STOCHASTIC SUBSPACE IDENTIFICATION
GUARANTEEING STABILITY AND MINIMUM PHASE

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Abstract: This paper presents a stochastic subspace identification algorithm to compute stable, minimum phase models from a stationary time-series data. The algorithm is based on spectral factorization techniques and a stochastic subspace identification method via a block LQ decomposition (Tanaka and Katayama, 2003c). Two Riccati equations are solved to ensure both stability and minimum phase property of resulting Markov models.

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1. INTRODUCTION

Stochastic subspace identification algorithms compute stochastic state space systems from a finite string of a time-series data (Van Overschee and De Moor (1993; 1996)), where the numerical operations include not only the singular value decomposition (SVD) and QR decomposition, but also computation of a stabilizing solution of an associated Riccati equation.

Lindquist and Picci pointed out that stochastic subspace identification algorithms (Aoki, 1990; Van Overschee and De Moor, 1993) may fail in solving the Riccati equation, since the failure is related to a non-trivial problem of positivity in the stochastic realization theory, where an essential part of the problem is equivalent to the covariance extension problem (Lindquist and Picci, 1996a). Some stochastic subspace identification methods therefore have been developed taking positive realness into account (Van Overschee and De Moor, 1996; Mari et al., 2000; Goethals et al., 2003).

In order to review the use of Riccati equation in the context of the stochastic realization theory, we have re-visited stochastic realization theory (Tanaka and Katayama, 2003a), and obtained a finite-interval realization method based on a block LQ decomposition (Tanaka and Katayama, 2003b). Furthermore, we have proved that an approximate innovation representation due to (Maciejowski, 1996) is of minimum phase under the idealized assumption that a finite complete covariance data is given (Tanaka and Katayama, 2004b). It should be noted that this fact implies that a minimum phase model is obtained without solving Riccati equations in the idealized case.

Adapting the finite-interval realization method via a block LQ decomposition (Tanaka and Katayama, 2003b) to a finite time-series data, we have presented a stochastic subspace identification algorithm (Tanaka and Katayama, 2003c). The algorithm, however, does not guarantee that the identified forward innovation representation is stable and of minimum phase. Based on a spectral factorization technique (SFT), we have developed a prototype algorithm to obtain a minimum phase model (Tanaka and Katayama, 2004a).

In this paper, using Riccati equations for Kalman filters, we give an explicit algorithm to obtain a stable,
minimum phase model. A numerical simulation result is also shown.

2. PROBLEM SETTING

2.1 Problem statement

Consider a second-order stationary process \{y_t, t = 0, \pm 1, \pm 2, \ldots\}, where \(y_t\) is a \(p\)-dimensional non-deterministic process with mean zero and covariance matrices

\[ A_k = E \{ y_{t+k} y_T^T \}, \quad k = 0, \pm 1, \pm 2, \ldots \]  

(1)

where a set of covariance matrices \(\{ A_k | k = 0, \pm 1, \ldots \} \) is a positive real sequence: \(\sum_{i,j} u_i^T A_{i-j} u_j > 0, u_i \neq 0\). We assume that there exists a finite dimensional realization for \(y_t\), so that the covariance matrix has a decomposition \(A_k = H F^{k-1} \Gamma, k = 1, 2, \ldots\), where \((F, \Gamma, H)\) is a minimal realization with \(F \in \mathbb{R}^{n \times n}\) stable. The spectral density of the stationary time series \(y_t\) is given by \(\varphi(z) = \sum_{j=-\infty}^{\infty} A_j z^{-j}\), and it has a canonical spectral factorization,

\[ \varphi(z) = \hat{W}(z) \hat{W}^T(z^{-1}), \]

where \(\hat{W}(z)\) is stable and of minimum phase.

Given a finite time series data \(\{y_0, y_1, \ldots, y_{n+2\tau-2}\}\) \((\tau > n)\), our problem is to estimate a forward innovation representation of \(y_t\), or estimate \(\hat{W}(z)\), where the model must be stable and of minimum phase.

2.2 Innovation representation

Define a vector space as

\[ \mathcal{Y} := \left\{ \sum a_T^T y_k | a_k \in \mathbb{R}^p, k = 0, \pm 1, \pm 2, \ldots \right\}, \]

which is a linear space spanned by all finite linear combinations of row vector of \(y_t\). Define a bilinear form (inner product) as

\[ \langle a_T^T y_i, b_T^T y_j \rangle := a^T E \{ y_i y_j^T \} b = a^T A_{i-j} b. \]  

(2)

By completing the vector space \(\mathcal{Y}\) with the norm induced by the inner product (2), we get a Hilbert space (Lindquist and Picci (1996a, 1996b)), which is also written as \(\mathcal{Y}\). For \(U \subseteq \mathcal{Y}\) and \(z \in \mathcal{Y}\), \(\hat{E}(z | U)\) expresses an orthogonal projection of \(z\) onto \(U\). The notation \(\hat{E}(z | U)\) is also used for a vector \(z = [z_1, z_2, \ldots]^T\), the symbol will then just denote the vector with components \(\hat{E}(z_j | U)\).

Define the past and future matrices, \(Y^-_t\) and \(Y^+_t\), respectively as

\[ Y^-_t = \begin{bmatrix} y_{t-1} \\ y_{t-2} \\ \vdots \\ y_{t-3} \end{bmatrix}, \quad Y^+_t = \begin{bmatrix} y_t \\ y_{t+1} \\ \vdots \\ y_{t+2} \end{bmatrix}. \]

We also define matrices \(\Phi := E\{ (Y^-_t)'(Y^-_t)^T \}, \Psi := E\{ (Y^+_t)'(Y^+_t)^T \}\) and \(\mathcal{H} := E(\{Y_t\}'(Y_t)^T)\). The block Hankel matrix \(\mathcal{H}\) has a decomposition \(\mathcal{H} = \mathcal{OC}\) such that \(C \Phi^{-1} C^T = C^T \Psi^{-1} \mathcal{O}\) (Desai et al., 1985).

The matrices \(\mathcal{O} \in \mathbb{R}^{\infty \times n}\) and \(\mathcal{C} \in \mathbb{R}^{\infty \times n}\) are extended observability and reachability matrices, respectively, which are described as

\[ \mathcal{O} = \left[ C^T \left( CA \right)^T \left( CA^2 \right)^T \cdots \right]^T, \]

\[ \mathcal{C} = \left[ G AG A^2 G A^3 G \cdots \right], \]

using stochastically balanced matrices \(A \in \mathbb{R}^{n \times n}\), \(G \in \mathbb{R}^{n \times p}\) and \(C \in \mathbb{R}^{p \times n}\) (Desai et al., 1985), where \(A_k = C A^{k-1} G\) holds.

Consider the following Riccati equation (Fauvre, 1976)

\[ P = APA^T + (G - APC)^T \times (A_0 - CPC T)^{-1} (G - APT)^T. \]  

(3)

Using the stabilizing solution of (3), define

\[ \hat{K} = (G - APT)(A_0 - CPC T)^{-1}, \]

\[ \hat{R} = A_0 - CPC T. \]

(4)

(5)

Defining \(\hat{x}_t = C \Phi^{-1} Y^-_t\), we have \(\hat{E}(Y^+_t | Y^-_t) = \mathcal{O} \hat{x}_t\), where \(\hat{E}(Y^+_t | Y^-_t) := \hat{E}(Y^+_t | \text{span}(Y^-_t))\), and \(\text{span}(Y^-_t)\) is a subspace of \(\mathcal{Y}\) spanned by row vectors of \(Y^-_t\).

**Proposition 1.** (Desai et al., 1985) The forward innovation representation of \(y_t\) is given by

\[ \hat{x}_{t+1} = A \hat{x}_t + \hat{K} \hat{v}_t, \]

\[ \hat{y}_t = C \hat{x}_t + \hat{v}_t, \]

(6a)

(6b)

where \(\hat{v}_t\) is stationary Gaussian defined as \(\hat{v}_t := y_t - C \hat{x}_t\), and its variance \(\hat{R} = E(\hat{v}_t \hat{v}_t^T)\).

From (6), we have a whitening filter of \(y_t\),

\[ \hat{x}_{t+1} = (A - \hat{K} C) \hat{x}_t + \hat{K} \hat{y}_t, \]

\[ \hat{v}_t = -C \hat{x}_t + \hat{y}_t. \]

(7)

(8)

It should be noted that \(\hat{v}_t\) is obtained from \(y_t\) via a stable whitening filter, since \(A - \hat{K} C\) is stable.

3. STOCHASTIC SUBSPACE IDENTIFICATION METHOD

We review a subspace identification method (Tanaka and Katayama (2003c; 2004a)).

3.1 Assumptions

Define a vector space spanned by all finite linear combinations of vectors \(\eta \in \mathbb{R}^{1 \times n}\) as \(\mathcal{Y}^\eta\). For \(\alpha\) and \(\beta \in \mathcal{Y}^\eta\), define an inner product as \(\langle \alpha, \beta \rangle = \hat{E}(\alpha \beta^T)\). The vector space equipped with the norm induced by the inner product \(\langle \cdot, \cdot \rangle\) is an inner product space, which is also written as \(\mathcal{Y}^\eta\). We extend \(\mathcal{Y}^\eta\) to \(\mathcal{Y}^{\mathbb{R}^n \times \mathbb{R}^n}\) so that matrices are included as its elements.
Define a matrix
\[ \tilde{y}_t := \begin{bmatrix} y_t & y_{t+1} & \cdots & y_{t+\nu-1} \end{bmatrix} \in \mathbb{R}^{\nu \times 1} \]
for \( t = 0, 1, \ldots, 2\tau - 1 \), and define matrices as
\[ \tilde{Y}_t = \begin{bmatrix} \tilde{y}_1 \\ \tilde{y}_2 \\ \vdots \\ \tilde{y}_t \end{bmatrix}, \quad \tilde{Y}_s^+ = \begin{bmatrix} \tilde{y}_{s-2} \\ \tilde{y}_{s-1} \\ \vdots \\ \tilde{y}_s \end{bmatrix} \tag{9} \]
for \( t = 1, \ldots, 2\tau \), and for \( s = 0, \ldots, 2\tau - 1 \). Define also incomplete covariance matrices as
\[ \tilde{\Phi}_t := \langle \tilde{Y}_s^+, \tilde{Y}_s^+ \rangle, \quad \tilde{\Psi}_s := \langle \tilde{Y}_s^+, \tilde{Y}_{s+1}^+ \rangle \]
and also define \( \tilde{\mathcal{H}}_t := \langle \tilde{Y}_s^+, \tilde{Y}_s^+ \rangle \). We assume \( \langle \tilde{Y}_0^+, \tilde{Y}_0^+ \rangle > 0 \) and rank \( \tilde{\mathcal{H}}_t = \tilde{n} < \tau \).

### 3.2 Identification algorithm

Compute the standard LQ decomposition
\[ \frac{1}{\sqrt{p}} \tilde{Y}_0^+ = LQ^T, \tag{10} \]
where \( \tilde{Y}_0^+ \) is defined in (9). Partition \( L \) as
\[ L = \begin{bmatrix} L_{0,0} & 0 \\ \vdots & \vdots \\ L_{2\tau-1,0} & L_{2\tau-1,2\tau-1} \end{bmatrix} = \begin{bmatrix} L_{pp} & 0 \\ L_{fp} & L_{ff} \end{bmatrix}, \]
where \( L_{i,j} \in \mathbb{R}^{p \times p} \) and \( L_{pp}, L_{fp}, L_{ff} \in \mathbb{R}^{p \times p} \).

Define a matrix as
\[ D_L := \text{block-diag}(L_{0,0}, \ldots, L_{2\tau-1,2\tau-1}), \]
where \( D_L \) is non-singular from assumptions.

Define matrices as
\[ \tilde{L}_0^+ := LD_L^{-1}, \tag{11} \]
\[ \tilde{R}_0^+ := DL_D L_T. \tag{12} \]

The following equations are then obtained
\[ \tilde{\psi}_t = L_{fp} L_{fp} + L_{ff} L_{ff}, \tag{13} \]
\[ \tilde{\psi}_0^+ = \text{block-diag}(\tilde{R}_0, \ldots, \tilde{R}_{2\tau-1}), \tag{14} \]
\[ \tilde{L}_0^+ = \begin{bmatrix} L_{0,0} & 0 \\ \vdots & \vdots \\ L_{2\tau-1,0} & L_{2\tau-1,2\tau-1} \end{bmatrix}, \tag{15} \]
where \( \tilde{R}_t \in \mathbb{R}^{p \times p}, \tilde{L}_{i,j} = L_{i,j}L_{i,j}^{-1}, \) and \( \tilde{L}_{i,i} = I_p \).

We summarize a stochastic subspace identification algorithm (Tanaka and Katayama, 2004a).

### A stochastic subspace identification algorithm

**Step 1:** Compute the standard LQ decomposition (10) and define \( \tilde{L}_0^+ \), \( \tilde{R}_0^+ \) and \( \tilde{\psi}_t \) as (11), (12) and (13), respectively.

**Step 2:** Calculate the SVD
\[ (\tilde{\psi}_t)^{\dagger} \tilde{L}_{ff} = \tilde{U}_t \tilde{\Sigma}_t \tilde{V}_t, \tag{16} \]
where \( \tilde{U}_t^T \tilde{U}_t = I, \tilde{V}_t^T \tilde{V}_t = I \) and rank \( \tilde{\Sigma}_t = \tilde{n} \).

Based on the SVD (16), define \( \tilde{\Omega}_t \) as
\[ \tilde{\Omega}_t = (\tilde{\psi}_t)^{\dagger} \tilde{U}_t \tilde{\Sigma}_t. \tag{17} \]

**Step 3:** Compute \( \tilde{C} \) and \( \tilde{A} \) by
\[ \tilde{C} = \tilde{\Omega}_t(1 : p), \]
\[ \tilde{A} = \tilde{\Omega}_t^T \tilde{\Omega}_t(p + 1 : p\tau,:), \]
where \( \tilde{\Omega}_t^{-1} := \tilde{\Omega}_t(1 : p\tau - 1,:), \) and \( (\cdot)^\dagger \) denotes the Moore-Penrose pseudo-inverse.

**Step 4:** Define \( \tilde{R}_t \) from (14), and compute \( \tilde{K}_t \) from
\[ \tilde{K}_t = \tilde{\Omega}_t^T \left[ \begin{array}{c} L_{t+1, t+1} \tilde{L}_{t+1, t+1}^T \\ L_{t+2, t+2} \tilde{L}_{t+2, t+2}^T \\ \vdots \\ L_{2\tau-1, 2\tau-1} \tilde{L}_{2\tau-1, 2\tau-1}^T \end{array} \right], \]
where \( L_{t+1, t+1}, L_{t+2, t+2}, \ldots, L_{2\tau-1, 2\tau-1} \) are found in the matrix \( D_L^2 \) in (15).

Define transfer functions
\[ \tilde{\gamma}_t(z) := \tilde{W}_t(z) \tilde{W}_t^T(z^{-1}), \tag{18} \]
\[ \tilde{W}_t(z) := (\tilde{C}(zI - \tilde{A})^{-1} \tilde{K}_t + I) \tilde{R}_t^T. \tag{19} \]

It can be shown that the transfer function \( \tilde{\gamma}_t(z) \) is positive real, and is a good approximation to the true spectral density \( \tilde{T}(z) \) for large \( \nu \) and \( \tau \) (Tanaka and Katayama (2003c; 2004a)). It is however not guaranteed that \( \tilde{W}_t(z) \) is stable and of minimum phase.

### 4. SPECTRAL FACTORIZATION TECHNIQUE

In this section, we summarize an SFT based on Riccati equations for Kalman filters.

Consider the following linear stochastic system
\[ x_{t+1} = Ax_t + w_t, \tag{20a} \]
\[ y_t = Cx_t + v_t, \tag{20b} \]
where \( A \in \mathbb{R}^{n \times n}, \) and variables \( w_t \) and \( v_t \) are stationary Gaussian with zero mean and variance
\[ \mathbb{E}\left( \begin{bmatrix} w_s \\ v_s \end{bmatrix} \begin{bmatrix} w_t \\ v_t \end{bmatrix}^T \right) = \begin{bmatrix} Q & S \\ S^T & R \end{bmatrix} \delta_{st} \]
with \( R > 0 \). We assume that covariance matrices of the system (20) is also given by \( \mathbb{E}\{y_{t+kh}y_t^T\} = A_k, \) and this assumption implies that \( A_k = CA_k^{-1}G \) holds, where \( G = \mathbb{E}\{x_{t+1}y_t^T\}, \) and that the spectral density function of \( y_t \) is given by \( \tilde{T}(z) = \sum_{j=-\infty}^{\infty} A_j z^{j-1}, \) which is positive real.

Assume that we can observe \( y_t \) and consider the problem of estimating \( x_t \) which minimize \( \mathbb{E}[\|x_t - \hat{x}_t\|^2], \) where \( \hat{x}_t \) is an estimate of \( x_t \) based on \( \{y_{t-1}, y_{t-2}, y_{t-3}, \ldots\}. \) It is well known that such a "\( \hat{x}_t \)" is given by the Kalman filter. Associated with this problem, consider the following Riccati equation for the Kalman filter
\[ \Xi = A \Xi A^T - (A \Xi C^T + S) \times (C \Xi C^T + R)^{-1}(A \Xi C^T + S)^T + Q. \tag{21} \]

Assume here that \( Q, S \) and \( R \) satisfies \( Q = SR^{-1}S^T. \)

This implies that \( \Xi = 0 \) is a solution of (21), and that there exists \( K \) such that
\[ \begin{bmatrix} Q & S \\ S^T & R \end{bmatrix} = \begin{bmatrix} K & I \\ I & K \end{bmatrix}^T. \]
We also assume that there are no eigenvalues $\lambda_i$ of $A - KC$, such that $\lambda_i \lambda_j = 1 \ (i \neq j)$ or $\lambda_i = 0$.

In order to solve the Riccati equation (21), define $M$ and $N$ as

$$ M := \begin{bmatrix} (A - SR^{-1}C)^T & 0 \\ SR^{-1}S^T - Q & I_n \end{bmatrix}, \quad N := \begin{bmatrix} I_n & C^T R^{-1}C \\ 0 & A - SR^{-1}C \end{bmatrix}. $$

**Proposition 2.** Consider an eigenvalue problem

$$ \lambda Nz = Mx. \tag{22} $$

Assume that $(C, A)$ is observable, and that $(A, Q)$ is stabilizable. If $\lambda$ satisfies (22), then there exists $z$ such that

$$ \lambda Mz = Nz \tag{23} $$

holds. This implies that $1/\lambda$ is also an eigenvalue of (22).

**Proposition 3.** (Arnold and Laub, 1984) Consider a generalized eigenvalue problem:

$$ M \begin{bmatrix} W_1 \\ W_2 \end{bmatrix} = N \begin{bmatrix} W_1 \\ W_2 \end{bmatrix} \Lambda, \tag{24} $$

where $\Lambda \in \mathbb{R}^{n \times n}$ has a Jordan form. Then, the solution of Riccati equation (21) is given by $W_2 W_1^{-1}$.

From Proposition 2 and assumptions, there exists $\Lambda$ whose every diagonal element satisfies $|\lambda_i| < 1$, and we define it as $\hat{\Lambda}$. Using $\Lambda$ in (24), we define a solution $\hat{\Xi}$, from Proposition 3.

**Proposition 4.** Define matrices $\hat{K}$ and $\hat{R}$ as

$$ \hat{K} = (A\hat{\Xi}C^T + S)(C\hat{\Xi}C^T + R)^{-1}, \quad \hat{R} = C\hat{\Xi}C^T + R. \tag{25} \tag{26} $$

Then, a set of every eigenvalue of $A - \hat{K}C$ coincides with the set of diagonal element of $\Lambda$.

Proposition 4 implies that $A - \hat{K}C$ is stable, and the matrix $\hat{\Xi}$ is a stabilizing solution of (21).

**Proposition 5.** Suppose that $\hat{\Xi}$ is a stabilizing solution, and define $\hat{K}$ as (25). Then, $A - \hat{K}C$ is stable.

**Proposition 6.** The transfer function $T(z)$ satisfies

$$ T(z) = W(z) W^T(z^{-1}) = \hat{W}(z) \hat{W}^T(z^{-1}), \tag{27} $$

where $W(z)$ and $\hat{W}(z)$ are given by

$$ W(z) := \begin{bmatrix} C(zI - A)^{-1}K + I \end{bmatrix} R^\frac{1}{2}, \quad \hat{W}(z) := \begin{bmatrix} C(zI - A)^{-1}\hat{K} + I \end{bmatrix} \hat{R}^\frac{1}{2}, \tag{28} \tag{29} $$

where $\hat{K}$ and $\hat{R}$ are given by (25) and (26), respectively.

Proposition 6 implies that a minimum phase factor $\hat{W}(z)$ is obtained from $W(z)$ satisfying (27) based on the stabilizing solution of the Riccati equation (21).

5. **STABLE, MINIMUM PHASE MODEL**

We obtain a stable, minimum phase model from $\hat{W}_r(z)$ in (19) based on the SFT.

5.1 **Enforcing stability**

Assume that $\hat{A}$ is unstable, $(\hat{A}^T, \hat{K}^T)$ is detectable, and that no eigenvalues of $\hat{A}$ are on the unit circle or on the origin in the complex plane. We derive a stable spectral factor, using the inverse of $\hat{T}_r(z)$,

$$ \hat{T}_r^{-1}(z) = W_{r}^{-T}(z^{-1})\hat{W}_r^{-1}(z), $$

where $W_{r}^{-T}(z^{-1})$ is given by

$$ W_r^{-T}(z^{-1}) = \left(-\hat{K}_r^T(zI - \hat{F}^T)^{-1}\hat{C}^T + I\right)\hat{R}_r^{-\frac{1}{2}}, \tag{30} $$

$$ \hat{F} := \hat{A} - \hat{K}\hat{C}. $$

Since zeros of $W_{r}^{-T}(z^{-1})$ are eigenvalues of $\hat{A}^T$, we can find a stable factor, applying Proposition 6 to $W_{r}^{-T}(z^{-1})$ as shown below.

Define matrices as

$$ \begin{bmatrix} \hat{Q} & \hat{S} \\ \hat{S}^T & \hat{R} \end{bmatrix} := \begin{bmatrix} \hat{C}^T \\ I \end{bmatrix} \hat{R}_r^{-1} \begin{bmatrix} \hat{C}^T \\ I \end{bmatrix}^T. \tag{31} $$

Then, the Riccati equation associated with (30) is obtained by

$$ \hat{z} = \hat{F}^T \hat{z} \hat{F} - (-\hat{F}^T \hat{z} \hat{K}_r + \hat{S}) \times (\hat{K}_r^T \hat{z} \hat{K}_r + \hat{R})^{-1}(-\hat{F}^T \hat{z} \hat{K}_r + \hat{S})^T + \hat{Q}. \tag{32} $$

**Lemma 1.** Define matrices $\hat{C}$ and $\hat{A}$ as

$$ \hat{C} := (\hat{K}_r^T \hat{z} \hat{K}_r + \hat{R})^{-1}(-\hat{F}^T \hat{z} \hat{K}_r + \hat{S})^T, \quad \hat{A} := \hat{F} + \hat{K}_r \hat{C}. \tag{33} \tag{34} $$

using a stabilizing solution of Riccati equation (32). Then, $\hat{A}$ is stable.

**Theorem 1.** A spectral factorization of $\hat{T}_r(z)$ is given by

$$ \hat{T}_r(z) = \hat{W}_r(z) \hat{W}_r^T(z^{-1}), $$

where $\hat{W}_r(z)$ is stable and given by

$$ \hat{W}_r(z) := (\hat{C}(zI - \hat{A})^{-1}\hat{K}_r + I)\hat{R}_r^\frac{1}{2}, \tag{35} $$

$$ \hat{R}_r := K_r^T \hat{z} \hat{K}_r + \hat{R}. \tag{36} $$

If $\hat{A}$ is stable, a stabilizing solution of (32) is given by $\hat{z} = 0$ from $\hat{C} = \hat{R}_r \hat{S}^T$, and hence we have

$$ \hat{A} := \hat{A}, \quad \hat{C} := \hat{C}, \quad \hat{R}_r := \hat{R}_r. \tag{37} $$

5.2 **Enforcing minimum phase**

Assume that $\hat{A} - \hat{K}_r \hat{C}$ is unstable, and that no eigenvalue of $\hat{A} - \hat{K}_r \hat{C}$ is on the unit circle or on the origin.
It should be noted that \((\hat{A}, \hat{C})\) is detectable, since \(\hat{A}\) is stable. Define matrices as
\[
\begin{bmatrix}
\hat{Q} & \hat{S} \\
\hat{S}^T & \hat{R}_r
\end{bmatrix} := \begin{bmatrix}
\hat{K}_r \\
I
\end{bmatrix} \hat{R}_r \begin{bmatrix}
\hat{K}_r \\
I
\end{bmatrix}^T.
\]
(38)
Consider the following Riccati equation
\[
\ddot{Z} = \hat{A} \dot{Z} \dot{Z}^T - (\hat{A} \dot{Z} \hat{C}^T + \dot{S})
\times (\hat{C} \dot{Z} \hat{C}^T + \hat{R}_r)^{-1}(\hat{A} \dot{Z} \hat{C}^T + \dot{S})^T + \dot{Q}.
\]
(39)
Lemma 2. Define \(\hat{K}_r\) as
\[
\hat{K}_r := (\hat{A} \dot{Z} \hat{C}^T + \dot{S})(\hat{C} \dot{Z} \hat{C}^T + \hat{R}_r)^{-1}.
\]
(40)
In terms of a stabilizing solution of Riccati equation (39). Then, \(\hat{A} - \hat{K}_r \hat{C}\) is stable.
Theorem 2. A spectral factorization of \(\hat{T}_r(z)\) is given by
\[
\hat{T}_r(z) = \hat{W}_r(z)\hat{W}_r^T(z)^{-1},
\]
where \(\hat{W}_r(z)\) is defined as
\[
\hat{W}_r(z) := (\hat{C}(z I - \hat{A})^{-1} \hat{K}_r + I)\hat{R}_r^{\frac{1}{2}},
\]
(41)
\[
\hat{R}_r := \hat{C} \dot{Z} \hat{C}^T + \hat{R}_r.
\]
(42)
The transfer function \(\hat{W}_r(z)\) is stable and of minimum phase.

If \(\hat{A} - \hat{K}_r \hat{C}\) is stable, a stabilizing solution of (39) is given by \(\dot{Z} = 0\) from \(\hat{K}_r = \hat{S} \hat{R}_r^{-1}\), and we therefore have
\[
\hat{K}_r := \hat{K}_r, \quad \hat{R}_r := \hat{R}_r.
\]
(43)

5.3 Stochastic subspace identification algorithm

We summarize a stochastic subspace identification algorithm which provides a stable, minimum phase model based on Theorems 1 and 2.

A new subspace identification algorithm

Steps 1-4: Compute Steps 1-4 in the stochastic subspace identification algorithm in Section 3.

Step 5: If \(\hat{A}\) is unstable, find a stabilizing solution of Riccati equation (32) to define \(\hat{C}_r\), \(\hat{A}\) and \(\hat{R}_r\) as (33), (34) and (36), respectively. If \(\hat{A}\) is stable, define \(\hat{C}_r\), \(\hat{A}\) and \(\hat{R}_r\) as (37).

Step 6: If \(\hat{A} - \hat{K}_r \hat{C}\) is unstable, find a stabilizing solution of Riccati equation (39) to define \(\hat{K}_r\) and \(\hat{R}_r\) as (40) and (42), respectively. If \(\hat{A} - \hat{K}_r \hat{C}\) is stable, define \(\hat{K}_r\) and \(\hat{R}_r\) as (43).

Steps 5 and 6 give a stable, minimum phase model \(\hat{W}_r(z)\), which can be used as an approximation to \(\hat{W}(z)\), by solving two Riccati equations (32) and (39). It should be noted that Steps 5 and 6 can be calculated by means of a Matlab function whose inputs are \((\hat{F}, \hat{C}^T, -\hat{K}_r, \hat{R}_r)\) and \((\hat{A}, \hat{K}_r, \hat{C}, \hat{R}_r)\), respectively.

The computational time of the proposed algorithm compares favorably with former stochastic subspace identification algorithms (Mari et al., 2000; Goethals et al., 2003); in fact, the computation tasks needed to guarantee stability and minimum phase property in Steps 5 and 6 are only solving Riccati equations, while the former algorithms (Mari et al., 2000; Goethals et al., 2003) use numerical optimization methods in order to take positive realizness into account.

6. NUMERICAL SIMULATION

We present a simulation result to explain feasibility of the subspace identification method proposed in this paper. Simulated data is generated by a system \(s_t = \hat{W}(z)e_t\), where \(e_t\) is a white noise with zero mean and unit variance, and \(\hat{W}(z)\) is given by
\[
\hat{W}(z) = \hat{W}_N(z)/\hat{W}_D(z),
\]
\[
\hat{W}_N(z) = 1.0 \times 10^{-3} + 0.0090z^{-1} + 0.0081z^{-2}
+ 0.0073z^{-3} + 0.0066z^{-4} + 0.0059z^{-5},
\]
\[
\hat{W}_D(z) = 1 - 2.6908z^{-1} + 4.3502z^{-2} - 4.2269z^{-3}
+ 2.5542z^{-4} - 0.8714z^{-5}.
\]
We estimated the system for 30 simulation runs carried out with different noise realizations where \(\tau = 12, \nu = 3,000\) and \(n = 7\). We confirmed that \(\hat{W}_r(z)\) computed through Steps 1-6 in Section 5 is stable and of minimum phase in every simulation, while we obtain only 6 stable, minimum phase models for \(\hat{W}_r(z)\) based on Steps 1-4 in Section 3.

Figure 1 shows Bode plots of the systems \(\hat{W}_r(z)\) estimated by the present method for 30 simulations. Bode plots of \(\hat{W}_r(z)\) are clustered around \(\hat{W}(z)\).

Figure 2 shows plots of poles and zeros of \(\hat{W}_r(z)\) and \(\hat{W}_r(z)\) in a sample of 30 simulations. We observe that

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\(\ddot{X} = \hat{A}\ddot{X}\ddot{X}^T + \dot{Q}\).

Define \(\hat{A}_0\) and \(\hat{G}\) as
\[
\hat{A}_0 = \hat{C} \hat{X} \hat{C}^T + \hat{R}_r, \quad \hat{G} = \hat{A} \hat{X} \hat{C}^T + \dot{S}.
\]
Find a stabilizing solution \(\hat{P}\) of the following Riccati equation:
\[
\hat{P} = \hat{A}\hat{P}\hat{A}^T + (\hat{G} - \hat{A} \hat{X} \hat{C}^T)
\times (\hat{X} - \hat{C} \hat{P} \hat{C}^T)^{-1}(\hat{G} - \hat{A} \hat{X} \hat{C}^T)^T
\]
(44)
Define \(\hat{K}_r\) and \(\hat{R}_r\) as
\[
\hat{K}_r = (\hat{G} - \hat{A} \hat{X} \hat{C}^T)(\hat{A}_0 - \hat{C} \hat{P} \hat{C}^T)^{-1},
\]
(45)
\[
\hat{R}_r = \hat{A}_0 - \hat{C} \hat{P} \hat{C}^T.
\]
(46)
Riccati equation (44) is always solvable from (38). Moreover, \(\hat{K}_r\) and \(\hat{R}_r\) in (45) and (46) coincide with the ones in (40) and (42), respectively, since stabilizing solutions for (39) and (44), \(\hat{P}\) and \(\hat{Z}\), satisfy \(\hat{P} = \hat{X} - \hat{Z}\).
only unstable poles and zeros of \( \tilde{W}_T(z) \) are reflected into the unit circle.

Fig. 1. Bode plots of \( W(z) \) and \( \tilde{W}_T(z) \) where the dotted line expresses the plots of \( \tilde{W}_T \) for 30 simulation runs.

Fig. 2. Poles and zeros of \( \tilde{W}_T(z) \) and \( \tilde{W}_T(z) \), where ”×” and ”+” in (a) express poles of \( W_T(z) \) and \( \tilde{W}_T(z) \), respectively, and where ”○” and ”◦” in (b) express zeros of \( \tilde{W}_T(z) \) and \( W_T(z) \), respectively.

7. CONCLUSIONS

We developed a stochastic subspace identification method which guarantees stability and minimum phase property. Two Riccati equations are solved to find a stable, minimum phase model \( \tilde{W}_T(z) \). A model \( W_T(z) \) is obtained without solving Riccati equations, though it is not always stable and of minimum phase.

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REFERENCES