DECOMPOSING COMPLETE GRAPHS INTO CUBES

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Abstract

This paper concerns when the complete graph on \( n \) vertices can be decomposed into \( d \)-dimensional cubes, where \( d \) is odd and \( n \) is even. (All other cases have been settled.) Necessary conditions are that \( n \) be congruent to 1 modulo \( d \) and 0 modulo \( 2^d \). These are known to be sufficient for \( d \) equal to 3 or 5. For larger values of \( d \), the necessary conditions are asymptotically sufficient by Wilson’s results. We prove that for each odd \( d \) there is an infinite arithmetic progression of even integers \( n \) for which a decomposition exists. This lends further weight to a long-standing conjecture of Kotzig.

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1. Introduction

A sequence \( H_1, H_2, \ldots, H_n \) of graphs with union \( G \) is called a decomposition of \( G \) if each edge of \( G \) is in \( H_i \) for exactly one \( i \), and in this case we write \( G = H_1 + H_2 + \cdots + H_n \). If in addition the subgraphs \( H_i \) are all isomorphic to \( H \), then we write \( G = nH \), and say that \( H \) divides \( G \). We call such a decomposition an \( H \)-decomposition of \( G \). If \( G_1 \) is a subgraph of \( G \) that includes all the vertices of \( G \) and each component of \( G_1 \) is isomorphic to \( H \), then we call \( G_1 \) an \( H \)-factor of \( G \). We denote the complete graph on \( n \) vertices by \( K_n \), and the complete bipartite graph with \( j \) vertices on one side and \( k \) on the other by \( K_{j,k} \). If \( m \leq n \) by \( K_n \setminus K_m \) we mean the complete graph on a set of \( n \) vertices with all edges internal to some subset of \( m \) vertices (called the hole) removed. By a \( k \)-set we mean a set with \( k \) elements.
The *d-cube*, denoted $Q_d$, is the graph whose vertices can be labelled with all the binary $d$-tuples, such that two vertices are adjacent if and only if they differ in a single coordinate. It is easy to see that $Q_d$ is $d$-regular, bipartite, and has $2^d$ vertices and $d2^{d-1}$ edges.

The decomposition of graphs is the focus of a great deal of research (see [2] for a thorough discussion of the subject). In particular, decompositions of $K_n$ into smaller complete graphs and decompositions of $K_n$ into cycles have received much attention. In 1979, Anton Kotzig initiated interest in $d$-cube decompositions of complete graphs by asking for which values of $d$ and $n$ there exists a $Q_d$-decomposition of $K_n$ (Problem 15 of [12]). In 1981 he established necessary conditions on $d$ and $n$ for the existence of $Q_d$-decompositions of $K_n$ for all $d$ and proved the sufficiency of these conditions for some cases [13]. Since $Q_d$ is $d$-regular with $2^d$ vertices and $d2^{d-1}$ edges, it is easy to see that the following are necessary conditions for the existence of a $d$-cube decomposition of $K_n$:

1. If $n > 1$ then $n \geq 2^d$,
2. $d \mid n - 1$, and
3. $d2^d \mid n(n - 1)$.

For a fixed $d$, these necessary conditions require that $n$ lies in certain congruence classes modulo $d$. In 1981, Kotzig [13] proved the following results.

**Theorem 1.** If there exists a $Q_d$-decomposition of $K_n$, then

(a) if $d$ is even, then $n \equiv 1 \pmod{d2^d}$;
(b) if $d$ is odd, then either
   (i) $n \equiv 1 \pmod{d2^d}$, or
   (ii) $n \equiv 0 \pmod{2^d}$ and $n \equiv 1 \pmod{d}$.

**Theorem 2.** There is a $Q_d$-decomposition of $K_n$ if $n \equiv 1 \pmod{d2^d}$.

These two theorems established the sufficiency of conditions (1) through (3) for the cases when $d$ is even and when $d$ is odd and $n$ is odd. Sufficiency of these conditions in the case $d = 3$ was shown by Maheo [14] in 1980. Recently, the case $d = 5$ was settled by Bryant *et al.* [4]. This however still leaves the following unsolved problem.

**Problem 1.** Let $d > 5$ be odd and let $n$ be such that $n \equiv 0 \pmod{2^d}$ and $n \equiv 1 \pmod{d}$. Show that $Q_d \mid K_n$. 
Although this problem has been cited often in the literature (see for example [2, 10, 11, 12]), little progress was made on the case $d$ odd and $n$ even until recently. Of course the well-known 1975 theorem of Wilson [15] implies that for each $d$ we have $Q_d \mid K_n$ for all sufficiently large $n$ satisfying conditions (1) through (3). A new technique for $Q_d$-decompositions using partitions of vector spaces into linearly independent sets was introduced in [6] in 1998. This technique was used in [8] to give, for each odd $d$, an explicit infinite sequence of even values of $n$ such that $Q_d \mid K_{2n}$.

**Theorem 3** [8]. Let $d$ be odd and let $s$ be the order of 2 (mod $d$). If $r$ is any integer with $r \geq d/s$, then $Q_d \mid K_{2rs}$.

Other articles dealing with various $d$-cube decompositions include [1, 3] and [9].

In this paper we prove that for each odd $d$ there is an infinite arithmetic progression of even integers $n$ for which a $Q_d$-decomposition of $K_n$ exists.

2. Preliminaries

Let $Z_2$ be the field of order 2. We denote $Z_2^m$, regarded as a vector space over $Z_2$, by $V_m$. Note that we can think of $V_m$ as the vertex set of $Q_m$. We denote by $\langle S \rangle$ the subspace of $V_m$ generated by $S \subseteq V_m$. For $a \in V_m$ and $A, B \subseteq V_m$ we define $a + B = \{a + b : b \in B\}$, we define $A + B = \bigcup_{a \in A}(a + B)$. If $A$ and $B$ are subsets of $V_m$ with $0 \notin B$, then let $G(A, B)$ be the graph with vertex set $A \cup (A + B)$ and edge set $\{a, a + b : a \in A, b \in B\}$.

The following is the $k = 2$ case of Lemma 1 of [6].

**Theorem 4.** Suppose $B$ is a linearly independent subset of $V_m$ with $d$ elements. Then $G(V_m, B)$ is a $Q_d$-factor of the complete graph on $V_m$.

The following somewhat more general result appears in [4], but we repeat the short proof here.

**Lemma 5.** Suppose $A, B \subseteq V_m$, with $A \supseteq A + B$, $|B| = d$, and $B$ linearly independent. Then $G(A, B)$ is a $Q_d$-factor of the complete graph on $A$.

**Proof.** Note that $G(\langle B \rangle, B) \cong Q_d$ by Theorem 4.

Now $A \supseteq A + B \supseteq (A + B) + B \supseteq \ldots$, and so $A \supseteq A + \langle B \rangle$, implying $A = A + \langle B \rangle$. Also if $a \in A$, then $G(a + \langle B \rangle, B) = a + G(\langle B \rangle, B) \cong Q_d$ by
the above. Furthermore the sets \( a + < B > \) for \( a \in A \) are cosets of \( < B > \), and so either identical or disjoint. Thus \( G(A, B) = G(A + < B >, B) = \bigcup_{a \in A} G(a + < B >, B) \), which is the vertex disjoint union of copies of \( Q_d \).

In [8] we prove a lemma (Lemma 3), which becomes the following when applied to \( V_m \).

**Theorem 6.** Let \( W \) be a subspace of \( V_m \), and let \( d_1, d_2, \ldots, d_t \) be integers with \( 1 \leq d_i \leq m \) for \( 1 \leq i \leq t \) and \( \sum_{i=1}^{t} d_i = |V_m \setminus W| \). Then \( V_m \setminus W \) can be partitioned into linearly independent sets \( X_1, X_2, \ldots, X_t \) such that \( |X_i| = d_i \) for \( 1 \leq i \leq t \).

Likewise Theorem 5 of [8] becomes the following when we take \( k = 2 \) and \( j = n = m \).

**Theorem 7.** Let \( d_1, d_2, \ldots, d_t \) be integers such that \( 1 \leq d_i \leq m \) for \( 1 \leq i \leq t \) and \( \sum_{i=1}^{t} d_i = 2^m - 1 \). Then \( K_{2^m} \) can be decomposed into a \( Q_{d_1} \)-factor, a \( Q_{d_2} \)-factor, \ldots, and a \( Q_{d_t} \)-factor.

### 3. Main Results

**Theorem 8.** Let \( d, a \) and \( b \) be integers with \( 0 < d \leq a < b \) such that \( 2^a - 1 \equiv 2^b - 1 \equiv r \pmod{d} \), where \( 0 \leq r < d \). Then \( K_{2^b} \setminus K_{2^a} \) can be written as a \( Q_r \)-factor on the non-hole vertices plus a graph divisible by \( Q_d \).

**Proof.** Let \( W \) be the subspace of \( V_b \) consisting of all vectors \((x_1, x_2, \ldots, x_b)\) such that \( x_1 = x_2 = \ldots = x_{b-a} = 0 \). Clearly \( W \) has \( 2^a \) vectors and is isomorphic to \( V_a \). We will take the vertex set of \( K_{2^b} \setminus K_{2^a} \) to be \( V_b \), with hole \( W \).

Let \( 2^a - 1 = qd + r \). By Theorem 6 we can partition \( W \setminus \{0\} \) into linearly independent sets \( B_1, B_2, \ldots, B_q, R \), with \( |B_i| = d \) for all \( i \) and \( |R| = r \), and partition \( V_b \setminus W \) into linearly independent \( d \)-sets \( C_1, C_2, \ldots, C_s \), where \( s = (2^b - 2^a)/d \).

Note that the hypotheses of Lemma 5 on \( A \) and \( B \) apply to each graph \( G(V_b \setminus W, R) \), \( G(V_b \setminus W, B_i) \), and \( G(V_b, C_i) \). Thus the graph \( G(V_b \setminus W, R) \) is a \( Q_r \)-factor of the complete graph on \( V_b \setminus W \), and the graphs \( G(V_b \setminus W, B_i) \), and \( G(V_b, C_i) \) are \( Q_d \)-factors of the complete graphs on \( V_b \setminus W \) and \( V_b \), respectively, for all appropriate \( i \).
Now we claim that the graph $K_{2^b} \setminus K_{2^a}$, interpreted as the complete graph on $V_b$ with all edges internal to $W$ removed, consists of the $r$-factor $G(V_b \setminus W, R)$ of $V_b \setminus W$ along with $\bigcup_{i=1}^q G(V_b \setminus W, B_i) \bigcup \bigcup_{i=1}^s G(V_b, C_i)$.

If $A$ and $B$ satisfy the hypotheses of Lemma 5, then the graph $G(A, B)$ contains $|A||B|/2$ edges. Thus $G(V_b \setminus W, R)$, $G(V_b \setminus W, B_i)$, and $G(V_b, C_i)$ contain $(2^b - 2^a)r/2$, $(2^b - 2^a)d/2$, and $2^b d/2$ edges, respectively. Then

$$G(V_b \setminus W, R) \bigcup \left( \bigcup_{i=1}^q G(V_b \setminus W, B_i) \right) \bigcup \left( \bigcup_{i=1}^s G(V_b, C_i) \right)$$

contains

$$\frac{(2^b - 2^a)r}{2} + \frac{(2^b - 2^a)d}{2} + \frac{2^b d}{2} = \frac{(2^b - 2^a)(2^a - 1)}{2} + \frac{(2^b - 2^a)2^b}{2}$$

$$= \frac{2^b(2^b - 1) - 2^a(2^a - 1)}{2}$$

edges, which is the correct number of edges in $K_{2^b} \setminus K_{2^a}$. Thus it suffices to show that if $x$ and $y$ are distinct elements of $V_b$, but not both in $W$, then the edge $\{x, y\}$ is included in the above union. We can assume that $x \notin W$.

First assume that $y - x \in W$. Then $y - x$ is in $R$ or $B_i$ for some $i$, and $\{x, y\}$ is an edge of $G(V_b \setminus W, R)$ or $G(V_b \setminus W, B_i)$, respectively.

Now assume that $y - x \notin W$. Then $y - x \in C_i$ for some $i$, and $\{x, y\}$ is an edge of $G(V_b, C_i)$.

The following is Theorem 4 of [7]

**Theorem 9.** There exists a $d$-cube decomposition of $K_{x, 2^d - 1, 2y, 2^d - 1}$ for all positive integers $x$, $y$, and $d$.

**Theorem 10.** Let $d$ and $a$ be integers with $d$ odd and $0 < d \leq a$ such that $2^d - 1 \equiv r \pmod{d}$, where $0 \leq r < d$. Let $s$ be the order of 2 modulo $d$ and set $b = a + s$. Then for any nonnegative integer $k$, $K_{2^a + k(2^b - 2^a)}$ can be decomposed into a $Q_r$-factor and a graph divisible by $Q_d$.

**Proof.** Let $2^a - 1 = dq + r$. Then by Theorem 7 the graph $K_{2^a}$ can be decomposed into a $Q_r$-factor and $q Q_d$-factors. Likewise by Theorem 8 the graph $K_{2^b} \setminus K_{2^a}$ can be written as a $Q_s$-factor on its nonhole vertices plus a graph divisible by $Q_d$. Let $2^a - 1 = dt$. Then by Theorem 9 with $x = y = 2^{a-d+1} k$ the graph $K_{2^{a-d} 2^b - 2^a}$ is divisible by $Q_d$. 


Now consider the vertex set of $K_{2^a+k(2^b-2^a)}$ to be partitioned into a $2^a$-set $X$ and $k \cdot (2^b-2^a)$-sets $Y_1, Y_2, \ldots, Y_k$. We can consider $K_{2^a+k(2^b-2^a)}$ as the union of the complete graph $K_{2^a}$ on $X$, $k$ complete graphs with holes $K_{2^b} \setminus K_{2^a}$ on the sets $X \cup Y_i$ with hole $X$, and $\binom{k}{2}$ complete bipartite graphs $K_{2^b-2^a, 2^b-2^a}$ with bipartite sets $Y_i$ and $Y_j$, $i \neq j$. By the previous paragraph these graphs taken together decompose into a $Q_r$-factor and a graph divisible by $Q_d$. 

Now we can show that if $d$ is odd there exists an infinite arithmetic progression of integers $n$ such that $Q_d$ divides $K_n$.

**Theorem 11.** Let $d$ be any odd positive integer, let $s$ be the order of $2$ modulo $d$ and let $t$ be the least integer not less than $d/s$. Then $Q_d$ divides $K_n$ where $n = 2^{st} + k(2^{st+s} - 2^{st})$.

**Proof.** We take $a = st$ in Theorem 10. Then $r = 0$ and so only $d$-cubes are involved in the decomposition. 

**References**


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