A database of rigorous and high-precision periodic orbits of the Lorenz model

Roberto Barrio\textsuperscript{a,c,*}, Angeles Dena\textsuperscript{b,c}, Warwick Tucker\textsuperscript{d}

\textsuperscript{a}Departamento de Matemática Aplicada and IUMA, University of Zaragoza, E-50009 Zaragoza, Spain.
\textsuperscript{b}Centro Universitario de la Defensa Zaragoza and IUMA. Academia General Militar, E-50090 Zaragoza, Spain.
\textsuperscript{c}Computational Dynamics group. University of Zaragoza, E-50009 Zaragoza, Spain.
\textsuperscript{d}Department of Mathematics, Uppsala University, Box 480 751 06, Uppsala, Sweden.

Abstract

A benchmark database of very high-precision numerical and validated initial conditions of periodic orbits for the Lorenz model is presented. This database is a “computational challenge” and it provides the initial conditions of all periodic orbits of the Lorenz model up to multiplicity 10 and guarantees their existence via computer-assisted proofs methods. The orbits are computed using high-precision arithmetic and mixing several techniques resulting in 1000 digits of precision on the initial conditions of the periodic orbits, and intervals of size $10^{-100}$ that prove the existence of each orbit.

Keywords: Computer-Assisted Proof, periodic orbits, validated numerics, high-precision, Lorenz model

1. Introduction

In computational physics and dynamics new developments in numerical techniques appear continuously. As a consequence, there is a need to validate the correctness and effectiveness of these new methods. Therefore, it is advisable to have a top level numerical database that may serve as a common benchmark for all these new studies. A general belief is that it is not possible to perform a reliable numerical simulation on a chaotic system, but this is clearly a misunderstanding. In fact, using suitable techniques and sufficiently high precision, it is possible to perform a very precise simulation for deterministic dynamical systems. So, a top and real challenge is to state correct and useful numerical data that may be used by everybody. Thus, this paper focuses on answering and providing results on the following question: is it possible to provide useful data for very high-precision simulations of deterministic chaotic systems? The answer is yes, and the most suitable set of data corresponds to information about some invariants of the system; in our case the set of unstable periodic orbits. This set has several advantages: first, it is clear how to use these data as a test of accuracy – simply try to follow one or several periodic orbits. In addition to this, during the construction of the set of benchmarks, we have reconfirmed some previous results on the proposed model, the Lorenz model.

The Lorenz model [1] is the most classical and paradigmatic low-dimensional chaotic problem since it is one of the first models with the presence of chaotic behavior and chaotic attractors. This nonlinear model has been analyzed by a large number of researchers, but it is still an important dynamical system to be studied. Based on a
more complicated model by Saltzman [2], Lorenz achieved his famous equations:

\[ \dot{x} = \sigma(y - x), \quad \dot{y} = -xz + r x - y, \quad \dot{z} = xy - bz, \]  

(1)

where \( \sigma \) (the Prandtl number), \( r \) (the relative Rayleigh number) and \( b \) are three dimensionless control parameters. It is well known that a good knowledge of the set of periodic orbits (POs) of the Lorenz model (unstable periodic orbits, UPOs, foliated to the attractor) provides some more general information about the system, and gives critical information in chaotic regions [3–10]. Therefore, having complete information of all UPOs of low-medium multiplicity is highly desirable. Some partial data have been already published in literature, but we focus on completing the references, giving at the same time a useful benchmark for analytical and numerical techniques in both dynamical systems analysis of low-dimensional chaotic systems, and in high-precision numerical methods for ODEs.

The location of UPOs has been an important and a well studied problem by physicists [11–15] and mathematicians using a vast number of numerical algorithms. Obtaining accurate information of UPOs is thus a very interesting task. Another interesting point is related to the question of computability of chaotic systems. As commented above, deterministic chaotic systems can be accurately numerically integrated, given sufficiently high precision; yet this is scarcely done in the literature. Moreover, some very recent publications state as a “computational challenge” the task of obtaining numerical solutions of the Lorenz system in some “long” time intervals [16–18]. The reported methods are extremely expensive, e.g. high-order implicit methods or very naive implementations of the Taylor series method. As such they require thousands of CPU-hours on massive parallel computers for a problem that with suitable techniques needs only a few hours on a standard laptop computer. Let us remark that this issue – high-precision numerical solutions of ODEs – nowadays is handled without any problem by several freely available softwares, such as TIDES\(^1\) [19] that uses a highly optimized Taylor-series method [20]. As an example, using this software, a periodic orbit (with 500 digits of precision) of the Lorenz system was shown in [21]. Of course, locating the initial conditions of the UPOs, and proving their existence with high precision becomes a much more complex problem. In this paper we have used a fast and accurate algorithm for the correction of approximate periodic orbits [22] that allows us to locate UPOs for any dynamical system up to any arbitrary precision and, in particular, to compute UPOs with 1000 precision digits for low-dimensional problems such as the Lorenz model. To our knowledge this is the only available algorithm and software (TIDES) capable of reaching arbitrary high precision (for instance 1000 digits) for ordinary differential equations (ODEs).

Another important application of the Taylor method is that it can be made to use interval arithmetic, which allows us to obtain validated numerical methods for differential equations. This is a corner stone of Computer-Assisted Proofs for proving the existence of periodic orbits. Therefore, using interval methods, we give rigor to the numerically obtained high-precision results. In other words, the results rigorously enclose the exact invariants in small sets. And therefore, we have not only some numerical results but, we will have a rigorous result that states the skeleton of UPOs of the system. This kind of information is an important complement to numerical studies as it provides rigor to some simulations [23, 24].

As a concrete benchmark, the values of the coordinates of nine periodic orbits (one per multiplicity) along their complete period – at fixed output times – are provided with 1000 precision digits for comparison purposes for computational dynamics tests.

The work reported here gives a complete database of high-precision and validated numerical data. We hope that this data can act as a serious benchmark for new numerical and analytical techniques aimed at dissipative chaotic systems.

The paper is organized as follows. In Section 2, we present the low-precision location of unstable periodic orbits in chaotic systems. In order to improve these, we explain in Section 3 the computation of high-precision initial conditions of the periodic orbits applied to the chaotic Lorenz system. Moreover, in this section we show the results of some numerical tests that illustrate good behavior of the method for the Lorenz model. Another important point that we deal with in Section 4 is the rigorous location of unstable periodic orbits in chaotic systems. In Section 5, we detail the contents of the developed database, which is available to the scientific community. Finally, we present the conclusions of this work, and in the Appendix A we show an example of the files of the database.

2. Low-precision location of unstable periodic orbits in chaotic systems

In this section, we describe how to locate low-precision unstable UPOs in the Lorenz model, for details see [25, 26].

The Lorenz system (1) is well understood in terms of geometric models [27]. It has been shown to be chaotic in the topological sense for the non-classical [28] and classical [29] parameter values. The existence of the Lorenz attractor has been verified using Computer-Assisted Proofs techniques in [4].

Given the Lorenz model (1), let \( x(0) = y \) be the initial conditions and

\[ x = x(t; y), \quad t \in \mathbb{R}, \quad x, y \in \mathbb{R}^3, \]  

(2)

the solution of the above autonomous differential system.

\(^1\)http://cody.unizar.es/software.html
A periodic orbit, which is characterized by the vector \( y \) of initial conditions and its period \( T \), verifies the periodicity condition

\[
x(T; y) - y = 0. \tag{3}
\]

for the direct flow. Instead, we use a variant of Newton’s method which brings contraction into play. More precisely, we will consider the global Poincaré map \( F: \Sigma^m \to \Sigma^m \) defined by

\[
F_k(z) = x_{(k+1 \mod m)} - P(x_k), \quad k = 1, \ldots, m \tag{4}
\]

where \( z = (x_1, \ldots, x_m) \in \Sigma^m \) (\( x_k \in \Sigma \)). Note that a zero of \( F \) corresponds to a multiplicity \( m \) (or period-\( m \)) orbit of \( P \). Applying Newton’s method on \( F \) makes the simple zeros of \( F \) super-attracting, and thus numerically stable.

In order to make our numerical computations rigorous, we use set-valued methods (sometimes known as interval analysis, see [30, 31]). In this framework, the interval Newton method becomes

\[
N([z]) = \hat{z} - [DF([z])^{-1}F(\hat{z})], \tag{5}
\]

where \([z] = ([x_1], \ldots, [x_m])\) is an interval vector, and \( \hat{z} \) is the midpoint of \([z]\). If \( N([z]) \subset [z] \) then \( F \) has a unique zero in \([z]\), and therefore \( P \) has a unique orbit of multiplicity \( m \), with each iterate \( x_k \) inside the rectangle \([x_k] \subset \Sigma \).

In order to find good candidate enclosures \([z]\) containing true periodic orbits, we use the fact that – for the Lorenz system – the periodic orbits are uniquely characterized by their symbolic dynamics. In effect, this means that we know exactly how many low-period orbits to expect, and roughly where to find them. Using a very long trajectory, we can search amongst its iterates for a best-approximate match for any particular periodic orbit. Applying Newton’s method to this approximation, followed by a small inflation into a set produces the desired candidate enclosure \([z]\). For details, see [26].

3. High-precision location of unstable periodic orbits in chaotic systems

This section reviews briefly the numerical algorithm that permits to compute periodic orbits with very high-precision.

In order to compute the roots of Eq. (3), equivalently, to find the initial conditions of a periodic orbit with high-precision, we use an iterative corrector of UPOs based on some modifications of the Newton method and the key use of an ODE solver able to solve differential systems with arbitrary precision. The Newton method begins with a set of approximated initial conditions \((y_i, T_i)\), obtained in the previous section, being \((y_i, T_i)\) at step \( i \) of the iterative process. Our aim is improve them, in such a way that

\[
||x(T_i + \Delta T_i; y_i + \Delta y_i) - (y_i + \Delta y_i)|| < ||x(T_i; y_i) - y_i||.
\]

For this purpose, we calculate the approximate corrections \((\Delta x, \Delta T_i)\), which are obtained by expanding

\[
x(T_i + \Delta T_i; y_i + \Delta y_i) - (y_i + \Delta y_i) = 0,
\]

in a multi-variable Taylor series up to the first order

\[
x(T_i; y_i) = y_i + \left( \frac{\partial x}{\partial y} - I \right) \Delta y_i + \frac{\partial x}{\partial t} \Delta T_i = 0. \tag{6}
\]
where $I$ is the identity matrix of order 3. The $3 \times 3$ matrix
\[ \frac{\partial x}{\partial y} \] is the fundamental matrix, i.e. the solution of the variational equations. This matrix evaluated at $(y, T)$ is an approximation $M$ of the monodromy matrix $M$. And, \[ \frac{\partial x}{\partial t} \] represents the derivative of the solution with respect to the time, i.e., $x = f(x)$. This vector, evaluated at the corrected initial conditions $(y, T)$, corresponds to $f(yT)$ where $yT = x(T, y)$.

In order to compute new values, the correction algorithm imposes an orthogonal displacement
\[ (f(y_i))^T \Delta y_i = 0. \] \tag{7}

In this way, the next $(n + 1) \times (n + 1)$ linear system is obtained
\[ \begin{pmatrix} M_i - I & f(yT) \\ (f(y_i))^T & 0 \end{pmatrix} \begin{pmatrix} \Delta y_i \\ \Delta T_i \end{pmatrix} = \begin{pmatrix} y_i - yT \\ 0 \end{pmatrix}. \] \tag{8}

The linear system (8) is solved using singular value decomposition (SVD) techniques which provide a stable numerical method [22], and this gives us the corrected initial conditions.

To be able to compute the correction we use an important tool, TIDES [19]. This software computes simultaneously the solution and the partial derivatives of the solution of (3), in double or multiple precision (using the multiple precision libraries $	ext{gmp}$ and $	ext{mpfr}$ [32]). The TIDES software is a key technique for computing the database as this is one of the few available softwares capable to solve ODEs in arbitrary precision. In [33], due to the lack of arbitrary precision numerical ODE solvers at that time, a much more cumbersome approach (based on the Lindstedt-Poincaré technique) is used to obtain high-precision periodic orbits.

The performance of the correction method can be seen in Table 1. Each row, which corresponds to a periodic orbit of multiplicity $m$ ($m = 2, \ldots, 10$), shows the CPU time in seconds versus the number of digits of the computational relative error (precision digits). All the numerical tests have been carried out using a personal computer PC Intel quad-core i7, CPU 860, 2.80 GHz under a 2.6.32-29-generic SMP x86 64 Linux system.

The behavior of the method in the determination of the periodic orbits of the Lorenz model is quite similar for all of them, as we obtain our goal of 1000 digits of precision in just 10 iterations. Therefore, we illustrate the process in Fig. 2 just for the LR and LLRLR periodic orbits. As expected, our algorithm is quadratically convergent since it is mainly based on the Newton method.

Having a database of periodic orbits of the Lorenz system has two important applications. The first one is to serve as benchmark of high-precision numerical ODE solvers. In the literature there are quite a few high-precision numerical integrations of chaotic dynamical systems that can be use to that purpose. Therefore, it is quite useful for that community to dispose of such an information, as to have correct data of initial conditions of periodic orbits permits to compare easily different numerical methods. For instance, in Fig. 3 we show some comparisons on the numerical integration of the LLRLR periodic orbit of the Lorenz model using the well established codes $	ext{dop853}$ (a Runge–Kutta code) and $	ext{odex}$ (an extrapolation code) developed by Hairer and Wanner [34], and the Taylor series method implemented on the TIDES code. We observe that the RK code $	ext{dop853}$ becomes the fastest option for low-precision requests. Nevertheless, in quadruple precision the $	ext{odex}$ code is far more efficient than the RK code because it is a variable order code, as TIDES. Finally, for very high-precision requests the Taylor series method is the only reliable method amongst the standard methods, and is capable to solve ODE systems up to thousands of precision digits in a reasonable CPU time. Note that the last few years some high-order implicit methods and fixed step-size methods have been announced, aimed

<table>
<thead>
<tr>
<th>Orbit</th>
<th>50</th>
<th>100</th>
<th>500</th>
<th>1000</th>
</tr>
</thead>
<tbody>
<tr>
<td>LR</td>
<td>0.04</td>
<td>0.22</td>
<td>22.29</td>
<td>251.49</td>
</tr>
<tr>
<td>LLR</td>
<td>0.08</td>
<td>0.31</td>
<td>33.32</td>
<td>372.27</td>
</tr>
<tr>
<td>LLLR</td>
<td>0.10</td>
<td>0.40</td>
<td>43.05</td>
<td>478.18</td>
</tr>
<tr>
<td>LLLLLR</td>
<td>0.13</td>
<td>0.47</td>
<td>50.38</td>
<td>579.47</td>
</tr>
<tr>
<td>LLLLLLLR</td>
<td>0.14</td>
<td>0.54</td>
<td>59.10</td>
<td>666.32</td>
</tr>
<tr>
<td>LLLLLLLLLR</td>
<td>0.17</td>
<td>0.63</td>
<td>66.95</td>
<td>760.35</td>
</tr>
<tr>
<td>LLLLLLLLLR</td>
<td>0.18</td>
<td>0.69</td>
<td>75.20</td>
<td>845.65</td>
</tr>
<tr>
<td>LLLLLLLLLL</td>
<td>0.20</td>
<td>0.77</td>
<td>85.17</td>
<td>929.85</td>
</tr>
</tbody>
</table>

Table 1: CPU time (seconds) for the computation of some UPOs depending on the precision digits.
at "precise" numerical integrations of the Lorenz system – but producing unreliable results and requiring extremely high CPU times [16, 18]. In our benchmark test, Fig. 3 it has been of great help to have as reference orbit the precise initial conditions and period of several periodic orbits.

We remark that these tests are also related to the computability of a deterministic chaotic system using a given precision (the round-off unit of the computations). The Lyapunov exponent $\lambda$ of a periodic orbit is defined as $\log (m_1)/T$, where $m_1$ is the magnitude of its leading characteristic multiplier and $T$ is its period. As an example, for the orbit LR we have $\lambda \approx 0.99465$. So, with this value we may estimate the number of laps that we may follow the periodic orbit with some precision. This total time $T_{\text{total}}$, the Lyapunov time that reflects the limits of the computability at the precision $u$ of the orbit LR of the Lorenz system, its Lyapunov time. The computability of the Lorenz system itself is also obtained in the same way, as it is already well known in dynamical systems literature.

Another application of having the database is to provide a computer-assisted verified topological template of the Lorenz attractor, whose existence was established in [4, 5]. Here, we just comment that, having a rigorous set of UPOs embedded in the attractor, we can guarantee the values of the linking matrix obtained considering the knots formed by the UPOs of the chaotic attractor. The topological structure of the Lorenz attractor [6–8] is described in terms of a paper-sheet model, called a template made of "normal" and twisted, like a Möbius band. A Möbius template can be quantified by a set of linking numbers – the local torsions. The torsions are, locally, the characteristic multiplier and $T$ is its period. As an example, for the orbit LR we have $\lambda \approx 0.99465$. So, with this value we may estimate the number of laps that we may follow the periodic orbit with some precision. This total time $T_{\text{total}}$, the Lyapunov time that reflects the limits of the computability at the precision $u$ of the orbit LR of the Lorenz system, its Lyapunov time. The computability of the Lorenz system itself is also obtained in the same way, as it is already well known in dynamical systems literature.

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This is just one example of the use of high-precision validated data (apart from the use also as a benchmark test of validated numerical ODE integrators).

4. Rigorous location of unstable periodic orbits in chaotic systems

The rigorous computations needed to validate the high-precision periodic orbits were carried out along the strategy outlined in Section 2. All computations were performed using the CAPD library, see [38], which uses gmp and mpfr libraries for its multiple precision. Given a specific high-precision approximation of the initial conditions at the $m$ different iterates of the Poincaré map of a periodic orbit $(x_1, \ldots, x_m)$, we begin by inflating the trajectory into an interval vector $[z] = ([x_1], \ldots, [x_m])$, each component having width $10^{-100}$. This will be our candidate enclosure for applying the interval Newton method for the global Poincaré map, as described earlier.
Theorem 1. For the Lorenz system (1) with the Saltzman parameter values ($b = 8/3, \sigma = 10, \rho = 28$) there exists a unique periodic orbit with symbolic notation LR (multiplicity $m = 2$) whose initial conditions are

\[
x_0 = \hat{x}_0 \pm 10^{-100},
\]

\[
y_0 = \hat{y}_0 \pm 10^{-100},
\]

\[
z_0 = 27,
\]

with

\[
\dot{x}_0 = -2.1473676319116125647657994834426364
\]

\[
53918312637730720608358273648286122
\]

\[
24089658325767107886028868
\]

\[
0.07804821146124900317478579765812352
\]

\[
432481250078273367551283626639574888
\]

\[
0.006207392602813065110324916
\]

Note that the above result is a theorem; it is rigorously proved via Computer-Assisted techniques. The 1000 digits in the files are very high-precision results checked numerically with a carefully done numerical study (they have been checked with more precision digits and we have some "guarantee" on the correctness of the presented digits, but all the digits have not proved theoretically). To our knowledge this is, by far, the most precise rigorous result on the location of periodic orbits in chaotic systems in literature, and the most precise numerical results also.

5. The Lorenz database

The goal of this work is to develop a database that consists of two kind of files. The complete database is provided as a complementary folder of this paper. In the first set of files, we provide the initial conditions of one periodic orbit per file with 1000 precision digits and the values of these coordinates validated with 100 digits that proves the existence of the periodic orbit. Recall that in all cases the $z$-coordinate has a fixed value, $z = 27$ (the same as the equilibria $P_\pm$).

<table>
<thead>
<tr>
<th>$m$</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
<th>9</th>
<th>10</th>
<th>TOTAL</th>
</tr>
</thead>
<tbody>
<tr>
<td>$nm$</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>2</td>
<td>3</td>
<td>5</td>
<td>9</td>
<td>16</td>
<td>28</td>
<td>51</td>
</tr>
</tbody>
</table>

Table 2: Number of periodic orbits $nm$ depending on their multiplicity $m$ and total number of computed UPOs.

In total there are 116 files with initial conditions (all the UPOs of the Lorenz attractor of multiplicity $m \leq 10$) as we show in Table 2, which specifies the number of UPOs, $nm$, depending on their multiplicity, $m$, and the total number of computed UPOs. These files are denoted by $\text{lor}_m\_\text{symb}.txt$ where $m$ is the multiplicity and $\text{symb}$ the symbolic sequence of the orbit. The format of these files is shown in the AppendixA. First of all, the individual number of the orbit ($\text{num}$) is specified; then the multiplicity ($\text{orbit-mult}$), the number of the orbit among all the orbits with the same multiplicity ($\text{num-same-mult}$) and the symbolic sequence of the orbit (\text{symb}). After that we find, with 1000 precision digits, the period $T$ and the initial conditions $x$ and $y$ of the orbit ($z = 27$ in all cases).
Finally, the rigorous intervals for the same variables ($x, y, z$) in high-precision are obtained just by taking the first 100 digits (these are the rigorously proved digits).

Besides, there are nine files with complete data of one periodic orbit each, one per each multiplicity ($2 \leq m \leq 10$), which are denoted $symb\_orbit.txt$. In these files, we give with 1100 digits the values of the coordinates of the orbit at fixed output times with time increment $h = 0.01$.

The format of the files is to give in each line one complete point, that is, the values of $t_i, x(t_i), y(t_i), z(t_i)$. Most of the shown digits are most likely correct (the computations have been done with an error tolerance of $10^{-1000}$ and each data has been carefully checked for the first 1000 digits), and again the first 100 digits are rigorously proved digits.

6. Conclusions

The goal of this paper is to present a high-precision and validated database of periodic orbits useful to scientific community. This consists on hundreds of approximated initial conditions (with 1000 digits of precision) of all the periodic orbits of the Lorenz attractor with multiplicities between 2 and 10. To obtain this database, we have combined two different methods: a corrector of periodic orbits algorithm in arbitrary precision, which allows us to obtain the initial conditions of UPOs of any dynamical system with the required precision, and Computer-Assisted techniques to prove the existence of these orbits within a tolerance of $10^{-100}$. This database is a “computational challenge” and it can be use as a benchmark for checking new numerical and theoretical techniques in computational physics and dynamics.

Acknowledgements

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Appendix A. File lor_2_LR.txt

<table>
<thead>
<tr>
<th>Format</th>
<th>Data of the orbit</th>
</tr>
</thead>
<tbody>
<tr>
<td>num</td>
<td>orbit-mult</td>
</tr>
<tr>
<td>1</td>
<td>2</td>
</tr>
<tr>
<td>0</td>
<td>0</td>
</tr>
</tbody>
</table>

According to the format, the first 100 digits are rigorously proved.

Figure 5: Periodic orbits of the Lorenz model of multiplicities between 2 and 5.
$$y = 2.0780482114612494003174785797658123524324$$

$$x = -2.14736763191811612564756759948344263645391$$


