rightarrow u_{i+1} \) by the triangle \( \{u_i, z_i, u_{i+1}\} \). A triangular extension of a path \( P \) is a path \( Q \) obtained from \( P \) by replacing some of the edges of \( P \) by triangles. We call \( P \) a triangular extension of itself as well. Let \( \Phi \) be a path property on \( G \). Then \( \Phi^\Delta \) is the path property defined by

\[
\Phi^\Delta = \{ Q \mid Q \text{ is a triangular extension of some path in } \Phi \}.
\]

Two important triangle path transit functions are derived from the transit functions \( I \) and \( J \); namely the transit functions \( I^\Delta \) and \( J^\Delta \). In [6] the convexity associated with \( J^\Delta \) has been discussed. That is, the characterization of the \( J^\Delta \)-convex hull, the classical convexity invariants like the Carathéodory, Helly and Radon numbers were given in [6]. Using the idea of transit functions, the notion of betweenness has been introduced by Mulder in [18], see also [16]. In this paper we attempt to study the betweenness properties of \( I^\Delta \) and \( J^\Delta \).

In Section 2 we formally define the transit functions \( I^\Delta \) and \( J^\Delta \) and introduce the betweenness axioms. In Sections 3 and 4, we deal with the betweenness of the \( I^\Delta \) and \( J^\Delta \) transit functions respectively, and in the last section we study pseudo-modular graphs using the transit function \( I^\Delta \). All the graphs in this paper are assumed to be finite, connected and simple.

2. Betweenness

We begin with the formal definition of transit function. A transit function on a non-empty set \( V \) is a function \( R: V \times V \to 2^V \) satisfying the following conditions, for every \( u, v \in V \)

\[
\begin{align*}
(t1) \quad u & \in R(u, v) \\
(t2) \quad R(u, v) & = R(v, u) \\
(t3) \quad R(u, u) & = \{u\}.
\end{align*}
\]

If \( G \) is a graph with vertex set \( V \) and \( R \) a transit function defined on \( V \), then we say that \( R \) is a transit function on \( G \). The triangle path transit functions derived from the functions \( I \) and \( J \) are \( I^\Delta \) and \( J^\Delta \), defined respectively as:

\[
I^\Delta(u, v) = \{ w \in V(G) \mid w \text{ lies on a shortest } u, v \text{-path in } G \text{ or is adjacent to two consecutive vertices in a } u, v \text{-shortest path} \}
\]

\[
J^\Delta(u, v) = \{ w \in V(G) \mid w \text{ lies on an induced } u, v \text{-path in } G \text{ or is adjacent to two consecutive vertices in a } u, v \text{-induced path in } G \}.
\]

Let \( R \) be any transit function on a non-empty set \( V \). The following are the betweenness axioms on \( R \).

\[
\begin{align*}
(b1) \quad x & \in R(u, v) \Rightarrow v \notin R(u, x) \\
(b2) \quad x & \in R(u, v) \Rightarrow R(u, x) \subseteq R(u, v) \\
(b3) \quad x & \in R(u, v), y \in R(u, x) \Rightarrow x \in R(y, v) \\
(b4) \quad x, y & \in R(u, v) \Rightarrow R(x, y) \subseteq R(u, v).
\end{align*}
\]

In this paper we slightly modify the \( b3 \) axiom as follows

\[
(b3') \quad x \in R(u, v), u, v \neq y \in R(u, x) \Rightarrow x \in R(y, v).
\]


The axioms $b_1$ and $b_2$ are called the natural betweenness axioms or simply betweenness. One can easily verify that $b_1$ and $b_3$ will imply $b_1$ and $b_2$ and hence the $b_3$ axiom is a stronger betweenness axiom than $b_2$. Also $b_3$ implies $b_3'$, but not conversely. We can see that the transit function $I$ satisfies all the betweenness axioms, including the $b_3'$ axiom. The axiom $b_4$ is known as the monotone axiom and $I$ does not satisfy this axiom in general, see [17], for more on monotone axiom on $I$.

If $G$ and $G'$ are two given graphs, we say that $G$ is $G'$-free if and only if $G$ does not contain $G'$ as an induced subgraph.

For the betweenness, the situation for the induced path transit function is different, we can easily see that $J$ need not satisfy the betweenness axioms in general. In [16], Mulder proved that $J$ is a betweenness if and only if $G$ is $HHDD$-free and satisfy the $b_3$ axiom if and only if $G$ is distance hereditary. Similarly for the monotone axiom the forbidden subgraphs are identified in [7] and a more elegant characterization of graphs satisfying the monotone axiom is done in [8]. We can see that the all-paths transit function $A$ satisfies $b_2$ and $b_4$ for any graph and satisfies $b_1$ and $b_3$ if and only if $G$ is a tree [9].

By a triangle we mean a complete graph on three vertices. By long even house we mean an induced even cycle on which a triangle is attached as seen in Fig. 1.

If $G$ contains a triangle, then $I^\Delta$ does not satisfy $b_1$ axiom.

**Observation 1.** If $G$ contains a triangle, then $I^\Delta$ does not satisfy $b_1$ axiom.

**Theorem 1.** $I^\Delta$ satisfies $b_2$ axiom if and only if $G$ is $K_4 - e$ and long even house free.

**Proof.** If $G$ contains $K_4 - e$ or a long even house, then choosing $u$, $v$, $x$, $y$ as in Fig. 1, we can see that $b_2$ axiom fails.

Conversely suppose that $G$ has no induced $K_4 - e$ or long even house. To show that $I^\Delta$ satisfies $b_2$ axiom. Suppose to the contrary, $I^\Delta$ do not satisfy $b_2$ axiom. Then there exist some $u$, $v$ $x$, $y$ $z$, and $w$ of $I^\Delta(u, v)$ such that $I^\Delta(u, x) \not\subseteq I^\Delta(u, v)$. Therefore there exists some $y \in I^\Delta(u, x)$ but $y \not\in I^\Delta(u, v)$. That is $y \not\in I(u, v)$ and is not adjacent to any two consecutive vertices in any shortest $u, v$-path in $G$.

**Case 1:** $uv \in E(G)$
\[ x \in I^\Delta(u, v) \Rightarrow ux, xv \in E(G). \] Since \( ux \in E(G) \) and \( y \in I^\Delta(u, x) \), we must have \( xy, yu \in E(G) \). Also since \( y \notin I^\Delta(u, v) \) and \( uv \in E(G) \), \( y \) cannot be adjacent to \( v \). So that the subgraph induced by \([u, v, x, y]\) is a \( K_4 - e \).

**Case 2:** \( uv \notin E(G) \)

If \( x \in I(u, v) \) and since \( I \) satisfies \( b2 \) axiom, we must have \( I(u, x) \subseteq I(u, v) \), then we must have \( I^\Delta(u, x) \subseteq I^\Delta(u, v) \). Hence we assume that \( x \notin I(u, v) \).

Therefore \( x \) is adjacent to two consecutive vertices \( x_1, x_2 \) in a shortest \( u, v \)-path, say \( P_1 \). Then we can easily see that \( u \rightarrow P_1 \rightarrow x_1 \rightarrow x \) is a shortest \( u, x \)-path. If \( u \rightarrow P_1 \rightarrow x_1 \rightarrow x \) is the only shortest \( u, x \)-path, then since \( y \in I^\Delta(u, x) \) and \( y \notin I^\Delta(u, v) \), we must have \( yx, yx_1 \in E(G) \). Now the subgraph induced by \([y, x, x_1, x_2]\) is a \( K_4 - e \) and we are done. Therefore assume that, there is another \( u, x \)-shortest path, say \( P_2 : u \ldots x \). Now we can see that the length of the path formed by \( u \rightarrow P_1 \rightarrow x_1 \rightarrow x \) and the path \( u \rightarrow P_2 \rightarrow x \) are of same length. Let \( u_1 \) be the last vertex common to \( P_1 \) and \( P_2 \). Now we get two internally disjoint vertex paths, \( u_1 \rightarrow P_1 \rightarrow x_1 \rightarrow x \) and \( u_1 \rightarrow P_2 \rightarrow x \). Therefore \( C = u_1 \rightarrow P_1 \rightarrow x_1 \rightarrow x \rightarrow P_2 \rightarrow u_1 \) is an even cycle. Suppose that \( C \) is an induced even cycle. Now the vertex \( x_2 \) cannot be adjacent to any internal vertex in \( u_1 \rightarrow P_2 \rightarrow x \). For, if it is adjacent to some vertex \( x_3 \), then \( u_1 \rightarrow P_2 \rightarrow x_3x_2 \) will be a path of length shorter than \( u_1 \rightarrow P_1 \rightarrow x_2 \) geodesic, which is not possible. Therefore \( C \) together with the triangle \([x, x_1, x_2]\) will form a long even house.

Suppose that \( C \) is not an induced even cycle. Then there can be chords from \( u_1 \rightarrow P_2 \rightarrow x \) to \( u_1 \rightarrow P_1 \rightarrow x_1 \rightarrow x \) which do not affect the length of \( P_1 \) and \( P_2 \). Let \( u_n u_n' \) be a chord from vertices of \( u_1 \rightarrow P_1 \rightarrow x_1 \rightarrow u_1 \rightarrow P_2 \rightarrow x \) respectively. We choose \( u_n \) as the vertex in \( u_1 \rightarrow P_1 \rightarrow x_1 \) closest to \( x_1 \) and \( u_n' \) in \( u_1 \rightarrow P_2 \rightarrow x \) as a vertex closest to \( x \) respectively. Then \( u_n \rightarrow u_n' \rightarrow P_2 \rightarrow P_2 \rightarrow x \rightarrow x_1 \rightarrow P_1 \rightarrow u_n \) is an induced even cycle \( C_1 \). As in the previous case \( C_1 \) together with the triangle \([x, x_1, x_2]\) will form a long even house.

Before we move onto prove \( b3' \) axiom we need the following observation and the figure.

**Observation 2.** For \( b3' \) axiom, if \( uv \in E(G) \), then \( x \in I^\Delta(u, v), y \in I^\Delta(u, x) \), then \( x, y, v \in I^\Delta(y, v) \). So for any two adjacent vertices \( u, v \) the \( b3' \) axiom is satisfied.

Now we give a theorem that gives us a condition that \( I^\Delta \) satisfies \( b3' \) axiom

**Theorem 2.** \( I^\Delta \) satisfies \( b3' \) axiom if and only if \( G \) is paw-free.

**Proof.** If \( G \) contains paw, then choosing \( u, v, x, y \) as in the figure of the paw (see Fig. 2), we can see that \( I^\Delta \) does not satisfy \( b3' \) axiom.

Conversely let \( G \) be paw-free. Owing to the Observation 2, we can assume that the graph contains at least 5 vertices and \( u \) and \( v \) are not adjacent. To show that \( I^\Delta \) satisfies \( b3' \) axiom, assume the contrary that \( I^\Delta \) does not satisfy \( b3' \) axiom. Therefore \( u, v \in V(G) \) and \( x \in I^\Delta(u, v) \) we have \( y \in I^\Delta(u, x) \) but \( x \notin I^\Delta(y, v) \), where \( y \neq u, v \).

**Case 1:** \( x \in I(u, v) \)

Since \( I \) satisfies \( b3 \) and \( b3' \), if \( y \in I(u, x) \) then \( x \in I(y, v) \) so that \( x \in I^\Delta(y, v) \). Therefore \( y \notin I(u, x) \). So \( y \) is adjacent to two consecutive vertices say \( y_1, y_2 \) in a shortest \( u, x \)-path. Since \( x \notin I^\Delta(y, v) \), we have \( x \notin I(y, v) \) and not adjacent to any two consecutive vertices in any shortest \( y, v \)-path.

If \( y \) is not adjacent to any vertex in \( I^\Delta(u, x) \), then taking a vertex \( y_3 \) either adjacent to \( y_1 \) or \( y_2 \) we can see that the subgraph induced by \([y, y_1, y_2, y_3]\) is a paw.

Suppose that \( y, y_3, y_3y_1 \in E(G) \). Then the path \( y_3 \rightarrow y \rightarrow y_2 \) is of length 2 which is same as the length of the subpath \( y_3 \rightarrow y_1 \rightarrow y_2 \) of a shortest \( u, x \)-path. Hence we can see that \( y \in I(u, x) \), which is a contradiction to our assumption.

Now if \( y_3 \) is not adjacent to \( y_1 \) or \( y_2 \) then we can see that the length of the path \( y_3 \rightarrow y \rightarrow y_2 \) is shorter than the subpath \( y_3 \ldots y_1 \rightarrow y_2 \) of a shortest \( u, x \)-path, which is also a contradiction.

Therefore we can see that \( y \) will not be adjacent to any vertex in \( I(u, x) \) other than \( y_1 \) and \( y_2 \) so that taking a vertex \( y_3 \) adjacent to \( y_1 \) or \( y_2 \), we can see that the subgraph induced by \([y, y_1, y_2, y_3]\) is a paw.

**Case 2:** \( x \notin I(u, v) \)

In this case, we have \( x \in I^\Delta(u, v) \). Therefore \( x \) is adjacent to two consecutive vertices say \( x_1, x_2 \) in a shortest \( u, v \)-path \( P \). We claim that \( x \) is not adjacent to any vertex in \( P \) other than \( x_1 \) and \( x_2 \). Suppose that \( x \) is adjacent to some vertex \( x_3 \) in \( P \). Without loss of generality we can assume that \( x_3 \) lies in \( u \rightarrow P \rightarrow x_2 \). If \( x_3 \) is adjacent to \( x_1 \) then \( x \) lies in some shortest \( u, v \)-path, contrary to our assumption. If \( x_3 \) is not adjacent to \( x_1 \), then the length of the
path $x_3 \rightarrow x \rightarrow x_2$ is less than the length of the subpath $x_3 \rightarrow P \rightarrow x_2$ of $P$, which is not possible and hence our claim. Then the subgraph induced by $\{x, x_1, x_2, x_4\}$, where $x_4$ is a vertex adjacent to $x_2$ or $x_1$ in $P$ is a paw, which completes the proof. $\square$

4. $J^\Delta$-transit function

In this section, we study the triangle induced path transit function $J^\Delta$. We mainly discuss the betweenness axioms $b1$ and $b2$. We identify the graphs for which the $J^\Delta$-transit function satisfies these axioms.

We write $J = J^0\Delta$ and $J^k\Delta = (J^{(k-1)\Delta})^\Delta$, for $k \geq 1$. In general $J^\Delta \subseteq J^2\Delta \subseteq J^3\Delta \subseteq \cdots \subseteq J^n\Delta$. For some $n$ one would expect that $J^n\Delta = J^{(n+1)\Delta}$. It is to be noted that $J^\Delta = J^n\Delta$ for $n \in N$ if and only if $G$ has no induced subgraph isomorphic to $K_4 - e$. Though $J^\Delta$ need not be equal to $J^n\Delta$ for $n > 1$, the convexity induced by $J^\Delta$ and $J^n\Delta$ on any connected graph coincide.

It is trivial to note that $J^\Delta$ transit function satisfies the betweenness axiom $b1$ only for triangle free graphs, in which case $J^\Delta$ transit function coincides with the induced path transit function $J$. Since $J$ satisfies $b1$ if and only if $G$ is $HHD$-free, we have the following observation.

Observation 3. The triangle induced path transit function $J^\Delta$ satisfies the betweenness axiom $b1$ if and only if $G$ is free from domino and cycles except the four cycle.

In the following theorem, we prove that $J^\Delta$ satisfies the betweenness axiom $b2$ if $G$ is $HHD$, $K_4 - e$-free. If $G$ is a house, a long cycle or domino, we can verify that the $b2$ axiom is not satisfied. First let us consider a long cycle (hole). Let $a$, $x$ and $y$ be 3 vertices of the long cycle so that $x$ and $y$ are adjacent vertices and $a$ is not adjacent to $x$ or $y$. Let $z$ be a vertex on the $a - y$ segment of the long cycle not containing $x$. Let $b$ be an extra vertex which is adjacent only to $z$ and $x$. Then $x \in J^\Delta(a, b)$, $y \in J^\Delta(a, x)$, but $y \notin J^\Delta(a, b)$. For a house and a domino the extra vertex $b$ and other vertices $a$, $x$ and $y$ are as shown in Fig. 3.

For a $K_4 - x$, choose the vertices $a$, $b$, $x$ and $y$ so that $b$ and $y$ are non-adjacent vertices. In all these cases we can see that $x \in J^\Delta(a, b)$, $y \in J^\Delta(a, x)$, but $y \notin J^\Delta(a, b)$.

Theorem 3. Let $G$ be a connected graph. Then the $J^\Delta$ satisfies the betweenness axiom $b2$ if $G$ is $HHD$ and $K_4 - e$-free.

Proof. Suppose that $G$ is $HHD$ and $K_4 - e$-free. We can prove that $J^\Delta(a, w) \subseteq J^\Delta(a, b)$ for any $a, b, w \in V(G)$ and $w \in J^\Delta(a, b)$. First we show that for any neighbor $w$ of $b$ in $J^\Delta(a, b)$, we have $J^\Delta(a, w) \subseteq J^\Delta(a, b)$. Assume that this is not true, and let $y$ be a vertex on $J^\Delta(a, w)$ and not in $J^\Delta(a, b)$ with $w$ neighbor of $b$ in $J^\Delta(a, b)$. Since $G$ is $HHD$-free, the induced path transit function $J$ on $G$ satisfies $b2$ axiom [16]. So we have $J(a, w) \subseteq J(a, b) \subseteq J^\Delta(a, b)$. Hence $y \in J^\Delta(a, w) / J(a, w)$. Without loss of generality, let us assume that there exists an induced $a - b$ path $P$ so that $w$ is on $P$ or $w$ is not on $P$ but adjacent to $b$ and the neighbor $w_1$, of $b$ on $P$ according as $w$ belongs to or not belongs to $J(a, b)$. Let $Q$ be an $a - w$ induced path so that $y$ is not on $Q$ but adjacent to two adjacent vertices $y_1$ and $y_2$ on $Q$ (see Fig. 4). Assume that $a \rightarrow Q \rightarrow y_1 \rightarrow y_2$. Since $y \notin J^\Delta(a, b)$ and $K_4 - e$ is avoided, $a$ cannot be adjacent to $y$.

Now $a \rightarrow Q \rightarrow w \rightarrow b$ cannot be induced; hence there are chords between $b$ and the internal vertices of $a \rightarrow Q \rightarrow y_2$. Let $bz$ be the chord from $b$ to $a \rightarrow Q \rightarrow y_2$ with $z$ closest to $a$. Then $a \rightarrow Q \rightarrow z \rightarrow b$ is an induced
Fig. 4. Forbidden subgraphs for pseudo-modular graphs.

Fig. 5.

Fig. 6.

Fig. 7.

\[ a - b \text{ path not containing } w. \]

We may choose \( a \) to be the common vertex of \( P \) and \( a \to Q \to z \) closest to \( w \) and \( z \), so that \( P \) and \( a \to Q \to z \to b \) are two internally disjoint \( a - b \) paths with no chord from \( w \) to \( a \to Q \to y_2 \). Since \( K_4 - e \) is forbidden, \( a \) cannot be adjacent to \( w \). Now we have the following two cases.

Case 1: \( w \in J(a, b) \)

So \( w \) is a vertex on \( P \). Hence \( P \) is of length at least 3. Since \( a \to Q \to z \to b \) is of length at least 2, there must be chords between the internal vertices of the two paths. Since there is no chord from \( v \), there must be a chord from \( z \) to the neighbor \( w_1 \) of \( w \) on \( a \to P \to w \); to avoid holes. Now to avoid the house, the domino or holes, there cannot be another chord from \( w_1 \) to \( a \to Q \to z \). Hence \( w_1 \to z \to Q \to a \) is an induced \( w_1 - a \) path. To avoid the house or the domino, this path together with the path \( a \to P \to w_1 \) cannot induce a triangle or a 4-cycle. But then there must be a chord between the internal vertices of the paths. Choose one with end \( p \) on \( P \) closest to \( w_1 \) and then with end \( q \) on \( Q \) closest to \( z \). Then \( b \to w \to w_1 \to P \to p \to q \to Q \to z \to b \) together with the chord \( w_1 z \) induces either a house or a domino; a contradiction (see Fig. 5).

Case 2: \( w \notin J(a, b) \)

Therefore \( w \) is not on \( P \); but adjacent with \( w_1 \) and \( b \). Hence \( P \) is of length at least 2.

Since \( a \to Q \to z \to b \) is of length at least 2, the cycle formed by the two paths is of length at least 4. Since \( w \notin J(a, b) \), \( w \) cannot be adjacent to any vertex on \( a \to P \to w_1 \), except \( w_1 \). Hence to avoid the house or hole, there must be a chord from \( z \) to \( w_1 \). Then the subgraph induced by \( w_1, b, w \) and \( z \) is isomorphic to a \( K_4 - e \) (see Fig. 6); again a contradiction.

These contradictions show that \( J^\Delta(a, w) \subseteq J^\Delta(a, b) \) for any neighbor \( w \) of \( b \) in \( J^\Delta(a, b) \). Let \( v \) be any vertex in \( J^\Delta(a, b) \). Choose an \( a - b \) triangle induced path \( P \) containing \( v \). Let \( P : a \to P \to v \to v_1 \to \cdots \to v_k \to b \). Then \( v_k \) is a neighbor of \( b \) in \( J^\Delta(a, b) \). Hence by the previous argument, we have \( J^\Delta(a, v_k) \subseteq J^\Delta(a, b) \). Similarly we find that \( J^\Delta(a, v) \subseteq J^\Delta(a, v_1) \subseteq \cdots \subseteq J^\Delta(a, v_k) \subseteq J^\Delta(a, b) \). This completes the proof. \( \square \)
5. Pseudo-modular graphs

In this section, we consider the function \( I^\Delta(u, v, w) \) for any triple of vertices \((u, v, w)\) on \( G \), where \( I^\Delta(u, v, w) = I^\Delta(u, v) \cap I^\Delta(v, w) \cap I^\Delta(w, u) \). The function \( I(u, v, w) = I(u, v) \cap I(v, w) \cap I(w, u) \) was first considered by Mulder in [17] for studying modular graphs and median graphs. Modular(median) graphs are precisely the graphs for which \( I(u, v, w) = \emptyset \) for every triple of vertices \( u, v, w \). Since pseudo-modular graphs are the generalizations of modular graphs we expect that using \( I^\Delta(u, v, w) \), we can characterize pseudo-modular graphs.

In this section we obtain a characterization of pseudo-modular graphs satisfying \( I^\Delta(u, v, w) \neq \emptyset \) together with a list of forbidden induced subgraphs.

Pseudo-modular graphs have been studied by various authors and is a well-established class of graphs in metric graph theory. See [1–4,11,19], for various aspects of pseudo-modular graphs. First, we give the definition of a pseudo-modular graph given in the literature.

Let \( u, v, w \) be three vertices of a graph \( G \). Then \( x, y, z \) form a pseudo-median of the triple \( u, v, w \), if the following distance equations are satisfied.

\[
\begin{align*}
    d(u, v) &= d(u, x) + d(x, y) + d(y, v) \\
    d(v, w) &= d(v, y) + d(y, z) + d(z, w) \\
    d(w, u) &= d(w, z) + d(z, x) + d(x, u) \\
    k &= d(x, y) = d(y, z) = d(z, x)
\end{align*}
\]

where \( k \geq 0 \) is minimal under these conditions. The number \( k \) is called the size of the corresponding pseudo-median. Note that pseudo-medians are not required to be unique. A graph \( G \) in which each triple of vertices has a pseudo-median of size ‘0’ is called a modular graph. A pseudo-modular graph is a graph in which each triple has a pseudo-median of size at most ‘1’.

Next we require three pertinent graphs:

For the graphs in Fig. 7, if we equally subdivide the edges \( ut \) and \( us \) and allowing some or all vertices in the resulting \( u, t \) and \( u, s \)-paths to be adjacent without affecting the length of the \( u, t \) and \( u, s \)-paths so that \( d(u, t) = d(u, s) \), then we get a family of graphs which we denote respectively by \( G_1, G_2 \) and \( G_3 \). We can verify that these classes of graphs are not pseudo-modular since \( I(u, v, w) = \emptyset \) for this family of graphs. In the main theorem of this section, we prove that these are the only forbidden subgraphs for pseudo-modular graphs. We first prove the following lemma.

**Lemma 1.** If \( x \in I^\Delta(u, v, w) \) and \( x \in I(u, v) \), \( x \in I(v, w) \), \( x \in I^\Delta(w, u) \setminus I(w, u) \) then \( x, y, z \) forms a pseudo-median of \( u, v, w \); where \( yz \in E(G) \) and \( y, z \in I(w, u) \).

**Proof.** \( x \in I^\Delta(u, v, w) \) and \( x \in I(u, v), x \in I(v, w), x \in I^\Delta(w, u) \setminus I(w, u) \). Since \( x \notin I(w, u) \), \( x \) is adjacent to two consecutive vertices \( y, z \) in a shortest \( w, u \)-path. Since \( x \in I(u, v) \), \( x \) is in a shortest \( u, v \)-path \( P_1 \). Now the paths defined by the subpath \( w \ldots x \) of a shortest \( w, v \)-path and the union of the subpath \( w \ldots y \) of a shortest \( w, u \)-path and the edge \( yx \) will be of the same length. Similarly the subpath \( u \ldots x \) of a shortest \( u, v \)-path and the union of the subpath \( u \ldots z \) of a shortest \( u, w \)-path and the edge \( zy \) will be of the same length. Now consider the triangle \( \{x, y, z\} \). The edge \( yz \) is in a shortest \( u, w \)-path, edge \( xy \) is in a shortest \( u, v \)-path and edge \( xz \) is in a shortest \( w, u \)-path so that \( x, y, z \) is a pseudo-median triangle for \( u, v, w \). 

Now we prove the main theorem of this section.

**Theorem 4.** Let \( G \) be any connected graph. Then \( G \) is pseudo-modular if and only if \( I^\Delta(u, v, w) \neq \emptyset \) and \( G \) does not contain any graph from the families \( G_1, G_2 \) and \( G_3 \) as induced subgraphs.

**Proof.** Suppose that \( G \) is pseudo-modular, then for the triple of vertices \( (u, v, w) \) in any of the family of graphs \( G_1, G_2 \) and \( G_3 \) described above, we can see that \( (u, v, w) \) does not have a pseudo median triangle. Therefore \( G \) doesn’t have any graph from the family \( G_1, G_2 \) and \( G_3 \) as induced subgraphs.

Next we show \( I^\Delta(u, v, w) \neq \emptyset \) for any \( u, v, w \in V(G) \). Since \( I(u, v) \subseteq I^\Delta(u, v) \) we have \( I(u, v, w) \subseteq I^\Delta(u, v, w) \). Therefore if \( I(u, v, w) \neq \emptyset \) then we have \( I^\Delta(u, v, w) \neq \emptyset \). Suppose that \( I(u, v, w) = \emptyset \), since \( G \) is a pseudo-modular graph and \( I(u, v, w) = \emptyset \); the triple \( u, v, w \) has a pseudo-median of size 1; say \( x, y, z \) so that
\[ d(x, y) = d(y, z) = d(z, x) = 1. \] Also \( x, y, z \in I(u, v), y, z \in I(v, w), z, x \in I(w, u) \) and \( xy, yz, zx \in E(G) \) we must have \( x, y, z \in I^A(u, v), x, y, z \in I^A(v, w) \) and \( x, y, z \in I^A(w, u) \) so that \( I^A(u, v, w) \neq \emptyset \).

Conversely suppose that \( I^A(u, v, w) \neq \emptyset \) and \( G \) does not contain any graph from the families \( G_1, G_2 \) and \( G_3 \) as induced subgraphs. By Lemma 1 we consider only the following cases.

\subsection*{Case 1:} \( x \in I(u, w) \) and \( x \not\in I(u, v) \cap I(v, w) \)

In this case, \( x \) is adjacent to two consecutive vertices \( x_1, x_2 \) in a shortest \( u, v \)-path and two consecutive vertices \( y_1, y_2 \) in a shortest \( v, w \)-path. Now consider the subgraph \( H \) induced by the section \( x_1 \rightarrow x_2 \rightarrow \cdots \rightarrow v \) of the shortest \( u, v \)-path. \( y_2 \rightarrow y_1 \rightarrow \cdots \rightarrow v \) of the shortest \( v, w \)-path. Note that the sections \( x_1 \rightarrow x_2 \rightarrow \cdots \rightarrow v \) and \( y_2 \rightarrow y_1 \rightarrow \cdots \rightarrow v \) are shortest \( x_1, v \) and \( y_1, v \)-paths. We prove that the shortest \( x_1, v \)-path and shortest \( y_1, v \)-paths are of the same length.

Suppose that \( d(x_1, v) < d(y_1, v) \). Since \( x_1, x_2 \) and \( y_1, y_2 \) are edges, we have \( d(x_2, v) < d(y_2, v) \). Assume that \( d(x_2, v) = d(y_2, v) = 1 \). Now, the path \( v \rightarrow x_1 \rightarrow x_2 \rightarrow \cdots \rightarrow v \) is of length equal to \( d(x_2, v) + 1 = d(y_2, v) + 1 = d(y_1, v) \), so that \( x \) lies in a shortest \( y_1, v \)-path. Since \( y_1 \in I(w, v) \), we have \( x \in I(y_1, v) \subseteq I(w, v) \), a contradiction. Thus, we have \( d(x_1, v) = d(y_1, v) \). If \( y_2 \) and \( x_2 \) are not adjacent then we can see that the subgraph \( H \) is a member of the family \( G_2 \) and if \( y_2 \) and \( x_2 \) are adjacent then \( H \) is in the family \( G_3 \), a contradiction.

\subsection*{Case 2:} \( x \not\in I(u, w) \cap I(u, v) \cap I(v, w) \)

\subsection*{Subcase 2.1:} \( d(u, v), d(u, w), d(w, v) = 1 \)

Since \( x \not\in I(u, v) \) we can find two adjacent vertices \( w_1, w_2 \) so that \( xw_1, xw_2 \in E(G) \). Since \( x \not\in I(u, v) \), we can find two adjacent vertices \( u_1, u_2 \) so that \( u_1, u_2 \in E(G) \). Now consider the subgraph \( H_1 \) induced by the shortest \( u \rightarrow u_1 \rightarrow u_2 \) path, \( x \) and shortest \( u_1 \rightarrow w_1 \rightarrow w_2 \) path. As in the previous case, we can show that \( d(w_2, u) = d(u_2, u) \). If \( u_1 \) is not adjacent to \( w_1 \), then \( H_1 \) belongs to the family \( G_2 \), otherwise \( H_1 \) belongs to \( G_3 \), a contradiction.

\subsection*{Subcase 2.2:} Suppose that \( d(u, w) = 1 \)

Since \( x \in I^A(u, w) \) we have \( ux, xw, uw \in E(G) \). Since \( x \in I^A(u, w) \), and \( u, x, w \) form a triangle, \( x \) is adjacent to \( u \) and \( w \) lying in a shortest \( u, v \)-path. Similarly \( y \in I^A(v, w) \) which is adjacent to \( w \) so that \( xw, xw_1, uw_1 \in E(G) \). Following similar arguments as in the previous case, we can show that \( d(u, v) = d(v, w) \). Now the subgraph induced by geodesics \( u, v; v, w \) and \( x \) is a member of the family \( G_1 \), a contradiction, which completes the proof. \( \square \)

Acknowledgement

The authors acknowledge the referee for their valuable comments and suggestions that helped to improve the presentation of the paper.

References