Abstract—The notion of Optimal System Design [12] holds that in order to ‘truly’ maximize/minimize an objective function, the feasible set needs to be optimized. Inspired by it, the attempt in our recent work [11] was to incorporate constraint-reduction in our earlier proposed procedures on dimensionality reduction of objectives [4], [10]. In that, while targeting constrained single-objective optimization problems (SOPs), we could arrive at a critical set of constraints and also their importance based rank-ordering. This information was used to study the shift from the constrained to the unconstrained optima. The methodology above was based on treating the a priori stated constraints as objectives besides the original-objective, and on applying [4], [10] to this combined objective set—but—without constraints. In this work, the endeavor is to extend the above notion to the realm of multi-objective optimization problems (MOPs). Towards it, while we hire much from the above methodology, we make a fundamental shift, in that, we retain the a priori stated constraints, while evaluating the combined objective set. The motivation for this shift lies, in that, it allows more effective realization of the notion of System Design than the approach in [11]. Reasonable effort has been spent on establishing this argument. Incorporating this change, a procedure for simultaneous reduction in objectives and constraints (for both SOPs, MOPs) is proposed, which also defines a realizable path towards Optimal System Design. Finally, the procedure is demonstrated on two test problems and one real world problem.

Keywords: Evolutionary Multi-objective Optimization (EMO), Multi-objective Optimization Problems (MOPs), Pareto-optimal front (P.O.F.), Principal Component Analysis (PCA), Computational (E)valuation, System Design.

I. INTRODUCTION

The real test of the maturity of the rapidly evolving field of evolutionary multi-objective optimization (EMO) would lie in its ability/ inability to address real world problems in business, science and engineering. It is easy to identify that many of the real world problems either involve large number of objectives or large number of constraints or both. However, given the EMO’s current state-of-the-art, addressal of such problems is a major challenge. We had demonstrated in [4] that the major difficulties associated with large number of objectives relate to stagnation of search process due to high percentage of non-dominated solutions, computational demands that rival any algorithm and difficulty in visualization of the objective space. Given this, we shifted focus to those large-objective (M) problems, which degenerate to possess a lower-dimensional (lower than M) P.O.F.. Towards targeting the collapse of the dimensionality at the P.O.F., we hired from the domain of dimensionality reduction and proposed both linear [4] and nonlinear [10] dimensionality reduction procedures (jointly representing block-I, in Figure1). The efficacy of these methods was tested up to 50-objective MOPs and results obtained were found to be promising.

Fig. 1. Link and comparison of present work with [4], [10], [11]

It was then deliberated, if the dimensionality reduction procedures for objectives could be generalized to accommodate constraint reduction, as well. Several issues that would call for an answer, if such a generalization were possible, were contemplated. For instance, as to what would a critical constraint set mean, in analogy to a critical objective set [4], [10]; what would possibly be the gain in such endeavor and what would be at stake. While the gain associated with a reduced representation of constraints could be seen in elimination of unnecessary checking and savings in storage space, the stake was far too enormous. So because, the solutions corresponding to the reduced constraint set could be infeasible w.r.t. the original problem and allowing any infeasibility is in stark contrast to the several approaches ([12]) on handling of constraints using evolutionary algorithms.

Given this, the question was, if at all the constraint reduction needs to be considered. However, we figured support for the possibility/need for constraint reduction, under two sets of arguments.

1) First, that (i) not all constraints may be hard (ii) not all constraints may equally important, that violation of any one be seen alike the violation of the rest. (iii) some of the constraints could just be approximate fixations of some preferences.

2) Second justification comes from the notion of System Design, which holds that in order to ‘truly’ maxi-
mize/minimize an objective function, the feasible set needs to be optimized. Clearly, it calls for exploring the gain in objective(s), by relaxing some constraints. Given this, a beginning was made with single-objective many-constraint problems and in [11], the existing procedures in [4], [10] were customized to accommodate constraints. Building up on it, the contribution of this work is that it describes a procedure for simultaneous reduction in objectives and constraints (for both SOPs, MOPs) and that it also defines a realizable path towards Optimal System Design (block-II, III, V in Figure1). The reason why [11] is not marked as representing block-II in Figure1, is where the approach in this work is different from [11] (not just an extension from SOPs to MOPs). While this issue is later discussed in details, let us mention that it relates to the different manner in which System Design has (a) been interpreted, (b) influenced the respective procedures.

II. COMPONENTS OF THE EARLIER PROPOSED DIMENSIONALITY REDUCTION ALGORITHMS

As stated above, the present work is based on generalizing the earlier proposed dimensionality reduction procedures. Hence, we present the main components of [4], [10] collectively, as follows:

1) Eigenvalue Analysis: This involves the eigen-decomposition of the correlation matrix (R) for linear dimensionality reduction [4]; correntropy matrix (V) or learned kernel matrix (K) for nonlinear dimensionality reduction [10].

2) Interpreting Multiple Principal Components: To make the dimensionality-reduction procedure effective and applicable to various scenarios, we suggest the following detailed procedure, which starts with analyzing the the first principal component and then proceed to analyzing the second principal component and so on, till all the significant components are considered. For this purpose, we pre-define TC—a threshold cut (details in [4],[10]) and when the cumulative contribution of the top principal components would exceed TC, we would not analyze any more principal components. For the first principal component, along with the objective corresponding to the most-positive element, we consider as important, any/all objectives which correspond to a negative component, howsoever small. If in some case, all the elements along PCA-1 are positive, we pick up the objectives corresponding to the first two most positive elements. For subsequent principal components, we first check if the corresponding eigenvalue is greater than 0.1 or not. If yes, we only choose the objective corresponding to the highest absolute element in the eigenvector. If yes and also if the cumulative contribution of eigenvalues is less than TC, we consider various cases. If all elements of the eigenvector are positive, we only choose the objective corresponding to the highest element. If all elements of the eigenvector are negative, we choose all objectives. Otherwise, if the value of the highest positive element (p) is less than the absolute value of the most-negative element (n), we check if \( p \geq 0.9|n| \). If yes, then we choose two objectives corresponding to p and n, otherwise we choose only the objective corresponding to n. Similarly, we also consider the possibility of the absolute value of the most-negative element (n) being less than the highest positive element (p), in which case we further check if \( |n| \geq 0.8p \). If yes, we choose both objectives corresponding to p and n, otherwise we only choose the objective corresponding to p.

3) Final Reduction Using the Selection Scheme for Reduced Correlation Matrix: Hopefully, the above procedure identifies many of the redundant objectives. To investigate the possibility of further reduction, we compose the reduced correlation matrix, by the columns and rows corresponding to objectives adjudged as important by Eigenvalue analysis, above. In this matrix, subsets of objectives are identified such that its any two members are positively correlated with each other and on top of it, these members have identical correlations with all the rest members. If such subsets exist, then any only member of them, could serve as the representative. Consider an identically correlated set of S objectives, each being represented by \( f_i \), \( i = 1 \ldots S \). Further, assume that V principal components were utilized for Eigenvalue analysis each being represented by \( v_j \), \( j = 1 \ldots V \) and each accounting for a proportion \( \epsilon_j \), \( j = 1 \ldots V \), towards the total variance of the data set. Let the contribution of \( f_i \) along \( v_j \) be represented by \( f_{ij} \). Then compute for each of the identically correlated objectives \( f_i \) the value \( c_i = \sum_{j=1}^{V} ||(f_{ij} \cdot e_j)|| \). Pick from the set \( \{c_i | i = 1 \ldots S\} \) of scalars the highest value, say \( c_k \). Then the objective \( f_k \) can be considered as a representative of the set of S objectives. However, if all values in set \( \{c_i\} \) turn to be equal, pick that objective as a representative, whose contribution along \( v_1 \) is highest. On the whole, this selection criterion physically implies that an objective which contributes most along the important principal components collectively, deems fit to be the representative.

III. ON TRANSLATING CONSTRAINTS TO OBJECTIVES AND SUBSEQUENT NOTATIONS, TERMINOLOGY USED

In Section II.-2, it is evident, how [4], [10] utilize the correlation-based notion of conflict towards interpreting the important principal components. It is this feature, that we will exploit for generalization of [4], [10] to accommodate constraint reduction.

A. Constraint-guided Multi-objectivization

It is easy to identify that any constraint which actually restrains the feasible search space, in effect prohibits the realization of the unconstrained P.O.F., for the stated objectives. Hence, if the constraints be treated as objectives besides the original-objectives, then:

1) corresponding to the constraints which restrain the feasible search space, the constraint-objectives will be in conflict with the original-objectives. This conflict will naturally show up along the principal components.

\(^1\)an addition only for constraint-objectives: if even \( v_1 \) fails to discriminate, then pick the constraint-objective with highest mean value, as it represents the constraint which restrains the search space most
2) if [4] or [10] be applied on the combined set of the original and the constraint-objectives, then a critical set comprising of all/some of both—the original-objectives and the constraint-objectives, could be arrived at.

3) further, if the importance of a constraint-objective can be interpreted in terms of the importance of its underlying constraint, then given a constrained multi-objective optimization problem, a critical objective set and a critical set of constraints could be identified.

Working along the same lines, our endeavor in [11] was to arrive at a reduced constraint set for SOPs. There, we had referred the redefining of constraints as objectives by the term multi-objectivization. However, in this paper, we take the opportunity to re-coin it as constraint-guided multi-objectivization, to allow its distinction with the term multi-objectivization coined by Knowles et al. [5].

### B. On notations and terminology used

We first mention the notations to be used, to represent the initial set of constraints and objectives. Here, it is assumed that the objectives are formulated towards minimization, while constraints, presented in \( \leq 0 \) form:

- **Initial constraint set:** \( \partial_0 = \{g_1, \ldots, g_I\} \)
- **Initial original-objective set:** \( \partial_0 = \{f_1, \ldots, f_M\} \)
- **Initial constraint-objective set:** \( \partial_0 = \{g_1, \ldots, g_J\} \)
- **Initial enhanced-objective set:** \( \partial_0 = \{g_1, \ldots, g_J\} \), where: \( \partial_0 = \{g_1, \ldots, g_J\} \)
- **Initial extended-objective set:** \( \partial_0 = \{g_1, \ldots, g_J\} \), where: \( \partial_0 = \{g_1, \ldots, g_J\} \)
- **Initial total-enhanced-objective set:** \( \partial_0 = \{g_1, \ldots, g_J\} \), where: \( \partial_0 = \{g_1, \ldots, g_J\} \)

### Critical constraint set (\( \partial_C \))

- Redundant constraint set (\( \partial_R \)): set of constraints which allow all such value assignments of the variables, that the objective functions achieve the best trade-offs possible over the entire variable space (or the absolute maxima/minima, in case of a single objective). For example, \( \{g_1\} \) in Figure 2(a).

### Critical constraint set (\( \partial_C \))

- Critical constraint set (\( \partial_C \)): set of constraints, whose corresponding constraint-objectives are found irreducible by the dimensionality reduction procedure. This set will essentially comprise of constraints which significantly confine the feasible search space, say \( \{g_1, g_2, g_3\} \) in Figure 2(a). However, given the prescription in Section II-3, \( \{g_2, g_3\} \) will be eliminated from this set by virtue of their correlation with \( g_1 \), w.r.t., \( f \). Hence only \( \{g_1\} \) will constitute the critical constraint set.

### Non-critical constraint set (\( \partial_N \))

- Non-critical constraint set (\( \partial_N \)): set of constraints, whose corresponding constraint-objectives are adjudged as unimportant by the dimensionality reduction procedure. For example, \( \{g_2, g_3, g_4\} \) in Figure 2(a).

### User-defined critical constraint set (\( \partial_U \))

- User-defined critical constraint set (\( \partial_U \)): set of constraints, which a user possessing some problem knowledge may assess as essential. Again, as is symbolically shown in figure 2, the critical constraint set itself may become the user’s set or the user may want to add/relax some more constraints over/from it.

### Tightest Constraint (\( g_r \))

- Tightest Constraint (\( g_r \)): one which restrains the search space the most, as compared to any other single constraint. Hence, most important too.

### Loosest Constraint (\( g_l \))

- Loosest Constraint (\( g_l \)): one which restrains the search space the least, as compared to any other single constraint. Hence, least important too.

### Feasible Space (\( X_F \))

- Feasible Space (\( X_F \)): the originally feasible search space, governed by all constraints

### Extended Space (\( X_E \))

- Extended Space (\( X_E \)): the entire variable space, governed by the variable bounds

Before leaving this section, let us highlight that while providing \( \partial_C \), the to-be-proposed procedure will also rank-order all the constraints from the tightest to the loosest. For instance, for the case in Figure 2(a), the rank-order will be \( \{g_1, g_2, g_3, g_4\} \). It will be shown in the later sections, how this rank-ordering information will help towards the controlled exploration of \( X_E \) starting with \( X_F \). Let the readers also note here [9], where the authors have employed similar rank-ordering for Constraint Satisfaction Problems (CSPs).

### IV. Procedure for Dimensionality Reduction for Objectives and Constraints: Towards System Design

We now build upon the basis laid in Sections II, III to present the procedure for simultaneous dimensionality reduction in objectives and constraints (for both SOPs and MOPs). The methodology employed paves a realizable3 way for System Design. Given an \( M \)-objective, \( J \)-constraint optimization problem, the procedure is, as follows:

1. **Step 1: Set** an iteration counter \( t = 0 \),

2. **Step 2: Set** an iteration counter \( t = 0 \),

3. **Step 3: Compute** the critical constraint set (\( \partial_C \))

4. **Step 4: Identify** the critical objective set (\( \partial_C \))

5. **Step 5: Rank-order** the constraints

6. **Step 6: Reduce** the search space

7. **Step 7: Solve** the reduced problem

8. **Step 8: Repeat** until convergence

### Fig. 2. Highlighting the basis of definitions of redundant, critical constraint and rank-ordering of constraints

Next, we introduce the terminology around which this work is based and also the corresponding notations. Reference to Figure 2(a), which represents a single-objective, four-constraint optimization problem, will make it clear that the presence of constraint \( g_1 \) is inconsequential. Further, each of \( g_2, g_3 \), if seen independent of others, restrict the feasible search space. Of them, \( g_1 \) confines the feasible search space most and hence prohibits most, the achievement of unconstrained minima (\( f \)). It is followed by \( g_2 \) and then \( g_3 \). However, given \( g_1 \), the effect of \( g_2 \) and \( g_3 \) ceases to exist. With this illustrative case, we define:

- **Critical objective set (\( \partial_C \))**: set of objectives which are found as irreducible by the dimensionality reduction procedure.

Footnotes:

3. note, they are constraints in \( \leq 0 \) form, to be minimized

4. a subject of discussion, in next section
the initial sets: $\hat{0}_0$, $U_0$, $G_0$, $I_0$ (Section III-B)
empty arrays: $M_0$ (of dimension $M$) and $I_0$
( of dimension $J$), to store the elements $\in \hat{0}_0$
and $\hat{0}_0$, respectively, in increasing order of their
importance.

Step 2: Initialize a random population for all objectives
in the set $I_0$, subjected\(^\text{a}\) to the constraint set $\hat{0}_0$; run an
EMO, and obtain a population $P_t$.

Step 3: Apply the dimensionality reduction procedure us-
ing $\parallel_t$ to deduce a reduced set of objectives $I_{t+1}$
using the predefined TC. Steps are as follows:

1) Compute the desired matrix: $R$ [4], $V$ or $K$
[10], depending on the nature of dimensionality
reduction being sought; find the eigenval-
ues and eigenvectors.

2) Reduction-I: Eliminate objectives based on the
proposed interpretation scheme for eigen-
values and eigenvectors (Section II--2).

3) Reduction-II: Of the objectives retained in
item 3.2, eliminate further, if possible, the
objectives, based on the proposed selection
scheme (Section II--3). Here, members of the
set $\parallel_t$ are to be reduced only for their
correlation with members of this set. Alike
holds for $G_t$. Else, errors\(^\text{b}\) will creep in.

4) Apply the proposed selection scheme on
collective set of objectives eliminated in
item 3.2, 3.3.

   Sort these objectives in increasing order of
importance; separately identify the original
and constraint-objectives, to obtain updated
$M_t$ and $J_t$, respectively.

Step 4: If $I_{t+1} = \parallel_t$: Apply the proposed selection
scheme on this irreducible objective set and update $M_{t+1}$
and $J_{t+1}$ Separately note the original and con-
straint objectives in this irreducible set, for they
are the critical ones. STOP
Else: Set $t = t + 1$ and go to Step 2.

V. ON INTERPRETING SYSTEM DESIGN: ENHANCED
OBJECTIVE SET—‘TO BE’ OR ‘NOT TO BE’ CONSTRAINED

Here, we compare the approach in [11] (Section III((4)-
Step 2)) with the approach adopted in the proposed procedure
(Section IV-Step 2). Notably, while the former treats the
enhanced-objective set $I_0$ as unconstrained, the latter subjects
$I_0$ to $\hat{0}_0$.

Before we justify this fundamental shift, let us have a
realistic view on how the notion of System Design can be realized in practice. This calls for answers to:

1) The reference with respect to which, the notion of
System Design can be explored: Clearly, any evalu-
ation of the gain in objectives against the violation
in constraints has to begin with the initially feasible
search space. Hence, the original-problem has to be
the reference.

2) The degree to which the constraint violations can be
accepted for the real world problems: It is easy to see
that relaxing too many constraints or even few—but-
tight constraints may lead to solutions which have no
practical significance. Hence, the degree is limited too.

3) The mode in which constraint relaxations can be
investigated: Even, relaxing all the constraints, one
by one in arbitrary order would see a transition from the
constrained to the unconstrained P.O.F. However,
studying here, the corresponding gain in objectives at
each step, would be nothing more than a meaningless
exercise. It naturally emphasizes how important it is
to observe a controlled exploration of $X_F$ starting with
$X_F$. It is here that the rank-ordering of constraints from
loosest to tightest as contained in $J_{t+1}$ (Section IV-Step
4) may bear utility, in the following ways:

   • The decision-maker (DM) may begin with $U_C$,
$\hat{0}_C$ and note the corresponding trade-off between
the gain in objectives ($\Delta f$) and the constraint
violations ($\nabla \overline{0}$).

   – if the DM assesses $\nabla \overline{0}$ as too large to be
allowed, then he could further restrain the
search space by incorporating more and more
constraints from the set $U_{C_I}$, by reducing order
of their importance, until an acceptable balance
between $\Delta \overline{0}$ and $\nabla \overline{0}$ is realized.

   – if at first place, the DM assesses $\nabla \overline{0}$ to be low
enough to allow further exploration, then he
could relax more and more constraints from the
set $U_{C_I}$, by increasing order of their importance,
until an acceptable balance between $\Delta \overline{0}$ and $\nabla \overline{0}$
is realized.

   • or as an alternative, the DM, beginning with the
removal of the worst ranked constraint, may re-
move more constraints, in increasing order of their
importance until an acceptable balance between
$\Delta \overline{0}$ and $\nabla \overline{0}$ is identified.

Now, let us identify the basis on which the set $I_0$ was treated
as unconstrained in [11]. Towards it, consider Figure 3,
where the total variable space $X_E$, is shown bounded by
the constraint-lines $gL$ and $gU$. Here, say, the original-
problem were to minimize $f_2$ within $X_F$, defined by the
lines $gL$ and $gU$. The notion of System Design would here call
for (a) relaxing the constraint $g_1$, towards exploring
$X_E$ (b) observing the gain in $f_2$, at the cost of violation
in $g_1$ (c) identifying a solution, where the gain in $f_2$
is considered worthy of accepting the corresponding violation
in $g_1$. Generalization of the above, for all the a priori stated
constraints in a problem, formed the basis of unconstraining
the set $I_0$ in [11].

\(^{\text{a}}\) a deviation from [11]
\(^{\text{b}}\) for (i) $g_a \in G_t$, eliminated for its correlation with an element in $\hat{0}_t$
fares poorly in constraint rank-ordering by virtue of being eliminated, even if
it were better, based on the selection criteria (ii) a $g_a \in G_t$ were eliminated
for its correlation with $f_{a} \in \hat{0}_t$ and if $f_{a}$ were eliminated in subsequent
termination.
However, deliberation on the approach in [11], while investigating its extension for MOPs, exposed an anomaly in it. This anomaly emanates from the fact that the results obtained from this approach are representative of \( \mathcal{X}_E \) and not necessarily of \( \mathcal{X}_F \). The above argument can be realized by considering the two objectives \( \{f_1, f_2\} \) in Figure 3. Clearly, the two objectives are in conflict in \( \mathcal{X}_F \), while they are positively correlated in \( \mathcal{X}_E \). Now, any inference made w.r.t \( \mathcal{X}_E \) (the case in [11]), will lead to elimination of one of them, even when they are strongly in conflict in \( \mathcal{X}_F \) (the space w.r.t which notion of System Design is to be explored). However, equally probable is that even the principal components for \( \mathcal{X}_E \) and \( \mathcal{X}_F \) will be different. In some cases, it may happen that two different sets of principal components and correlations, for \( \mathcal{X}_E \) and \( \mathcal{X}_F \), respectively, may lead to the same \( \mathcal{O}_C, \mathcal{O}_C \) and even the same rank-orders on objectives and constraints, denoted by \( \mathcal{M}_{t+1} \) and \( \mathcal{J}_{t+1} \), respectively. While we illustrate this possibility in Section VI-A, it is something that cannot be guaranteed, which we show in Section VI-B. Hence, [11] was not shown to represent Block-II in Figure 1, even when it dealt with single-objective constrained problems.

VI. TEST PROBLEMS

Now we demonstrate the proposed procedure (Section IV) on two test problems and a real world problem. It had been established in [10], that MVU-PCA-NSGA-II based on learned kernel matrix (K), with the parameter k=M-1, is the most reliable version among our earlier proposed procedures. Hence, the learned kernel matrix K has been used for Step 3.1 in the proposed procedure. Further, to account for the randomness influences, towards Step 2 in the proposed procedure, three independent runs with NSGA-II (different random seed) were performed. Each time, a population size of 600 and 10,000 generations were employed (to provide a reasonable computational effort). The SBX crossover [3] with a probability of 0.9 and index 5 and polynomial mutation [3] with a probability of 0.1 and index of 20, were employed. It is worth mentioning that for the problems considered here, the results were found to be consistent across the different runs.

A. Test Problem-1: Comparing the two approaches

First we consider a simple two variable, two-objective and two-constraint problem [3], given by Equation 1. The intent here, is to highlight that despite the differences, the two approaches could lead to the same results.

\[
\begin{align*}
\text{Minimize} & \quad f_1(x) = x_1 \\
\text{Minimize} & \quad f_2(x) = (1 + x_2)/x_1 \\
\text{Subject to:} & \quad g_1(x) = 9x_1 + x_2 - 6 \geq 0 \\
& \quad g_2(x) = 9x_1 - x_2 - 1 \geq 0 \\
\text{where:} & \quad 0.1 \leq x_1 \leq 1, \quad 0.0 \leq x_2 \leq 5
\end{align*}
\]

Given Section IV, \( \mathcal{M}_0 = \{F_1, F_2, F_3, F_4\} \) where \( F_1 = f_1, F_2 = f_2, F_3 = G_1 \) and \( F_4 = G_2 \).

1) Based on the present approach (Section IV): Adopting the procedure in Section IV, it can be observed from the Table I(a) that based on eigenvalue analysis, the first three objectives are assessed important. Further, the Table I(b) shows that no further reduction is possible as the three objectives are uncorrelated. It implies that \( \mathcal{O}_C = \{f_1, f_2\} \) and \( \mathcal{O}_C = \{g_1\} \), while the rank-ordering of objectives and constraints is given by Tables I(c), I(d), respectively.

2) Based on the approach in [11], extended for MOPs: The approach in [11] on being extended for MOPs, gives the following results. The eigenvalue analysis, as shown in Table II(a) assesses all four objectives as important. Further, the Table II(b), interpreted on the basis of Section IV.3.3, highlights that while no further reduction is possible for original objectives, the constraint-objective \( g_2 \) is to be eliminated. To summarize, \( \mathcal{O}_C = \{f_1, f_2\} \) and \( \mathcal{O}_C = \{g_1\} \), while the rank-ordering of objectives and constraints is given by Tables II(c), II(d), respectively.

It can be seen that the results obtained from the above two approaches--\( \mathcal{O}_C, \mathcal{O}_C, M_0 \) and \( J_1 \) are the same. This may lead one to contemplate that two different sets of principal components and corresponding correlations, may still lead to the same results. We believe, that it may hold for some cases but the limitation in the approach in [11] would still remain in place, only to be reflected in some other problem (as we show in Section VI-B).

![Fig. 3. Highlighting: (i) the basis-of and the anomaly-in [11] (ii) need for distinction between correlations in \( \mathcal{X}_F \) and \( \mathcal{X}_E \)](image)
TABLE I
ITER. I. BASED ON THE APPROACH IN SECTION IV

(a) (i) Eigenvalue Analysis

<table>
<thead>
<tr>
<th>Iter. 1: PCA-1 (≈ 92.56% variance)</th>
<th>F₁</th>
<th>F₂</th>
<th>F₃</th>
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<tbody>
<tr>
<td>PCA-2 (≈ 97.42% variance)</td>
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(b) (ii) Red. Cor. Analysis

<table>
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<tr>
<th>F₁</th>
<th>F₂</th>
<th>F₃</th>
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<td>+</td>
<td>-</td>
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<td>+</td>
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<td>-</td>
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</table>

(c) (iv) M array

<table>
<thead>
<tr>
<th>M₁</th>
<th>f₁</th>
<th>f₂</th>
</tr>
</thead>
</table>

TABLE II
ITER. I. BASED ON THE APPROACH IN SECTION IV

(a) (i) Eigenvalue Analysis

<table>
<thead>
<tr>
<th>Iter. 1: PCA-1 (≈ 99.83% variance)</th>
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<th>F₂</th>
<th>F₃</th>
<th>F₄</th>
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<tr>
<td>PCA-2 (≈ 99.85% variance)</td>
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(b) (ii) Red. Cor. Analysis and Sel. Criteria

<table>
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<th>F₁</th>
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<th>F₃</th>
<th>F₄</th>
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<td>+</td>
<td>+</td>
<td>0.15</td>
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</table>

(c) (iv) M array

<table>
<thead>
<tr>
<th>M₁</th>
<th>f₁</th>
<th>f₂</th>
</tr>
</thead>
</table>

B. Test Problem OSY: Exploring the notion of Optimal System Design

First being considered here, is a problem used by Osyczka and Kundu [7], also discussed by Deb [3]. The Pareto-optimal front for this problem, defined by Equation 2, as shown in Figure 2(a). It turns out that the two objectives have no effect. The eigenvalue analysis in Iteration 1 (Table III(a)), suggests that all eight objectives are important. However, the correlation matrix (Table III(b)), shows two sets of correlated objectives. While, the first of these comprises of \{F₁, F₃, F₅, F₆, F₇\}, the second contains \{F₂, F₄, F₈\}. Given Section IV Step3(3), (a) F₅ and F₆ stand eliminated in Iteration 1 and are rank-ordered as shown in (Table III(d)) (b) the rest qualify for the next Iteration, making I₄ = \{F₁, F₂, F₃, F₄, F₅, F₆, F₇\}.

In Iteration 2, again the eigenvalue analysis (Table IV(a)) suggests all objectives to be important. Further, given Section IV Step3(3), (a) any further reduction based on the reduced correlation matrix (Table IV(b)), is negated. (b) rank-ordering arrays are updated as shown in Table IV(c), IV(d). Hence, the results can be summarized as \(\overline{\partial \mathbf{c}} = \{f₁, f₂\}\), \(\overline{\partial \mathbf{c}} = \{g₁, g₂, g₄, g₆\}\), with grading arrays being \(M₂ = \{f₂, f₁\}\) and \(J₂ = \{g₆, g₄, g₅, g₆, g₄\}\), respectively.

TABLE III
ITERATION 1

(a) (i) Eigenvalue Analysis

<table>
<thead>
<tr>
<th>Iter. 1: PCA-1 (≈ 94.54% variance)</th>
<th>F₁</th>
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<th>F₃</th>
<th>F₄</th>
<th>F₅</th>
<th>F₆</th>
<th>F₇</th>
<th>F₈</th>
<th>cᵢ</th>
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<tbody>
<tr>
<td>PCA-2 (≈ 95.45% variance)</td>
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(b) (ii), (iii) Reduced Correlation Analysis and Selection Criteria

<table>
<thead>
<tr>
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<th>F₂</th>
<th>F₃</th>
<th>F₄</th>
<th>F₅</th>
<th>F₆</th>
<th>F₇</th>
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(c) (iv) M array

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<th>f₂</th>
<th>f₃</th>
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TABLE IV
ITERATION 2

(a) (i) Eigenvalue Analysis

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<th>Iter. 2: PCA-1 (≈ 90.95% variance)</th>
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<th>F₇</th>
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(b) (ii), (iii) Reduced Correlation Analysis and Selection Criteria

<table>
<thead>
<tr>
<th>F₁</th>
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<th>F₃</th>
<th>F₄</th>
<th>F₅</th>
<th>F₆</th>
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(c) (iv) M array

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<tr>
<th>M₁</th>
<th>f₁</th>
<th>f₂</th>
<th>f₃</th>
<th>f₄</th>
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</thead>
</table>

1) Analysis of Results: A nice beginning point for the analysis of the results obtained, could be to compare the P.O.F. for the original problem with that corresponding to \(\overline{\partial \mathbf{c}}\). As highlighted in Figure 4(a), the P.O.F. for these two cases is the same. It suggests that the non-critical constraint set (\(\partial \mathbf{x}\)) has no restraining influence on the feasible search space. It could mean that either of the constraints in \(\overline{\partial \mathbf{c}}\) nullifies their effect (just as the constraint \(g₁\) did to \(g₂\), \(g₃\) in figure 2(a)). It turns out that the later is true. As can be seen from Figure 5(b), the effect of the constraints \(g₃\), \(g₅\) and \(g₆\) on the search space is the same. And, as \(g₇\) is already a part of the \(\overline{\partial \mathbf{c}}\), the rest two have no effect.

Further, towards assessing the meaningfulness of the constraint set inferred as \(\overline{\partial \mathbf{c}}\), the two objectives \{\(f₁, f₂\)\} were subjected to each of the six constraints—one at a time. It could be seen from Figure 4(c) and Figure 5, that

- the P.O.F. corresponding to \(g₂\), is the closest to the P.O.F. for the original problem, as compared to any other
constraint if applied alone. It suggests that $g_2$ is the constraint which restrains the search space, the most. This inference is confirmed by the fact that $g_2$ while constituting $C_5$, is also the best ranked constraint.

- the restraining effect of the six constraints, applied one a time, assumes one of the four different forms.

It is noteworthy that in $C_5$, each of these forms is represented.

2) Exploring the notion of System Design: The various modes in which the above notion could be explored were discussed in Section V. In this problem, it is perceived that even the case with only two most critical constraints, as shown in Figure 4(b), does not offer much gain in objectives. Hence, we investigate case with only the most critical constraint, namely, $g_2$. Some of the points that may interest the decision maker have been numbered from $\{1, \ldots, 9\}$ in Figures 4(c) and the corresponding points on the original P.O.F. are marked as $\{j, \ldots, o\}$. The trade-off between $\Delta(j)$ and $(\nabla o)$ can be realized from the Table V. For instance, compare the point 4 on the front P-Q-R-S-T with the point m on the front A-B-C-D-E-F. For the given $f_2$, a significant gain of approximately 170 can be made in $f_1$ at the cost of violation in $g_4$ and $g_6$, as shown in the table. Same holds for other points. Hence, if the DM may consider this gain as significant enough to allow the corresponding violations, then in that case the problem reduces to a two-objective, single-constraint problem.

3) Comparison with the approach in [11], extended for MOPs: Without discussing the details, let us mention the results corresponding to the approach in [11]. It leads to $C_5 = \{f_1, f_2\}$, $C_6 = \{g_1, g_4, g_6\}$, with grading arrays being $M_2 = \{f_2, f_1\}$ and $J_2 = \{g_1, g_4, g_6, g_4, g_2\}$, respectively. It can be seen that, while the two most important constraints ($g_2, g_4$) are correctly identified for the set $C_5$, $g_1$ whose distinct effect can be seen in Figure 5(c) is left out. Moreover, $g_3$ and $g_6$ which can be seen to be fully correlated to each other (Figure 5(b)) are both adjudged critical at the same time. This points to the anomaly we highlighted in Section V, in that these results are representative of $\mathcal{X}_E$ and not of $\mathcal{X}_F$ and hence, are not realizable.

C. Water Problem: Simultaneous Reduction in Objectives and Constraints

It is a five-objective, seven-constraint problem which relates to optimal planning for a storm drainage system in an urban area. This problem was described originally by Musselman and Talavage [6] and further attempted by Cheng and Li [1] and Ray et al. [8]. Its concise description, is as given below:
In Figure 6, we compare the value plots [3] obtained with five-objective NSGA-II run with seven constraints and then, without. It is easy to see that the constraints are such, that they only restrict the objective values at the higher end; making them redundant, as the objective optimization.

1) Examining the Redundancy of Constraints: Visual Assessment—Value plot: In Figure 6, we compare the value plots [3] obtained with five-objective NSGA-II run with seven constraints and then, without. It is easy to see that the constraints are such, that they only restrict the objective values at the higher end; making them redundant, as the objectives are being minimized. What is interesting to note is that, when the variable set resulting from the NSGA-II run with the claimed critical objectives, namely \( f_3 \) and \( f_4 \) is used for reconstruction of the other objectives, the resulting value plot is in good conformance with that of five-objective, seven-constraint problem. This establishes that the three objectives (\( f_1, f_2, f_3 \)) and the seven constraints are actually redundant.

![Fig. 6. Highlighting the Redundancy of three objectives and seven constraints](image)

TABLE VI

<table>
<thead>
<tr>
<th>(a) (i) Eigenvalue Analysis</th>
<th>(b) (ii) Red. Cor. Analysis</th>
</tr>
</thead>
<tbody>
<tr>
<td>PCA-1 (99.98% variance)</td>
<td>( F_3 )</td>
</tr>
<tr>
<td>( F_1 )</td>
<td>( F_4 )</td>
</tr>
<tr>
<td>( F_3 )</td>
<td>+</td>
</tr>
<tr>
<td>( F_4 )</td>
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</table>

VII. CONCLUSIONS

In this paper, a generic procedure for simultaneous dimensionality reduction of objectives and constraints, for both single and multi-objective optimization problems, is proposed. One of the key highlights here is that a potential source of error in our recent work [11] is identified and also rectified. The proposed procedure is demonstrated on two test problems and one real world problem. While it is encouraging to find the procedure having accurately and effectively reduced these problems, it remains important to maintain caution on inferences drawn on the constraint set. Barring the elimination of truly redundant constraints, allowing relaxation of any single constraint is a subjective issue and may necessarily call for the decision maker’s intervention. Nonetheless, it is hoped that this work lays a pathway for the meaningful realization of the notion of System Design and its call for optimizing the feasible set or designing the optimal.

ACKNOWLEDGEMENTS

The authors appreciate Arnab Sinha, research scholar at Department of Electrical Engineering, IIT Kanpur, for fruitful discussions with the first author. Authors also acknowledge the support provided by STMicroelectronics, Italy, Singapore and India for performing this study.

REFERENCES