Optimal control for control moment gyros — center-stable manifold approach

Kazuo Ishikawa and Noboru Sakamoto

Abstract—This paper proposes a framework for optimal control of mechanical systems possessing conserved quantities such as control moment gyros (CMGs). It will be shown that output regulation theory is appropriate to handle such constraints and that the center-stable manifold method for solving the associated Hamilton-Jacobi equation is effective for computing the optimal feedback control. With this framework, it is possible to design robust and high performance controller for CMGs. Simulations show the effectiveness of the proposed method.

Keywords: Optimal control, control moment gyro, conserved quantity, Hamilton-Jacobi equation, input saturation

I. INTRODUCTION

Among attitude control devise of spacecraft, there are external actuators such as thrusters or electrical jets and there are internal actuators such as reaction wheels (RWs) or control moment gyroscopes (CMGs) (see, e.g., [1]). Internal actuators are considered to be more suitable for accurate attitude control than external actuators. Especially, control moment gyroscopes (CMGs) are effective device for controlling the attitude of spacecraft for its ability to immediately provide large angular momentum to spacecraft. CMGs generate larger driving torque by tilting the angle of a gimbal on which a spinning wheel at high speed is mounted. This mechanism is called the gyro effect and can be explained by conservation of angular momentum. In real applications, CMGs are used as an attitude control device for large scale spacecraft, for example Skylab, space station Mir and International Space Station (ISS).

Due to its strong nonlinearity, controlling CMGs poses several challenges in nonlinear control theory (see, e.g., [2]). One of them is that a nonholonomic constraint (conserved quantity) exists in the CMG system and it is an example for which asymptotic stabilization by continuous feedback cannot be achieved. Another challenge in control of CMGs is controller design that guarantees better energy effectiveness. Since CMGs are used in isolated environments, where energy supply is limited, improvement of energy consumption in the control system leads to longer lifetime of the whole system. Another research direction of CMGs is the study of singular configurations of CMG arrays, where no control torque is generated along a certain direction [3], [4], [5]. It is known that every CMG array, irrespective of the number of CMGs, has singular configurations.

In this paper, we consider an optimal control design problem for nonlinear control systems that possess conserved quantities or constraints. We first propose a general framework for such systems, nonetheless, specific applications for CMGs and other mechanical systems are intended. Existing works for controlling this class of systems consider only specific initial configurations to fix the value of conserved quantity [6], [7], [2].

The main contribution of the paper is to show that the nonlinear output regulation framework [8], [9], [10], [11] is more appropriate to handle the class of systems by considering the conserved quantity as a signal from an exosystem. In this framework, signals from the exosystem are arbitrary in magnitude, allowing arbitrary value of the conserved quantity for our problem. The second contribution of the paper is to apply the center-stable manifold method [12] for solving the Hamilton-Jacobi equation for optimal control design, where center part corresponds to the exosystem or conserved quantities. If one reduces the system using the conservation law, a regular Hamilton-Jacobi equation arises and the approximation methods, e.g., the Taylor series or stable manifold methods [13], [14], [15], can be used, but, the resulting controller works only for the fixed value of the conserved quantity. It should be noted, however, that optimal control for CMGs is not found in the literature even with this reduction approach due to the difficulty of obtaining (approximate) solutions for Hamilton-Jacobi equation. The third contribution is to design high performance controllers incorporating input saturation. Input saturation is ubiquitous in mechanical control systems and can be a cause of windup phenomenon and one often decreases feedback gain to avoid such undesirable effects, resulting in low performance control systems. It is known that the stable manifold method [15] for solving Hamilton-Jacobi equation is able to handle non-analytic nonlinearities such as input saturation [16], [17] and we apply the center-stable manifold method, which is a generalization of [15], for the optimal control of CMGs. The current paper significantly extends the report [18] in that a rigorous problem formulation is established, controller performance is improved and that the robustness for friction torque is newly guaranteed.

II. OPTIMAL CONTROL FOR SYSTEMS WITH CONSERVED QUANTITY

A. Optimal partial stabilization

Consider a nonlinear system of the form

$$\dot{y} = \bar{f}(y) + \bar{G}(y)u, \quad y(t) \in \mathbb{R}^n, \quad u(t) \in \mathbb{R}^m,$$  (1)
where \( f : \mathbb{R}^n \rightarrow \mathbb{R}^n, \ G : \mathbb{R}^n \rightarrow \mathbb{R}^{n \times m} \) are smooth functions with \( f(0) = 0 \). Suppose that there exists a smooth function \( C : \mathbb{R}^n \rightarrow \mathbb{R}^p, \ p < n \) such that for (1), it holds that \( \frac{\partial}{\partial t} C(y(t)) = 0 \) for any \( u(t) \) and all \( t \in \mathbb{R} \). Denote \( \Omega_\mu = \{ y \in \mathbb{R}^n | C(y) = \mu \} \) for \( \mu \in \mathbb{R}^p \).

Assumption 2.1: There exist a set of indexes \( \{ i_1, \ldots, i_l \} \subset \{ 1, \ldots, n \} \) such that the intersection of the hyperplane \( H = \{ y_{i_1} = \cdots = y_{i_l} = 0 \} \) and \( \Omega_\mu \) defines a non-empty connected manifold for all \( \mu \in \mathbb{R}^p \) and \( H \cap \Omega_\mu \) is invariant for system (1) with \( u = 0 \).

This assumption means that one can expect merely partial stabilization due to the conserved quantity, but it is possible to formulate an optimal control problem as follows.

Definition 2.2 (Partial optimal stabilization problem): For system (1), find a control input such that the states of the closed loop system remain in a bounded neighborhood of the origin for all \( t \geq 0 \) and the performance index

\[
J = \int_0^\infty v^T Q v + u^T R u \, dt,
\]

is minimized, where \( v = (y_{i_1}, \ldots, y_{i_l}), \ Q \geq 0, \ R > 0 \) with dimensions \( l \times l, m \times m \), respectively.

Assumption 2.3: There exist \( n - p \) smooth functions \( \varphi_1, \ldots, \varphi_n-p : \mathbb{R}^n \rightarrow \mathbb{R} \) around the origin such that \( \Phi(y) = (\varphi_1(y) \cdots \varphi_{n-p}(y) \ c_1(y) \cdots c_p(y)), \) where we denote \( C = (c_1, \ldots, c_p), \) is a diffeomorphism around the origin.

With Assumption 2.3, in the new coordinates \( (x, w), \ x = (\varphi_1(y), \ldots, \varphi_{n-p}(y)), \ w = C(y), \) system (1) is represented as follows

\[
\begin{align*}
\dot{x} &= f(x, w) + G(x, w) u \\
\dot{w} &= 0.
\end{align*}
\]  (2)

We will consider the second equation in (2) as an exosystem and \( w \) an signal, which is actually a step signal, with arbitrary magnitude depending on initial conditions. For the new system (2), the performance index \( J \) is rewritten as

\[
J = \int_0^\infty v(x, w)^T Q v(x, w) + u^T R u \, dt.
\]

Let us denote the linearized matrices of the system (2) and \( J \) by

\[
A = \frac{\partial f}{\partial x}(0, 0), \quad P = \frac{\partial f}{\partial u}(0, 0), \quad B = G(0, 0),
\]

\[
V_x = \frac{\partial C}{\partial x}(0, 0), \quad V_w = \frac{\partial C}{\partial w}(0, 0).
\]

Assumption 2.4: There exists a matrix \( \Pi \in \mathbb{R}^{(n-p) \times p} \) satisfying

\[
0_{(n-p) \times p} = \Pi P + (n-p) \times 0 = V_x \Pi + V_w.
\]  (4)

B. Hamilton-Jacobi equation with input saturation

In this subsection, we will include input saturation into the system (1) (consequently into (2)) and derive a Hamilton-Jacobi equation for the optimal partial stabilization problem.

First, the input \( u \) in (1) and (2) is replace with

\[
sat(u) = \begin{bmatrix} sat_1(u_1) \\ sat_2(u_2) \\ \vdots \\ sat_m(u_m) \end{bmatrix}
\]

where

\[
sat_i(u) = \begin{cases} u_i^{\max} & (u_i^{\max} \leq u) \\ u_i^{\min} & (u_i^{\min} < u < u_i^{\max}) \\ u_i^{\min} & (u \\ leq u_i^{\min}) \\ u_i^{\max} & (i = 1, 2, \cdots, m) \end{cases}
\]  (5)

and \( u_i^{\max}, u_i^{\min} \) are the maximum and minimum values of the input channel \( i \).

The dynamic programming is applied to (2) with saturated input and the cost function \( J \) for Hamiltonian \( H_D \)

\[
H_D(x, w, p_x, p_w) = p_x^T (f + G sat(u)) + p_w^T \times 0 + v^T Q v + sat(u)^T R sat(u),
\]

where \( p_x, p_w \) are co-vectors corresponding to \( x, w \) parts, respectively. The minimizing input vector for \( H_D \) is given by

\[
\tilde{u}(x, w, p_x) = sat \left( \frac{1}{2} R^{-1} G^T p_x \right),
\]

and the Hamilton-Jacobi equation is derived as follows

\[
p_x^T \{ f + G sat(\frac{1}{2} R^{-1} G^T p_x) \} - v^T Q v \\
+ sat(\frac{1}{2} R^{-1} G^T p_x)^T R sat(\frac{1}{2} R^{-1} G^T p_x) = 0
\]  (6)

The associated Hamiltonian system with (6) is now

\[
\begin{align*}
\dot{x} &= Ax - Pw - RBp_x + N_1(x, w, p_x) \\
\dot{w} &= 0 \\
\dot{p}_x &= -2V_x^T QV_x x - 2V_x^T QV_w w - AT p_x \\
+ N_2(x, w, p_x) \\
\dot{p}_w &= -2V_w^T QV_x x - 2V_w^T QV_w w - PT p_x \\
+ N_3(x, w, p_x),
\end{align*}
\]  (7)

where \( R_B = \frac{1}{2} BR^{-1} B^T \) and \( N_1(x, w), N_2(x, w) \) and \( N_3(x, w) \) are higher order nonlinear terms appropriately computed. For example, the linear, nonlinear decomposition for \( sat(u) \) is based on the relation \( sat(u) = u + (sat(u) - u) = u + dz(u) \). The second term corresponds to nonlinear part, which sometimes can be denoted by the dead-zone function \( dz \).

With appropriate linear transformation, (7) can be put into the form with block-diagonalized linear parts. Then, Theorem 2.1 in [12] gives a parameterization of the center-stable manifold. In practice, the iteration in [12] is applied finite times and the parameterized manifold is represented via, for instance, multi-dimensional polynomial fitting or interpolation and we obtain the manifold

\[
p_x = p_x(x, w)
\]  (8)
in an approximate sense. This computation method has been applied for several systems with experimental verifications (see, for detail, [16], [12], [17]). The resultant optimal feedback controller is, using (8), given by

\[ u = \text{sat} \left( -\frac{1}{2}R^{-1}G^T(x,w)p(x,w) \right) \]  

(9)
in the \((x,w)\)-coordinates.

III. OPTIMAL STABILIZATION OF CONTROL MOMENT GYRO

A. Model of control moment gyro

We first describe the control moment gyro (CMG) unit by Educational Control Products (ECP), the schematic representation of which is depicted in Fig. 1. The CMG consists of 4 bodies, A, B, C and D, allowing 4 degree-of-freedom experiments. Body A is mounted on a platform in order that it rotates around the axis a. Body B is outer gimbal that rotates with respect to body A around the axis b. Inner gimbal is body C whose axis of rotation is the axis c. Finally, the roter, body D, spins around axis d. q1 is angle of the wheel whose initial value is 0, q2 is angle of between vertical axis and internal gimbal, q3 is angle of between horizontal axis and external gimbal, q4 is angle of body A with respect to the platform whose initial value is 0 and The angular velocity \( \omega_i \) \((i = 1, 2, 3, 4)\) is defined as \( \omega_i = \frac{dq}{dt} \).

In this paper, the relative position of body B to body A is locked, which means \( q_3 = 0 \). The roter spin torque \( T_1 \) is provided by a dc-motor and another dc-moter generate torque \( T_2 \) to body C with respect to body B.

Our control problem for this CMG is to control the position of Body C and angular velocity of Body A. Note that it is not possible to control Bodies A, C and D at the same time due to the conserved angular momentum around axis a. In other words, we wish to suppress the rotation of Body A and to drive the position of Body C to a given value while letting the velocity of Body D (roter speed) converge to the value that is determined from the conserved quantity depending on initial values of the whole system. Specifically, we consider the stabilization of \( q_2 = \frac{\pi}{2}, \omega_2, \omega_4 \).

Next, we derive the state equation including the conservation law. First, let us indicate Bodies A, C, D with suffixes a, c, d, respectively. Let \( I_a, J_a, K_c \) be the moments of inertia around axis \( \nu_1, \nu_2, \nu_3 \) \((\nu = a, c, d)\) respectively. For example, denote by \( J_d \) the moment of inertia of Body D around \( d_2 \) axis. Then, the Lagrangian \( L \) is written as follows

\[ L = \frac{1}{2} \left\{ J_d \omega_4^2 + (I_c + I_d) \omega_2^2 + (J_2 - J_1 \cos^2(q_2)) \omega_4^2 \right\} + J_d \omega_1 \omega_4 \sin(q_2), \]

where \( J_1 = J_c + J_d - K_c - I_d, J_2 = K_a + K_b + J_c + J_d \). The Euler-Lagrange equation is

\[
\frac{d}{dt} \left( \frac{\partial L}{\partial \dot{q}} \right) - \frac{\partial L}{\partial q} + T = 0. 
\]  

(10)

where \( T = [T_1 \ T_2 \ 0]^T \). The state space equation corresponding to (1) is derived by setting \( \dot{q}_i = \omega_i, i = 1, 2, 4 \) and solving (10) with respect to \( \dot{\omega}_i \). Since we are interested only in \( q_2, \omega_1, \omega_2, \omega_4 \) and \( q_1, q_4 \) does not appear in the equation, a 4-dimensional state space model corresponding to (1) is obtained.

On the other hand, one can see that \( L \) does not include \( q_4 \). This is the well-known condition for the existence of conserved quantity and the coordinate \( q_4 \) is called cyclic coordinate. Then, the corresponding generalized momentum \( p_4 \) is the conserved quantity. However, this quantity is hidden in the state space equation derived from (10). To take this quantity into the state equation, we employ the Legendre transformation to work in the framework of Hamiltonian mechanics. The generalized coordinates are \( q_1, q_2, q_4 \), which are CMG angle coordinate, and the generalized momentum \( p_1, p_2, p_4 \) corresponding to the generalized coordinates \( q = [q_1 \ q_2 \ q_4]^T \) are given by

\[
P = \begin{bmatrix}
p_1 \\
p_2 \\
p_4 \\
\end{bmatrix} = \begin{bmatrix}
\frac{\partial L}{\partial \dot{q}_1} \\
\frac{\partial L}{\partial \dot{q}_2} \\
\frac{\partial L}{\partial \dot{q}_4} \\
\end{bmatrix} = \begin{bmatrix}
J_d \omega_1 + J_d \omega_4 \sin(q_2) \\
(I_c + I_d) \omega_2 - \omega_4 \cos(q_2) \\
(J_2 - J_1 \cos^2(q_2)) \omega_4 + J_d \omega_1 \sin(q_2) \\
\end{bmatrix}.
\]

In addition, \( \dot{q} \) is written as follows using \( p_i, q_i (i = 1, 2, 4) \)

\[
\begin{bmatrix}
\dot{q}_1 \\
\dot{q}_2 \\
\dot{q}_4 \\
\end{bmatrix} = \begin{bmatrix}
\frac{\omega_1}{J_1} \\
\frac{\omega_2}{J_2} \\
\frac{\omega_4}{J_d} \\
\end{bmatrix} + \begin{bmatrix}
\frac{\sin(q_2)}{J_2} p_1 + p_4 J_1 \cos^2(q_2) - p_4 J_d \\
J_2 (J_1 \cos^2(q_2) - J_2 + J_d - J_d \cos^2(q_2)) + \frac{p_4}{J_d} I_d - \frac{p_4}{J_d} + p_4 \sin(q_2) p_3 \\
J_2 \cos^2(q_2) - J_2 + J_d - J_d \cos^2(q_2) \\
\end{bmatrix}.
\]

We define the Hamiltonian \( H_{cmg} \) as follows

\[ H_{cmg} = p^T \dot{q} - L \]  

(11)

and the Hamilton’s canonical equation is

\[
\begin{bmatrix}
\dot{q} \\
\dot{p} \\
\end{bmatrix} = \begin{bmatrix}
\frac{\partial H_{cmg}}{\partial q} \\
-\frac{\partial H_{cmg}}{\partial p} + T \\
\end{bmatrix}.
\]
i.e.,
\[
\begin{bmatrix}
\dot{q}_1 \\
\dot{q}_2 \\
\dot{q}_4 \\
\dot{p}_1 \\
\dot{p}_2 \\
\dot{p}_4
\end{bmatrix} = 
\begin{bmatrix}
\sin(q_2)J_d p_4 + p_1 J_1 (\cos(q_2))^2 - p_1 J_2 \\
J_d (J_1 (\cos(q_2))^2 - J_3 + J_4 - J_d (\cos(q_2))^2) \\
\frac{-p_4 + \sin(q_2) p_4}{J_1 (\cos(q_2))^2 - J_3 + J_4 - J_d (\cos(q_2))^2} \\
0 \\
0 \\
0
\end{bmatrix}
\begin{bmatrix}
f_5 \\
0 \\
0
\end{bmatrix}
+ 
\begin{bmatrix}
O_{3 \times 2} \\
I_2 \\
O_{1 \times 2}
\end{bmatrix}
\begin{bmatrix}
T_1 \\
T_2
\end{bmatrix},
\]

where
\[
f_5 = \cos(q_2) f_{51},
\]
\[
f_{51} = -\sin(q_2) \left\{ (J_4 - K_c) p_4^2 + J_3 p_4^2 \right\} + \left\{ J_3 + J_4 - K_c + (K_c - J_1) \cos^2(q_2) \right\} p_1 p_4, \\
f_{52} = \left\{ \cos^2(q_2) (K_c - J_4) - J_2 + J_d \right\}^2,
\]
and \( J_3 = K_a + K_b + K_c + K_d, J_4 = -J_c + K_b + K_c + K_d \).

Now, since we are not interested in \( q_1, q_4 \) in our control problem for this system, we obtain a 4-dimensional state equations
\[
\begin{bmatrix}
\dot{q}_2 \\
\dot{p}_1 \\
\dot{p}_2 \\
\dot{p}_4
\end{bmatrix} = 
\begin{bmatrix}
\frac{-p_4 + \sin(q_2) p_4}{J_1 (\cos(q_2))^2 - J_3 + J_4 - J_d (\cos(q_2))^2} \\
0 \\
0 \\
0
\end{bmatrix}
\begin{bmatrix}
f_5 \\
0 \\
0 \\
0
\end{bmatrix}
+ 
\begin{bmatrix}
O_{3 \times 2} \\
I_2 \\
O_{1 \times 2}
\end{bmatrix}
\begin{bmatrix}
T_1 \\
T_2
\end{bmatrix},
\]

(12)

which explicitly contains the conserved quantity and corresponds to (2) in the previous section. Our control problem is the stabilization of \( q_2 - \frac{\pi}{2}, \omega_2, \omega_4 \), therefore, we take the function \( v \) in Definition 2.2 as follows,
\[
v = \begin{bmatrix}
q_2 - \frac{\pi}{2} \\
\frac{-p_4 + \sin(q_2) p_4}{J_1 (\cos(q_2))^2 - J_3 + J_4 - J_d (\cos(q_2))^2}
\end{bmatrix}
\]

and equilibrium \( x_e \) is taken
\[
x_e = \begin{bmatrix}
q_{2e} \\
p_{1e} \\
p_{2e}
\end{bmatrix} = \begin{bmatrix}
\frac{\pi}{2} \\
0 \\
0
\end{bmatrix}.
\]

IV. SIMULATION RESULTS

Simulations are undertaken to see the effectiveness of the proposed controller and to compare the results with the standard linear optimal control. The simulation model is based on actual experimental apparatus shown in Fig. 2 and its physical parameters are in Table I.

\begin{table}[h]
\centering
\begin{tabular}{|c|c|}
\hline
\textbf{Symbol} & \textbf{Value} \\
\hline
\hline
\( J_1 \) & 0.020 kg \cdot m^2 \\
\hline
\( J_2 \) & 0.015 kg \cdot m^2 \\
\hline
\hline
\( J_3 \) & 0.024 kg \cdot m^2 \\
\hline
\( J_4 \) & 0.031 kg \cdot m^2 \\
\hline
\hline
\( J_2 \) & 0.162 kg \cdot m^2 \\
\hline
\end{tabular}
\caption{Physical parameters}
\end{table}

A. The model for simulation

In the actual experimental device, frictional torques exist on the rotating bodies and controllers must have certain robustness for the frictions. In our CMG, the friction torque along the \( \alpha_3 \) axis is evident and to incorporate this, the model is modified with
\[
\dot{p}_4 = -\mu \omega_4
\]
in stead of \( \dot{p}_4 = 0 \). The value of \( \mu \) identified by experiment and \( \mu = 0.4 \) and the response of the simulation model well coincides with experiments (see Figs 3, 4).

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{fig3.png}
\caption{\( p_4 \) response (experiment)}
\end{figure}

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{fig4.png}
\caption{\( p_4 \) response (simulation)}
\end{figure}

B. Controller evaluations

1) Optimal partial stabilizing controller: The optimal stabilizing controller that takes \( v \) in (13) to zero and minimizing the cost function (3) is designed with
\[
Q = \begin{bmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 10
\end{bmatrix}, \quad R = \begin{bmatrix}
1 & 0 \\
0 & 1
\end{bmatrix}.
\]

The center-stable manifold for the associated Hamiltonian system is computed applying the iteration in [12] 20 times \( (k = 20) \) after block-diagonalizing the Hamiltonian system.
The computed center-stable manifold \( p_x = p_x(x, w) \) is represented by third order polynomials of \( q_2, p_1, p_2 \) and \( p_4 \) and the feedback controller is given by (9).

2) Linear optimal controller: For comparison, a linear optimal controller is designed that is robust for the value of conserved quantity. The standard optimal servo controller design is used for the linearized system of (12) at the equilibrium, which is written as

\[
\dot{x} = Ax + Bu \\
\dot{w} = 0 \\
\bar{v} = V_x x + V_w w,
\]

where \( x = (q_2, p_1, p_2) \), \( w = p_4 \), \( \bar{v} = (q_2 - \pi/2, -p_4 + p_1 \sin(q_2))^T \) and the right side of the third equation in (15) is the linearization of \( \bar{v} \). Introducing \( z = \int_0^t v \, d\tau \), we have an augmented system with states \((x, z)\). The linear quadratic method is used for designing a feedback controller

\[
u = K_x x + K_z z.
\]

The gains \( K_x, K_z \) in (16) are computed as

\[
K_x = \begin{bmatrix} 0.0000 & -3.1623 & 0.0000 \\ -1.0000 & 0.0000 & -7.5593 \end{bmatrix} \\
K_w = \begin{bmatrix} 3.1623 \\ 0.0000 \end{bmatrix}.
\]

3) Comparison results: We show the simulation results for initial condition

\[
\begin{bmatrix} q_2(0) \\ p_1(0) \\ p_2(0) \end{bmatrix} = \begin{bmatrix} -1.0472 \\ -0.1073 \\ 0.0000 \end{bmatrix}, \quad p_4(0) = -0.1925,
\]

in which Body A, D are rotating at the beginning while the gimbal (Body C) is stationary with respect to Body A. The control objective is to regulate \( q_2 \to \pi/2, \omega_4 \to 0 \) in an optimal way within the limitation of input saturations.

In ideal situation where no frictional torque exists, the controller with fixed \( p_4 \) at the initial value would work well. In Figs. 5, 6, 7 and 8, the responses by the controller with \( p_x = p_x(x, p_4(0)) \) are shown. \( q_2 \) does not converge to \( \pi/2 \) and the control objective is not achieved. This shows that there is a possibility for most nonlinear control methods in the literature that fix the value of conserved quantity would not work well due to uncertainties and disturbances in actual systems.

To adapt the controller to arbitrary value of the conserved quantity or to gain the robustness for uncertainties and disturbances, the center-stable manifold of 4-dimension should be used for feedback control (note that the above controller with \( p_x = p_x(x, p(0)) \) corresponds to 3 dimensional manifold). Figs. 9, 10, 11 and 12 show the responses of the controllers by the center-stable method (NLOC) and linear optimal servo design (Linear). Both controllers are robust for the friction at \( q_4 \) and achieve the regulation of \( q_2 \to 0, \omega_4 \to 0 \), however, significant differences in their performances are evident. The convergences of \( q_2, \omega_4 \) are faster in NLOC (see Fig. 11). This is due to the use of maximum torques allowed in the system for longer duration (see Fig. 12). This design is possible because we have (numerically) solved the Hamilton-Jacobi equation (6) including saturations.

V. CONCLUSIONS

In this paper, we proposed a framework, using the theories of output regulation and center-stable manifolds, for optimal control of systems with conserved quantities. This class of systems and the proposed method are especially useful for controlling control moment gyroscopes (CMGs). For the design of optimal feedback control, the center-stable manifold method in [12] is shown to be effective in solving the associated Hamilton-Jacobi equation, where center part corresponds to the conserved quantity with arbitrary values. We have also considered the input saturation in the system allowing the controller to use the maximum torque without causing undesired responses such as windup. The proposed controller
exhibits robustness to frictions and excellent performance compared with linear optimal control.

REFERENCES


