A Complete Logic for Non-Deterministic Database Transformations

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Abstract

Database transformations provide a unifying framework for database queries and updates. Recently, it was shown that non-deterministic database transformations can be captured exactly by a variant of ASMs, the so-called Database Abstract State Machines (DB-ASMs). In this article we present a logic for DB-ASMs, extending the logic of Nanchen and Stärk for ASMs. In particular, we develop a rigorous proof system for the logic for DB-ASMs, which is proven to be sound and complete. The most difficult challenge to be handled by the extension is a proper formalisation capturing non-determinism of database transformations and all its related features such as consistency, update sets or multisets associated with DB-ASM rules. As the database part of a state of database transformations is a finite structure and DB-ASMs are restricted by allowing quantifiers only over the database part of a state, we resolve this problem by taking update sets explicitly into the logic, i.e. by using an additional modal operator $[X]$, where $X$ is interpreted as an update set $\Delta$ generated by a DB-ASM rule. The DB-ASM logic provides a powerful verification tool to study properties of database transformations.

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1. Introduction

Database transformations comprise algorithms for queries and updates, the two fundamental types of computations in databases capturing the capability to retrieve and update data. In database theory the integration of queries and updates has always been a challenging research problem \cite{1}. The rising trend of new application areas such as service-oriented architectures and NoSQL database solutions has further increased the importance of this problem since these new application areas are increasingly complicated and their data is
more schema-less, heterogeneous, redundant, inconsistent and frequently modified, from which arises the need for specifying and verifying queries and updates in a unified manner. To address these challenges, a theoretical framework for database transformations is required.

In most of the database theory literature database transformations are defined as binary relations on database instances over an input schema and database instances over an output schema which should satisfy certain criteria, e.g., well-typedness, effective computability, genericity and functionality as discussed in [2, 4]. Previous investigations into database transformations have yielded meaningful findings in the case of queries, e.g., computable queries [11], determinate transformations [2], semi-deterministic transformations [35], constructive transformations [33, 36], etc. Unfortunately, extending these results to updates is by no means straightforward. The authors of [34] stated that queries and updates may have some fundamental distinctions and the question of whether there exists a theoretical framework unifying both queries and updates has since been open.

An important observation is that the usual notion of query equivalence, in which queries are considered as computable functions which preserve isomorphisms and where two queries are equivalent if they compute the same function, is not appropriate for updates and also not suited for the context of query refinement. Take for example two updates $u_1$ and $u_2$ such that, for all database instance $I$, the new instances $u_1(I)$ and $u_2(I)$ resulting of applying the updates $u_1$ and $u_2$ to $I$, respectively, are never identical but still isomorphic by an isomorphism other than the identity. Clearly, for every database instance $I$, no computable query can distinguish between $u_1(I)$ and $u_2(I)$ and thus $u_1$ and $u_2$ should be consider equivalent updates even though they compute different functions. A similar problem arises if we look at query refinement, for instance at the level of the computer code needed to evaluate a query. In these contexts, the stronger notion of equivalence as used in the field of behavioural theory of algorithms [6] becomes relevant. Under this notion, equivalent algorithms have the same runs. Thus behavioural equivalence can be applied across the different formalisms for computing queries as well as updates, even though they are usually bound to different levels of abstraction [24].

Inspired by the sequential Abstract State Machine (ASM) thesis capturing sequential algorithms [17], Schewe and Wang investigated database transformations from an algorithmic point of view defining database transformations by five intuitive postulates [24]. That is, in the same way as Gurevich distinguishes an algorithm from the (computable) function it computes we shifted the focus on database transformations from looking only at the relationship between input and output databases to the algorithms that produce the output from the input. On one hand this does not exclude the study of declarative logical formalisms to express algorithms or at least parts of them. On the other hand it permits refinements of database transformations down to the lowest level of abstraction within the same theoretical framework. Furthermore, they developed a general computation model for database transformations, called Database Abstract State Machines (DB-ASMs), a variant of ASMs. It could be proven that DB-ASMs satisfy the five postulates for database transformations, and most importantly, all computations stipulated by the postulates for
database transformations can also be simulated step-by-step by a behaviourally equivalent DB-ASM [24]. This in a sense establishes the database analogue of Gurevich’s sequential ASM thesis [17]. As a database transformation is behaviourally equivalent to a DB-ASM and vice versa, a rigorous logic for DB-ASMs can thus offer vast advantages for reasoning about database transformations, such as verifying the correctness of specification, deriving static or dynamic properties, determining the equivalence of programs, comparing the expressive power of computation models, etc.

Contributions
In this article we investigate the logical implications of the DB-ASM thesis which leads us to defining a logic for DB-ASMs. Our work builds on the logic of Nanchen and Stärk for ASMs [30], which it extends by handling non-determinism and meta-finite structures. However, it also preserves the restriction of this logic dealing only with properties of a single step of an ASM, not with properties of whole ASM runs. The advantage is that this restriction to a one-step logic allows us to define a Hilbert-style proof theory and to show its completeness, whereas for a logic dealing with properties of whole ASM runs (and even more so, whole DB-ASMs runs) can hardly be expected to be complete.

Our first contribution is to characterise states of DB-ASMs by the logic of meta-finite structures [13] which is then incorporated into the logic for DB-ASMs. In database theory, states of a database transformation are predominantly regarded as finite structures. However, when applying algorithmic toolboxes in database-related problems, the finiteness condition on states turns out to be too restrictive for several reasons. First, database transformations may deal with new elements from countably infinite domains, e.g. counting queries produce natural numbers even if no natural number occurs in a finite structure. Second, finite structures may have invariant properties that possibly have infinite elements implied in satisfying them, such as, numerical invariants of geometric objects or database constraints. Third, each database transformation either implicitly or explicitly lives in a background that supplies all necessary information relating to the computation and usually exists in the form of infinite structures. Thus, we consider a state of database transformation as a meta-finite structure consisting of a database part, which is a finite structure, an algorithmic part, which may also be an infinite structure, and a finite number of bridge functions between these parts. Characterising states of database transformations by using the logic of meta-finite structures provides sufficient expressive power to reason about aggregate computations that commonly exist in database applications.

Our second contribution is the handling of bounded non-determinism in the logic of DB-ASMs. This was also the most challenging problem we faced in this study. It is worth to mention that non-deterministic transitions manifest themselves as a very difficult task in the logical formalisation for ASMs. Nanchen and Stärk analysed potential problems to several approaches they tried by taking non-determinism into consideration and concluded [30]:

Unfortunately, the formalisation of consistency cannot be applied directly to non-deterministic ASMs. The formula Con(R) (as defined in Sect. 8.1.2 of
expresses the property that the union of all possible update sets of $R$ in a given state is consistent. This is clearly not what is meant by consistency. Therefore, in a logic for ASMs with choose one had to add $\text{Con}(R)$ as an atomic formula to the logic.

However, we observe that this conclusion is not necessarily true, as finite update sets can be made explicit in the formulae of a logic to capture non-deterministic transitions. In doing so, the formalisation of consistency defined in [30] can still be applied to such an explicitly specified finite update set $\Delta$ yielded by a rule $r$ in the form of the formula $\text{con}(r, X)$ where the second-order variable $X$ is interpreted by $\Delta$, as will be discussed in Subsection 5.1. Thus we solve this problem by the addition of the modal operator $[X]$ for an update set generated by a DB-ASM rule. The approach works well, because DB-ASMs are restricted to have quantifiers only over the database part of a state which is a finite structure and update sets (or multisets) yielded by DB-ASM rules are thus restricted to be finite. Hence, the logic for DB-ASMs is able to capture non-deterministic database transformations.

Our third contribution is the development of a proof system for the logic for DB-ASMs, which extends the proof system for the logic for ASMs [30] in several aspects:

- DB-ASMs can collect updates yielded in parallel computations under the multiset semantics, i.e. update multisets, and then aggregate updates in an update multiset to an update set by applying so-called location operators. Our proof system can capture this by incorporating the axioms for both the predicate of update multisets and the predicate of update sets. The axioms also specify the interaction between update multisets and sets in association with same DB-ASM rules.

- A DB-ASM rule may be associated with a set of different update sets. Applying different update sets may lead to a set of different successor states to the current state. As the logic for DB-ASMs includes formulae denoting explicit update sets and multisets and second-order variables that are bound to update sets or multisets, our proof system allows us to reason about the interpretation of a formula over all successor states or over some successor state after applying a DB-ASM rule over the current state.

- In addition to capturing the consistency of an update set yielded by a DB-ASM rule, our proof system also develops two notions of consistency for a DB-ASM rule (i.e. weak version and strong version). When a DB-ASM rule is deterministic, these two notions coincide.

Our last contribution is a proof of the completeness of the logic for DB-ASMs. Due to the importance of non-determinism for enhancing the expressive power of database transformations and for specifying database transformations at flexible levels of abstraction, DB-ASMs take into account choice rules. Consequently, the logic for DB-ASMs has to
handle all the issues related to non-determinism which have been identified as the source of problems in the completeness proof of the logic for ASMs \cite{30}. Nevertheless, we can prove that, in spite of the inclusion of second-order formulae in the logic for DB-ASMs, the finiteness of domains that quantifiers are restricted to allows us to translate the logic for DB-ASMs into a definitional extension of a complete many-sorted first-order logic in which set membership is treated as a predicate under a fixed interpretation. This is the approach we use to prove the completeness of the logic for DB-ASMs.

Structure of the Article

The remainder of the article is structured as follows. Section 2 discusses related work on logical characterisations of database transformations. In Section 3 we discuss and exemplify the central concept of DB-ASM. As states of a database transformation are meta-finite structures, in Section 4 we define a logic for DB-ASMs that is built upon the logic of meta-finite structures. Subsequently, a detailed discussion of basic properties of the logic for DB-ASMs, such as consistency, update sets and multisets, along with the formalisation of a proof system is presented in Section 5. In Section 6 we present some interesting properties of the logic for DB-ASMs which are implied by the axioms and rules of the proof system introduced in Section 5. We discuss the completeness proof of the logic for DB-ASMs in Section 7. We conclude the article with a brief summary in Section 8.

2. Related Work

It is widely acknowledged that a logic-based perspective for database queries can provide a yardstick for measuring the expressiveness and complexity of query languages. To extend the application of mathematical logics from database queries to database updates, a number of logical formalisms have been developed providing the reasoning for both states and state changes in a computation model \cite{8,27}. A popular approach was to take dynamic logic as a starting point and then to define the declarative semantics of logical formulae based on Kripke structures. It led to the development of the database dynamic logic (DDL) and propositional database dynamic logic (PDDL) \cite{27,26,28}. DDL has atomic updates for inserting, deleting and updating tuples in predicates and for functions, whereas PDDL has two kinds of atomic updates: passive and active updates. Passive updates change the truth value of an atom while active updates compute derived updates using a logic program. In \cite{29} Spruit, Wieringa and Meijer proposed regular first-order update logic (FUL), which generalises dynamic logic towards specification of database updates. A state of FUL is viewed as a set of non-modal formulae. Unlike standard dynamic logic, predicate and function symbols rather than variables are updateable in FUL. There are two instantiations of FUL. One is called relational algebra update logic (RAUL) that is an extension of relational algebra with assignments as atomic updates. Another one is DDL that parameterizes FUL by two kinds of atomic updates: bulk updates to predicates and assignment updates to functions. It was shown that DDL is also “update complete”
in relational databases with respect to the update completeness criterion proposed by Abiteboul and Vianu in [3].

As we explained before ASMs turn out to be a promising approach for specifying database transformations. The logical foundations for ASMs have been well studied from several perspectives. Groenboom and Renardel de Lavalette presented in [15] a logic called modal logic of creation and modification (MLCM) that is a multimodal predicate logic intended to capture the ideas behind ASMs. On the basis of MLCM they developed a language called formal language for evolving algebras (FLEA) [16]. Instead of values of variables, states of an MLCM are represented by mathematical structures expressed in terms of dynamic functions. The work in [23] generalises MLCM and other variations from [12] to modification and creation logic (MCL) for which there exists a sound and complete axiomatisation. In [25] Schönegge presented an extension of dynamic logic with update of functions, extension of universes and simultaneous execution (called EDL), which allows statements about ASMs to be directly represented. In addition to these, a logic complete for hierarchical ASMs (i.e., ASMs that do not contain recursive rule definitions) was developed by Nanchen and Stärk in [30]. This logic for ASMs differs from other logics in two respects: (1) the consistency of updates has been accounted for; (2) modal operators are allowed to be eliminated in certain cases. As already remarked, the ASM logic of Nanchen and Stärk permits reasoning about ASM rules, but not about ASM runs, which is the price to be paid for obtaining completeness. In this article we will extend this logic for ASMs towards database transformations, in which states are regarded as meta-finite structures and a bounded form of non-determinism is captured.

It was Chandra and Harel who first observed limitations of finite structures in database theory [11]. They proposed a notion of an extended database that extends finite structures by adding another countable, enumerable domain containing interpreted features such as numbers, strings and so forth. The intention of their study was to provide a more general framework that can capture queries with interpreted elements. Another extension of finite structures was driven by the efforts to solve the problem of expressing cardinality properties [4, 10, 14, 20, 21, 22, 31, 32]. For example, Grädel and Otto developed a two-sorted structure that adjoins a one-sorted finite structure with an additional finite numerical domain and added the terms expressing cardinality properties [14]. They aimed at studying the expressive power of logical languages that involve induction with counting on such structures. A promising line of work is meta-finite model theory. Grädel and Gurevich in [13] defined meta-finite structures consisting of a primary part that is a finite structure, a secondary part that may be a finite or infinite structure, and a set of weight functions from the primary part into the secondary part, and further extended a logic suitable for finite structures (e.g., first-order logic, fixed point logic, the infinitary logic, etc.) to a logic of meta-finite structures. Based on the work presented in [13], Hella et al. in [18] studied the logical grounds of query languages with aggregation, which is closely related to our work presented in this article. However, the logic for DB-ASMs covers not only database queries with aggregation but also database updates. Put it in another way, it is a logical characterisation for database transformations including aggregate computing
3. A Computation Model for Database Transformations

In this section we present the definitions of Database Abstract State Machines (DB-ASMs) [24, 38] along with an illustrative example that describes how DB-ASMs can capture database transformations.

3.1. Preliminaries

States of a DB-ASM are meta-finite structures [13]. Each state consists of a finite database part and a possibly infinite algorithmic part linked via bridge functions, in which actual database entries are treated merely as surrogates for the real values. This permits the database to remain finite while allowing database entries to be interpreted in possibly infinite domains such as the natural numbers with arithmetic operations. A signature Υ of states comprises a sub-signature Υ_{db} for the database part, a sub-signature Υ_{a} for the algorithmic part and a finite set \mathcal{F}_{b} of bridge function names. The base set of a state \( S \) is a nonempty set of values \( B = B_{db} \cup B_{a} \), where \( B_{db} \) is finite, and \( B_{a} \) contains natural numbers, i.e., \( \mathbb{N} \subseteq B_{a} \). Function symbols \( f \) in \( \Upsilon_{db} \) and \( \Upsilon_{a} \), respectively, are interpreted as functions \( f^{S} \) over \( B_{db} \) and \( B_{a} \), and the interpretation of a \( k \)-ary function symbol \( f \in \mathcal{F}_{b} \) defines a function \( f^{S} \) from \( B_{db}^{k} \) to \( B_{a} \). For every state over \( \Upsilon \), the restriction to \( \Upsilon_{db} \) results in a finite structure.

As in ASMs we distinguish between updatable dynamic functions and static functions which cannot be updated. Let \( S \) be a state over \( \Upsilon \), \( f \in \Upsilon \) be a dynamic function symbol of arity \( n \) and \( a_{1}, \ldots, a_{n} \) be elements in \( B_{db} \) or \( B_{a} \) depending on whether \( f \in \Upsilon_{db} \cup \mathcal{F}_{b} \) or \( f \in \Upsilon_{a} \), respectively. Then \( (f,(a_{1}, \ldots, a_{n})) \) is called a location of \( S \). An update of \( S \) is a pair \((\ell,b)\), where \( \ell \) is a location and \( b \in B_{db} \) or \( b \in B_{a} \) depending on whether \( f \in \Upsilon_{db} \) or \( f \in \Upsilon_{a} \cup \mathcal{F}_{b} \), respectively, is the update value of \( \ell \). To simplify notation we write \((f,(a_{1}, \ldots, a_{n}),b)\) for the update \((\ell,b)\) with the location \( \ell = (f,(a_{1}, \ldots, a_{n})) \). The interpretation of \( \ell \) in \( S \) is called the content of \( \ell \) in \( S \), denoted by \( \text{val}_{S}(\ell) \). An update set \( \Delta \) is a set of updates; an update multiset \( \Delta \) is a multiset of updates. A location operator \( \rho \) is a multiset function that returns a single value from a multiset of values, e.g. AVERAGE, COUNT, SUM, MAX and MIN used in SQL.

Let \( \Upsilon = \Upsilon_{db} \cup \Upsilon_{a} \cup \mathcal{F}_{b} \) be a signature of states. Fix a countable set \( X_{db} \) of first-order variables, denoted with standard lowercase letters \( x, y, z, \ldots \), that range over the primary database part of the states (i.e., the finite set \( B_{db} \)). The set of first-order terms \( T_{\Upsilon,X_{db}} \) of vocabulary \( \Upsilon \) is defined in a similar way than in meta-finite model theory [13]. That is, \( T_{\Upsilon,X_{db}} \) is constituted by the set \( T_{db} \) of database terms and the set \( T_{a} \) of algorithmic terms. The set of database terms \( T_{db} \) is the closure of the set \( X_{db} \) of variables under the application of function symbols in \( \Upsilon_{db} \). The set of algorithmic terms \( T_{a} \) is defined inductively: If \( t_{1}, \ldots, t_{n} \) are database terms in \( T_{db} \) and \( f \) is an \( n \)-ary bridge function symbol in \( \mathcal{F}_{b} \), then \( f(t_{1}, \ldots, t_{n}) \) is an algorithmic term in \( T_{a} \); if \( t_{1}, \ldots, t_{n} \) are algorithmic
terms in $T_a$ and $f$ is an $n$-ary function symbol in $\Upsilon_a$, then $f(t_1, \ldots, t_n)$ is an algorithmic term in $T_a$; nothing else is an algorithmic term in $T_a$.

Let $S$ be a meta-finite state of signature $\Upsilon$. A valuation or variable assignment $\zeta$ is a function that assigns to every variable in $\chi_{db}$ a value in the base set of the database part $B_{db}$ of $S$. The value $\text{val}_{S,\zeta}(t)$ of a term $t \in T_{\Upsilon,\chi_{db}}$ in the state $S$ under the valuation $\zeta$ is defined as usual in first-order logic. The first-order logic of meta-finite states is defined as the first-order logic with equality which is built up from equations between terms in $T_{\Upsilon,\chi_{db}}$ by using the standard connectives and first-order quantifiers. Its semantics is defined in the standard way. The truth value of a first-order formula of meta-finite states $\varphi$ in $S$ under the valuation $\zeta$ is denoted as $[\varphi]_{S,\zeta}$.

In our definition of DB-ASM rule, we use the fact that function arguments can be read as tuples. Thus, if $f$ is an $n$-ary function and $t_1, \ldots, t_n$ are arguments for $f$, we write $f(t)$ where $t$ is a term which evaluates to the tuple $(t_1, \ldots, t_n)$, instead of $f(t_1, \ldots, t_n)$. This is not strictly necessary, but it greatly simplifies the presentation of the technical details in this paper. Let $t$ and $s$ denote terms in $T_{\Upsilon,\chi_{db}}$, $f$ a dynamic function symbol in $\Upsilon$ and let $\varphi$ denote a first-order formula of meta-finite states of vocabulary $\Upsilon$. The set of DB-ASM rules over $\Upsilon$ is inductively defined as follows:

- **assignment rule**: update the content of $f$ at the argument $t$ to $s$;
  \[ f(t) := s \]
- **conditional rule**: execute the rule $r$ if $\varphi$ is true; otherwise, do nothing;
  \[ \text{if } \varphi \text{ then } r \text{ endif} \]
- **forall rule**: execute the rule $r$ in parallel for each $x$ satisfying $\varphi$;
  \[ \text{forall } x \text{ with } \varphi \text{ do } r \text{ enddo} \]
- **choice rule**: choose a value of $x$ that satisfies $\varphi$ and then execute the rule $r$;
  \[ \text{choose } x \text{ with } \varphi \text{ do } r \text{ enddo} \]
- **parallel rule**: execute the rules $r_1$ and $r_2$ in parallel;
  \[ \text{par } r_1 \text{ r}_2 \text{ endpar} \]
- **sequence rule**: first execute the rule $r_1$ and then execute the rule $r_2$;
  \[ \text{seq } r_1 \text{ r}_2 \text{ endseq} \]
- **let rule**: assign the location operator $\rho$ to the location $l = (f, t)$ and then aggregate all update values of $l$ yielded by the rule $r$;
  \[ \text{let } (f, t) \rightarrow \rho \text{ in } r \text{ endlet} \]

Notice that all variables appearing in a DB-ASM rule are database variables that must be interpreted by values in $B_{db}$. A rule $r$ is **closed** if all variables of $r$ are bounded by forall and choice rules.
3.2. Update Sets and Multisets

The semantics of DB-ASM rules is defined in terms of update multisets and update sets. More specifically, each DB-ASM rule is associated with a set of update multisets, which then “collapses” to a set of update sets. Thus, if \( r \) is a DB-ASM rule of signature \( \Upsilon \) and \( S \) is a state of \( \Upsilon \), then we associate a set \( \Delta(r, S, \zeta) \) of update sets and a set \( \tilde{\Delta}(r, S, \zeta) \) of update multisets with \( r \) and \( S \), respectively, which depend on a variable assignment \( \zeta \).

Let \( \zeta[x \mapsto a] \) denote the variable assignment which coincides with \( \zeta \) except that it assigns the value \( a \) to \( x \). We formally define the sets of update sets and sets of update multisets yielded by DB-ASM rules in Figures 1 and 2, respectively. Assignment rules create updates in update sets and multisets. Choice rules introduce non-determinism. Let rules aggregate updates to the same location into a single update by means of location operators. All other rules only rearrange updates into different update sets and multisets.

An update set \( \Delta \) is called consistent if it does not contain conflicting updates, i.e., for all \((\ell, b), (\ell, b') \in \Delta\) we have \( b = b' \). If \( \Delta \) is consistent, then there exists a unique state \( S + \Delta \) resulting from updating \( S \) with \( \Delta \). We have

\[
\text{val}_{S+\Delta}(\ell) = \begin{cases} 
  b & \text{if } (\ell, b) \in \Delta \\
  \text{val}_S(\ell) & \text{otherwise}
\end{cases}
\]

If \( \Delta \) is not consistent, then \( S + \Delta \) is undefined.

From our definitions we obtain the following straightforward result, which nonetheless is key for the development of the DB-ASM logic in this article.

**Lemma 3.1.** For each state \( S \), each DB-ASM rule \( r \) and each variable assignment \( \zeta \) from \( X_{db} \) to the base set \( B_{db} \) of the database part of \( S \), the following holds:

1. \( \Delta(r, S, \zeta) \) and \( \tilde{\Delta}(r, S, \zeta) \) are finite sets.
2. Each \( \Delta \in \Delta(r, S, \zeta) \) is a finite update set.
3. Each \( \tilde{\Delta} \in \tilde{\Delta}(r, S, \zeta) \) is a finite update multiset.

**Proof.** (Sketch). We use structural induction on \( r \). The case of the assignment rule is obvious, as a single update will be created.

The conditional rule either produces exactly the same update sets and multisets as before or a single empty update set and multiset, respectively. For the forall rule the set \( V = \{ a \in B_{db} \mid [\varphi]_{S, \zeta[x \mapsto a]} = \text{true} \} \) is finite, because \( x \) ranges over the finite set \( B_{db} \).

The stated finiteness then follows by induction, as \( \Delta(r, S, \zeta) \) and \( \tilde{\Delta}(r, S, \zeta) \) are finite sets, all \( \Delta \in \Delta(r, S, \zeta) \) and all \( \tilde{\Delta} \in \tilde{\Delta}(r, S, \zeta) \) are finite, and the new update sets and update multisets are built by set and multiset unions, respectively, that range over the finite set \( V \).

For all other rules, the individual update sets and multisets are built by \( \cup \), \( \uplus \), \( \ominus \) and aggregation with location operators applied to finite update sets and multisets, which gives the statements 2 and 3. Furthermore, the sets of update sets and update multisets,
\[ \Delta(f(t) = s, S, \zeta) = \{(f, a, b)\} \text{ where } a = \text{val}_{S, \zeta}(t) \text{ and } b = \text{val}_{S, \zeta}(s) \]

\[ \Delta(\text{if } \varphi \text{ then } r \text{ endif}, S, \zeta) = \begin{cases} \Delta(r, S, \zeta) & \text{if } [\varphi]_{S, \zeta} = \text{true} \\ \emptyset & \text{otherwise} \end{cases} \]

\[ \Delta(\forall x \text{ with } \varphi \text{ do } r \text{ enddo}, S, \zeta) = \bigcup_{a_i \in B_{db}} \{ \Delta(r, S, \zeta[x \mapsto a_i]) \mid [\varphi]_{S, \zeta[x \mapsto a_i]} = \text{true} \} \]

\[ \Delta(\text{choose } x \text{ with } \varphi \text{ do } r \text{ enddo}, S, \zeta) = \bigcup_{a_i \in B_{db}} \{ \Delta(r, S, \zeta[x \mapsto a_i]) \mid [\varphi]_{S, \zeta[x \mapsto a_i]} = \text{true} \} \]

\[ \Delta(\text{let } (f, t) \rightarrow \rho \text{ in } r \text{ endlet}, S, \zeta) = \{(\ell, a) \mid a = \rho(\{a_i \mid (\ell, a_i) \in \Delta\}) \} \cup \{(\ell', b) \in \tilde{\Delta} \mid \ell' \neq \ell \} \mid \tilde{\Delta} \in \tilde{\Delta}(r, S, \zeta) \]

where \( \ell = (f, \text{val}_{S, \zeta}(t)) \)

Figure 1: Sets of update sets of DB-ASM rules

respectively, are built by comprehensions that range over finite sets. Hence they are finite as well, which gives statement 1 and completes the proof.

\hfill \square

A Database Abstract State Machine (DB-ASM) \( M \) over signature \( \Upsilon \) consists of

- a set \( S \) of states over \( \Upsilon \), non-empty subsets \( S_I \subseteq S \) of initial states and \( S_F \subseteq S \) of final states,

- a closed DB-ASM rule \( r \) over \( \Upsilon \), and
• \( \tilde{\Delta}(f(t) := s, S, \zeta) = \{ \{ (f, a, b) \} \} \) where \( a = val_{S, \zeta}(t) \) and \( b = val_{S, \zeta}(s) \)

• \( \tilde{\Delta}(\text{if } \varphi \text{ then } r \text{ endif}, S, \zeta) = \begin{cases} \tilde{\Delta}(r, S, \zeta) & \text{if } [\varphi]_{S, \zeta} = \text{true} \\ \{ \{ \} \} & \text{else} \end{cases} \)

• \( \tilde{\Delta}(\text{forall } x \text{ with } \varphi \text{ do } r \text{ enddo}, S, \zeta) = \begin{cases} \tilde{\Delta}(r, S, \zeta) & [\varphi]_{S, \zeta[x \mapsto a_i]} = \text{true} \\ \{ \{ \} \} & \text{else} \end{cases} \)

• \( \tilde{\Delta}(\text{choose } x \text{ with } \varphi \text{ do } r \text{ enddo}, S, \zeta) = \bigcup_{a_i \in B_{db}} \{ \tilde{\Delta}(r, S, \zeta[x \mapsto a_i]) \} \)

• \( \tilde{\Delta}(\text{par } r_1 r_2 \text{ endpar}, S, \zeta) = \{ \tilde{\Delta}_1 \uplus \tilde{\Delta}_2 \mid \tilde{\Delta}_1 \in \tilde{\Delta}(r_1, S, \zeta) \) and \( \tilde{\Delta}_2 \in \tilde{\Delta}(r_2, S, \zeta) \} \)

• \( \tilde{\Delta}(\text{seq } r_1 r_2 \text{ endseq}, S, \zeta) = \{ \tilde{\Delta}_1 \uplus \tilde{\Delta}_2 \mid \tilde{\Delta}_1 \in \tilde{\Delta}(r_1, S, \zeta) \) is consistent and \( \tilde{\Delta}_2 \in \tilde{\Delta}(r_2, S + \Delta_1, \zeta) \} \)

• \( \tilde{\Delta}(\text{let } (f, t) \mapsto \rho \text{ in } r \text{ endlet}, S, \zeta) = \{ \{ (\ell, a) \mid a = \rho(\{ a_i \mid (\ell, a_i) \in \tilde{\Delta} \}) \} \uplus \{ (\ell', b) \mid \ell' \neq \ell \} \mid \tilde{\Delta} \in \tilde{\Delta}(r, S, \zeta) \} \)

where \( \ell = (f, val_{S, \zeta}(t)) \)

\[ \Delta = \{ (S, S + \Delta) \mid \Delta \in \Delta(r, S) \} \text{ consistent} \],

where the set \( \Delta(r, S) \) (\( \zeta \) is omitted from \( \Delta(r, S, \zeta) \) since \( r \) is closed) of update sets yielded by rule \( r \) over the state \( S \) defines the successor relation \( \delta \) of \( M \). A run of \( M \) is a finite sequence \( S_0, \ldots, S_n \) of states with \( S_0 \in S_I, S_n \in S_F, S_i \notin S_F \) for \( 0 < i < n \), and \( (S_i, S_{i+1}) \in \delta \) for all \( i = 0, \ldots, n - 1 \).
3.3. Examples

In our examples we abuse the notation by allowing more than one variable to be associated to one forall rule. This allows us to present shorter and more readable DB-ASM rules, but such rules can always be replaced by well formed ones (i.e., that respect our formal definition) by using nested forall rules.

Our first example illustrates the synchronisation of parallel branches by using a let rule.

**Example 3.1.** Consider the relation $\text{Route} = \{\text{FromCid}, \text{ToCid}, \text{Distance}\}$ in Figure 3, and the two DB-ASMs presented in Figure 4.

<table>
<thead>
<tr>
<th>FromCid</th>
<th>ToCid</th>
<th>Distance</th>
</tr>
</thead>
<tbody>
<tr>
<td>$c_1$</td>
<td>$c_2$</td>
<td>$d_1$</td>
</tr>
<tr>
<td>$c_1$</td>
<td>$c_5$</td>
<td>$d_4$</td>
</tr>
<tr>
<td>$c_2$</td>
<td>$c_3$</td>
<td>$d_2$</td>
</tr>
<tr>
<td>$c_1$</td>
<td>$c_4$</td>
<td>$d_1$</td>
</tr>
</tbody>
</table>

Figure 3: A relation $\text{Route}$

\[
\text{let } (\ell_{\text{num}},()) \rightarrow \text{SUM in}
\forall x_1, x_2 \text{ with } \exists x_3 (\text{ROUTE}(x_1, x_2, x_3)) \text{ do}
\ell_{\text{num}} := 1
\text{ enddo}
\text{endlet}
\]

(a) First DB-ASM

\[
\forall x_1, x_2 \text{ with } \exists x_3 (\text{ROUTE}(x_1, x_2, x_3)) \text{ do}
\ell_{\text{num}} := 1
\text{ enddo}
\]

(b) Second DB-ASM

Figure 4: Two DB-ASMs

The first DB-ASM in Figure 4(a) computes the total number of routes in the relation $\text{ROUTE}$. Here $\text{SUM}$ is a location operator assigned to the location $(\ell_{\text{num}},())$. In a state containing the relation $\text{ROUTE}$ in Figure 3 the forall sub-rule yields the update multiset $\{((\ell_{\text{num}},()),1),((\ell_{\text{num}},()),1),((\ell_{\text{num}},()),1),((\ell_{\text{num}},()),1)\}$ and the update set $\{(\ell_{\text{num}},1)\}$. In turn the let rule (and thus the DB-ASM) yields the corresponding update set $\{(\ell_{\text{num}},4)\}$, which results from the aggregation produced by the location operator $\text{SUM}$ of the four updates to the location $(\ell_{\text{num}},())$ that appear in the multiset produced by the forall rule.
Since the second DB-ASM in Figure 4(b) has no location operator associated with the location \((\ell_{num},())\) and the forall rule yields the same update multiset and update set as before, this second DB-ASM produces the update set \(\{(\ell_{num},(),1)\}\) instead.

Our following example illustrates how database transformations can be captured by DB-ASMs and how a logic for DB-ASMs can be used for verifying database transformations (one of the potential applications of the logic for DB-ASMs).

**Example 3.2.** Consider a relational database which stores information regarding pairs of cities connected with a direct route and their distances. Let the relation schemata of the database be City = \{Cid, Name\} and Route = \{FromCid, ToCid, Distance\}. Assume that for all pairs of cities \((c_1,c_2)\) in a valid database instance, \((c_1,c_2,d)\in Route\) iff \((c_2,c_1,d)\in Route\). Note that every instance of Route induces a corresponding undirected graph in which the nodes represent cities and the edges represent direct routes. Assume that these graphs are always connected. Let \(Q_1(c)\) be the query “find a shortest path tree rooted at city \(c\)”, which consists on finding a spanning tree \(T\) with root node \(c\) such that the path distance from \(c\) to any other node \(c'\) in \(T\) is the shortest path distance from city \(c\) to \(c'\) in the graph induced by Route. The DB-ASM rule in Figure 6 (of the signature \(\Upsilon_G\) described next), which corresponds to the famous Dijkstra’s algorithm, expresses Query \(Q_1(c)\).

\[
\Upsilon_G = (\Upsilon_{db}, \Upsilon_a, F_b), \text{ where}
\]
- \text{City, Route, Visited, Result, c, True, False, Initial } \in \Upsilon_{db};
- \text{Infinity, Zero, MDist } \in \Upsilon_a;
- \text{Val, Dist } \in F_b; \text{ and}
- \text{Min is a location operator.}

Figure 5: A signature

Let \(\Upsilon_G\) be the signature in Figure 5. Apart from City and Route, \(\Upsilon_G\) includes Visited = \{Cid\} and a bridge function Dist : Cid \rightarrow \mathbb{N} to keep track of cities visited during a computation and their corresponding shortest distances to \(c\), respectively, and Result = \{ChildCid, ParentCid\} to store the shortest path tree as a child-parent node relationship. We assume that in every initial state the relations Visited and Result are empty and that Initial = True. We also assume that the constant symbol Infinity is interpreted by a natural number which is strictly greater than the sum of all the distances in Route, that Zero is interpreted by the value 0, and that the values interpreting the constant symbols True and False are different. A state in which every city has been visited, i.e., a state in which Visited contains every city id in the database, is considered as a final state. Notice that entries in the database part of the states which correspond to the distances between adjacent nodes in Route are surrogates for the actual distances.
which are natural numbers in the algorithmic part of the state. The bridge function $\text{Val}$ maps these surrogate values to the actual natural numbers in the algorithmic part.

1 par
2 if INITIAL then
3 par
4 forall $x$ with $\exists y(\text{City}(x, y))$ do
5 par
6 if $x = c$ then $\text{Dist}(x) := \text{Zero}$ endif
7 if $x \neq c$ then $\text{Dist}(x) := \text{Infinity}$ endif
8 endpar
9 enddo
10 INITIAL := false
11 endpar
12 endif
13 if $\neg$INITIAL then
14 seq
15 let $(\text{MDist},()) \rightarrow \text{Min}$ in
16 forall $x$ with $\exists y(\text{City}(x, y) \land \neg \text{Visited}(x))$ do
17 $\text{MDist} := \text{Dist}(x)$
18 enddo
19 endlet
20 choose $x$ with $\text{Dist}(x) = \text{MDist} \land \neg \text{Visited}(x)$ do
21 par
22 $\text{Visited}(x) := \text{True}$
23 forall $y, z$ with $\text{Route}(x, y, z) \land \neg \text{Visited}(y) \land$
24 $\text{MDist} + \text{Val}(z) < \text{Dist}(y)$ do
25 par
26 $\text{Dist}(y) := \text{MDist} + \text{Val}(z)$
27 $\text{Result}(y, x) := \text{True}$
28 forall $x'$ with $x' \neq x \land \text{Result}(y, x')$ do
29 $\text{Result}(y, x') := \text{False}$
30 enddo
31 endpar
32 enddo
33 endpar
34 enddo
35 endseq
36 endif
37 endpar

Figure 6: A DB-ASM
The DB-ASM proceeds in two stages:

- The first stage is described by Lines 2-12. The DB-ASM starts with an initial state in which \( \text{Initial} = \text{True} \), then assigns (in parallel) to every city a tentative distance value (i.e., 0 for the city \( c \) and \( \text{Infinity} \) for all other cities), and ends with \( \text{Initial} = \text{False} \).

- The second stage is described by Lines 13-36. The shortest paths to reach other cities from the city \( c \) are repeatedly calculated and stored in \( \text{Result} \) until a final state in which every city has been visited is reached.

At Line 15 of the ASM rule in Figure 6 the location operator \( \text{Min} \) is assigned to the location \( (\text{MDist},()) \), and thus \( \text{MDist} \) is updated to the shortest distance among the collection of distances between the city in consideration and all its unvisited neighbor cities. At Line 20, we can see that the DB-ASM is non-deterministic because a city is arbitrarily chosen from the non-visited cities whose shortest paths are equally minimum at each step of the computation process. This indeed exemplifies the importance of non-determinism for specifying database transformations at a high-level of abstraction.

Let us now suppose that we want to know whether the properties P1 and P2 described next, are satisfied by the DB-ASM corresponding to the DB-ASM rule in Figure 6 over certain states of signature \( \text{\textsc{U}}_G \). Clearly, the use of a logic to specify such properties of DB-ASMs can contribute significantly to the verification of the correctness of database transformations expressed by means of DB-ASMs. Although the logic proposed for DB-ASMs in this paper can only reason about such properties within one-step of computation, it nevertheless provides a useful tool which is a first step towards developing a logic that can reason about properties of whole DB-ASMs runs.

P1 In every non-initial state of a run, each city in the child/parent node relationship encoded in \( \text{Result} \) has exactly one parent city, except for \( c \) which has none. In other words, \( \text{Result} \) encodes a tree with root node \( c \).

\[
\neg\text{Initial} \rightarrow \\
\forall xy(\text{Result}(x, y) \rightarrow x \neq c \land \forall z(z \neq y \rightarrow \neg\text{Result}(x, z))) \land \exists x(\text{Result}(x, c))
\]

P2 In every state of a run, if a city not yet visited (by the algorithm) is a neighbour city of one already visited, then the calculated (shortest so far) distance from \( c \) to that city is strictly less than \( \text{Infinity} \) already.

\[
\forall xy(\text{Visited}(x) \land \neg\text{Visited}(y) \land \exists z(\text{Route}(x, y, z)) \rightarrow \text{Dist}(y) < \text{Infinity})
\]

4. A Logic for DB-ASMs

In this section we present the logic for DB-ASMs. States of DB-ASMs are characterised by the logic of meta-finite structures [13].
4.1. Syntax

Let $\Upsilon = \Upsilon_{db} \cup \Upsilon_a \cup \mathcal{F}_b$ be a signature of states and let $\Lambda$ denote a set of location operators. Fix a countable set $\mathcal{X} = \mathcal{X}_{db} \cup \mathcal{X}_a$ of first-order variables. Variables in $\mathcal{X}_{db}$, denoted with standard lowercase letters $x, y, z, \ldots$, range over the database part of meta-finite states (i.e., the finite base set $B_{db}$), whereas variables in $\mathcal{X}_a$, denoted with typewriter-style lowercase letters $x, y, z, \ldots$, range over the algorithmic part of meta-finite states (i.e. the possible infinite base set $B_a$). As discussed later in this section, we allow the presence of second-order variables in the logic for DB-ASMs. Without loss of generality, a variable assignment $\zeta$ as previously defined for first-order variables that range over $B_{db}$, can be extended to first-order variables that range over $B_a$ as well as to second-order variables that range over finite sets. We use $fr(t)$ to denote the set of (both first-order and second-order) free variables occurring in $t$.

Definition 4.1. The set of terms in the logic for DB-ASMs is constituted by the set $T_{db}$ of database terms and the set $T_a$ of algorithmic terms expressed as follows:

- $x \in T_{db}$ for $x \in \mathcal{X}_{db}$ and $fr(x) = \{x\}$;
- $x \in T_a$ for $x \in \mathcal{X}_a$ and $fr(x) = \{x\}$;
- $f(t) \in T_{db}$ for $f \in \Upsilon_{db}$, $t \in T_{db}$ and $fr(f(t)) = fr(t)$;
- $f(t) \in T_a$ for $f \in \mathcal{F}_b$, $t \in T_{db}$ and $fr(f(t)) = fr(t)$;
- $f(t) \in T_a$ for $f \in \Upsilon_a$, $t \in T_a$ and $fr(f(t)) = fr(t)$;
- $\rho_x(t | \varphi(x, \bar{y})) \in T_a$ for a location operator $\rho \in \Lambda$, a formula $\varphi(x, \bar{y})$ of the logic for DB-ASMs (see Definition 4.2 below), $x$ a variable in $\mathcal{X}_{db}$, $\bar{y}$ a tuple of arbitrary variables, and $t \in T_a$ with $fr(t) \subseteq fr(\varphi(x, \bar{y})) = \{x_i | x_i = x$ or $x_i$ appears in $\bar{y}\}$ and $fr(\rho_x(t | \varphi(x, \bar{y}))) = fr(\varphi(x, \bar{y})) - \{x\}$.

We use the notion $\rho$-term for a term of the form of $\rho_x(t | \varphi(x, \bar{y}))$ and pure term for a term that does not contain $\rho$-terms, i.e., a term that does not contain any formulae. $\rho$-terms are built upon formulae; on the other hand they can also be used for constructing formulae. This gives $\rho$-terms great expressive power to specify database transformations.

The following example illustrates how $\rho$-terms are able to express aggregate queries.

Example 4.1. Consider the relation Route in Example 3.1 again. The following two aggregate queries are expressible.

- **Q1**: Calculate the total number of direct routes.
  \[
  \text{COUNT}_{x}(x \mid \exists yz(\text{Route}(x, y, z)))
  \]
  In an SQL database, Q1 can be expressed by the following SQL statement:
SELECT \( count(*) \) FROM Route

- Q2: Find the maximum number of direct connections of any city in the database.

\[
\max_x (\text{COUNT}_x(y' \mid \exists z (\text{Route}(x, y', z))) \mid \exists y z (\text{Route}(x, y, z)))
\]

In a similar way, Q2 can be expressed by the following SQL statement:

SELECT \( \text{max}(\text{NumofConnections}) \)
FROM (SELECT Cid, \( count(*) \) as NumofConnections
FROM Route
GROUP BY Cid)

**Definition 4.2.** The formulae of the logic for DB-ASMs are those generated by the following grammar:

\[
\varphi, \psi ::= s = t \mid s_a = t_a \mid \neg \varphi \mid \varphi \land \psi \mid \forall x(\varphi) \mid \exists x(\varphi) \mid \forall M(\varphi) \mid \forall X(\varphi) \mid \forall X'(\varphi) \\
\mid \forall \tilde{X}(\varphi) \mid \forall \tilde{X}(\varphi) \mid \forall F(\varphi) \mid \forall G(\varphi) \mid \text{upd}(r, X) \mid \text{upm}(r, \tilde{X}) \mid M(s, t_a) \\
\mid X(f, t, t_0) \mid \tilde{X}(f, t, t_0, s) \mid \tilde{X}(f, t, t_0, t_a) \mid \tilde{X}(f, t, t_0, t_a, s) \\
\mid F(f, t, t_0, t_a, t', t_0', t_a', s) \mid G(f, t, t_0, t_a, t', t_0', t_a', s) \mid [X] \varphi
\]

where \( s, t \) and \( t' \) denote terms in \( \mathcal{T}_db \), \( s_a, t_a \) and \( t'_a \) denote terms in \( \mathcal{T}_a \), \( x \in \mathcal{X}_db \) and \( x \in \mathcal{X}_a \) denote first-order variables, \( M, X, X', \tilde{X}, \tilde{X}', F \) and \( G \) denote second-order variables, \( r \) is a DB-ASM rule, \( f \) is a dynamic function symbol in \( \mathcal{Y}_db \cup \mathcal{F}_b \), and \( t_0 \) and \( t'_0 \) denote terms in \( \mathcal{T}_db \) or \( \mathcal{T}_a \) depending on whether \( f \) is in \( \mathcal{Y}_db \) or \( \mathcal{F}_b \), respectively.

In the logic, disjunction \( \lor \), implication \( \rightarrow \), and existential quantification \( \exists \) are defined as abbreviations in the usual way. \( \forall M(\varphi), \forall X(\varphi), \forall X'(\varphi), \forall \tilde{X}(\varphi), \forall \tilde{X}(\varphi), \forall F(\varphi) \) and \( \forall G(\varphi) \) are second-order formulae in which \( M, X, X', \tilde{X}, \tilde{X}', F \) and \( G \) range over finite relations as defined in Section 4.2. When applying forall and parallel rules of a DB-ASM, updates yielded by parallel computations may be identical. Thus, we need the multiset semantics for describing a collection of possible identical updates. This leads to the inclusion of \( \text{upm}(r, \tilde{X}) \) and \( \tilde{X}(f, t, t_0, t_a) \) in the logic. \( \text{upd}(r, X) \) and \( \text{upm}(r, \tilde{X}) \) respectively state that a finite update set represented by \( X \) and a finite update multiset represented by \( \tilde{X} \) are generated by a rule \( r \). \( X(f, t, t_0) \) describes that an update \( (f, t, t_0) \) belongs to the update set represented by \( X \), while \( \tilde{X}(f, t, t_0, t_a) \) describes that an update \( (f, t, t_0) \) occurs at least \( n \) times in the update multiset represented by \( \tilde{X} \). If \( (f, t, t_0) \) occurs \( n \)-times in the update multiset represented by \( \tilde{X} \), then there are \( n \) distinct \( a_1, \ldots, a_n \in B_a \) such that \( (f, t, t_i, a_i) \in \tilde{X} \) for every \( 1 \leq i \leq n \) and \( (f, t, t_0, a_j) \notin \tilde{X} \) for every \( a_j \) other than \( a_1, \ldots, a_n \). We use \([X] \varphi \) to express the evaluation of \( \varphi \) over a state after executing the
update set represented by $X$ on the current state. The second-order variables $X$ and $\dot{X}$ are used to keep track of the parallel branches that produce the update sets and multisets, respectively, in a way which becomes clear later on. Finally, we use $M$ to denote binary second-order variables which are used to represent the finite multisets in the semantic interpretation of $\rho$-terms, and $F$ and $G$ to denote second-order variables which encode bijections between update multisets. Again, the need for these types of variables becomes clear later in the paper.

A formula of the logic for DB-ASMs is pure if it does not contain any $\rho$-term and is generated by the following restricted grammar:

$$\varphi, \psi ::= s = t | s_a = t_a | \neg \varphi | \varphi \land \psi | \forall x(\varphi) | \forall x(\varphi)$$

As defined before the formulae occurring in conditional, forall and choice rules of a DB-ASM are pure formulae of this logic. A formula or a term is static if it does not contain any dynamic function symbol.

**Example 4.2.** Consider Example 3.2, and the following formulae that specify some properties for the DB-ASM depicted in Figure 6. We denote the rule of the DB-ASM as $r$ and the current state as $S$.

1. If the rule $r$ over $S$ yields an update set $\Delta$ containing an update $(\text{Visited}, x, \text{True})$, then for every neighbour city $y$ of $x$, the (current) shortest distance in state $S + \Delta$ (calculated by the algorithm) between $y$ and $c$ is no longer Infinity. Representing $\Delta$ by the second-order variable $X$, we obtain:

$$\exists X x \left( \text{upd}(r, X) \land X(\text{Visited}, x, \text{True}) \rightarrow [X] \forall y z (\text{Route}(x, y, z) \rightarrow \text{Dist}(y) \neq \text{Infinity}) \right)$$

2. If in the current state $S$, the distance between a non-visited (by the algorithm) city $x$ and $c$ is minimal among the non-visited cities, then there is an update set $\Delta$ yielded by the rule $r$ in state $S$ which updates the status of $x$ to visited. Representing $\Delta$ by the second-order variable $X$, we obtain:

$$\exists x (\neg \text{Visited}(x) \land \forall y (\neg \text{Visited}(y) \rightarrow \text{Dist}(x) \leq \text{Dist}(y)) \rightarrow \\
\exists X (\text{upd}(r, X) \land [X] \text{Visited}(x))$$

3. If the current state $S$ is not an initial nor a final state and $\Delta$ is an update set yielded by the rule $r$ in $S$, then the value of $\text{MDist}$ in the successor state $S + \Delta$ equals the distance between $c$ and the closest unvisited (by the algorithm in state $S$) city. Representing $\Delta$ by the variable $X$ and using a $\rho$-term with location operator $\text{Min}$, we obtain:
4. If the current state $S$ is not an initial nor a final state and $\Delta$ is an update set yielded by the rule $r$ in $S$, then every update multiset $\Delta'$ yielded by $r$ in $S$ contains at least one update $(\text{MDist}, (), a_i)$ such that $a_i$ coincides with the value stored in the location $\text{MDist}, ()$ in the successor state $S + \Delta$ and $a_i \leq a_j$ for every update $(\text{MDist}, (), a_j)$ in $\Delta$. Representing $\Delta$ and $\Delta'$ by the variables $X$ and $\bar{X}$, respectively, we obtain:

$$\forall X \left( \neg \text{Initial} \land \exists xy(\text{City}(x,y) \land \neg \text{Visited}(x)) \land \text{upd}(r, X) \rightarrow 
\right.$$  

$$[X] \text{MDist} = \text{MIN}_x (\text{Dist}(x) \ | \ \exists y(\text{City}(x,y) \land \neg \text{Visited}(x)))) \right)$$

4.2. Semantics

Let $S$ be a state of signature $\Upsilon$ with the base set $B = B_{db} \cup B_a$, $\mathcal{F}_{dyn}$ be the set of all dynamic function symbols in $\Upsilon$ and $\zeta$ be a variable assignment. We define the semantics of the logic for DB-ASMs as follows.

Terms:

- $\text{val}_{S,\zeta}(x) = \zeta(x)$ for $x \in \mathcal{X}_{db}$ with $\zeta(x) \in B_{db}$
- $\text{val}_{S,\zeta}(x) = \zeta(x)$ for $x \in \mathcal{X}_a$ with $\zeta(x) \in B_a$
- $\text{val}_{S,\zeta}(f(t)) = f^S(\text{val}_{S,\zeta}(t))$ and
  - if $f \in \mathcal{Y}_{db}$, then $\text{val}_{S,\zeta}(t) \in B_{db}$ and $f^S(\text{val}_{S,\zeta}(t)) \in B_{db}$
  - if $f \in \mathcal{Y}_{a}$, then $\text{val}_{S,\zeta}(t) \in B_a$ and $f^S(\text{val}_{S,\zeta}(t)) \in B_a$
  - if $f \in \mathcal{F}_b$, then $\text{val}_{S,\zeta}(t) \in B_{db}$ and $f^S(\text{val}_{S,\zeta}(t)) \in B_a$
- $\text{val}_{S,\zeta}(\rho_x(t)[\varphi(x,\bar{y})]) = \rho(\text{val}_{S,\zeta}[x \mapsto a_i](t)|a_i \in B_{db} \text{ and } \text{[\varphi(x,\bar{y})]}_{S,\zeta}[x \mapsto a_i] = \text{true})$

Formulae:

The truth value $\text{[\varphi]}_{S,\zeta}$ of a formula $\varphi$ on state $S$ under variable assignment $\zeta$ is assumed to be either true or false.

- $[s = t]_{S,\zeta} = \text{true}$ iff $\text{val}_{S,\zeta}(s) = \text{val}_{S,\zeta}(t)$
- $[s_a = t_a]_{S,\zeta} = \text{true}$ iff $\text{val}_{S,\zeta}(s_a) = \text{val}_{S,\zeta}(t_a)$
- $[M(t_a, s)]_{S,\zeta} = \text{true}$ iff $(\text{val}_{S,\zeta}(t_a), \text{val}_{S,\zeta}(s)) \in \text{val}_{S,\zeta}(M)$
- $[X(f, t, t_0)]_{S,\zeta} = \text{true}$ iff $(f, \text{val}_{S,\zeta}(t), \text{val}_{S,\zeta}(t_0)) \in \text{val}_{S,\zeta}(X)$. 

19
\[ [\mathcal{X}(f, t, t_0, s)]_{S, \zeta} = \text{true} \text{ iff } (f, \text{val}_{S, \zeta}(t), \text{val}_{S, \zeta}(t_0), \text{val}_{S, \zeta}(s)) \in \text{val}_{S, \zeta}(\mathcal{X}). \]

\[ [\tilde{\mathcal{X}}(f, t, t_0, t_0)]_{S, \zeta} = \text{true} \text{ iff } (f, \text{val}_{S, \zeta}(t), \text{val}_{S, \zeta}(t_0), \text{val}_{S, \zeta}(t_0)) \in \text{val}_{S, \zeta}(\tilde{\mathcal{X}}). \]

\[ [\tilde{\mathcal{X}}(f, t, t_0, a, s)]_{S, \zeta} = \text{true} \text{ iff } (f, \text{val}_{S, \zeta}(t), \text{val}_{S, \zeta}(t_0), \text{val}_{S, \zeta}(a), \text{val}_{S, \zeta}(s)) \in \text{val}_{S, \zeta}(\tilde{\mathcal{X}}). \]

\[ [F(f, t, t_0, t_0', t_0', t_0', a, s)]_{S, \zeta} = \text{true} \text{ iff } (f, \text{val}_{S, \zeta}(t), \text{val}_{S, \zeta}(t_0), \text{val}_{S, \zeta}(t_0'), \text{val}_{S, \zeta}(t_0'), \text{val}_{S, \zeta}(a), \text{val}_{S, \zeta}(s)) \in \text{val}_{S, \zeta}(F). \]

\[ [G(f, t, t_0, t_0, t_0', t_0', t_0', a, s)]_{S, \zeta} = \text{true} \text{ iff } (f, \text{val}_{S, \zeta}(t), \text{val}_{S, \zeta}(t_0), \text{val}_{S, \zeta}(t_0'), \text{val}_{S, \zeta}(t_0'), \text{val}_{S, \zeta}(a), \text{val}_{S, \zeta}(s)) \in \text{val}_{S, \zeta}(G). \]

\[ \lbrack\neg \varphi \rbrack_{S, \zeta} = \text{true iff } \lbrack\varphi \rbrack_{S, \zeta} = \text{false}. \]

\[ \lbrack\varphi \land \psi \rbrack_{S, \zeta} = \text{true iff } \lbrack\varphi \rbrack_{S, \zeta} = \text{true and } \lbrack\psi \rbrack_{S, \zeta} = \text{true}. \]

\[ \lbrack\forall x (\varphi) \rbrack_{S, \zeta} = \text{true iff } \lbrack\varphi \rbrack_{S, \zeta[x \mapsto a]} = \text{true for all } a \in B_{db}. \]

\[ \lbrack\forall x (\varphi) \rbrack_{S, \zeta} = \text{true iff } \lbrack\varphi \rbrack_{S, \zeta[x \mapsto a]} = \text{true for all } a \in B_{a}. \]

\[ \lbrack\forall M (\varphi) \rbrack_{S, \zeta} = \text{true iff } \lbrack\varphi \rbrack_{S, \zeta[M \mapsto P]} = \text{true for all finite } P \subseteq B_{db} \times B_{a}. \]

\[ \lbrack\forall X (\varphi) \rbrack_{S, \zeta} = \text{true iff } \lbrack\varphi \rbrack_{S, \zeta[X \mapsto P]} = \text{true for all finite } P \subseteq \mathcal{F}_{\text{dyn}} \times B_{db} \times B_{a}. \]

\[ \lbrack\forall \tilde{X} (\varphi) \rbrack_{S, \zeta} = \text{true iff } \lbrack\varphi \rbrack_{S, \zeta[\tilde{X} \mapsto P]} = \text{true for all finite } P \subseteq \mathcal{F}_{\text{dyn}} \times B_{db} \times B_{a} \times B_{db}. \]

\[ \lbrack\forall \tilde{X} (\varphi) \rbrack_{S, \zeta} = \text{true iff } \lbrack\varphi \rbrack_{S, \zeta[\tilde{X} \mapsto P]} = \text{true for all finite } P \subseteq \mathcal{F}_{\text{dyn}} \times B_{db} \times B \times B_{db} \times B_{a} \times B_{db}. \]

\[ \lbrack\forall F (\varphi) \rbrack_{S, \zeta} = \text{true iff } \lbrack\varphi \rbrack_{S, \zeta[F \mapsto P]} = \text{true for all finite } P \subseteq \mathcal{F}_{\text{dyn}} \times B_{db} \times B \times B_{db} \times B \times B_{db} \times B_{a} \times B_{db}. \]

\[ \lbrack\forall G (\varphi) \rbrack_{S, \zeta} = \text{true iff } \lbrack\varphi \rbrack_{S, \zeta[G \mapsto P]} = \text{true for all finite } P \subseteq \mathcal{F}_{\text{dyn}} \times B_{db} \times B \times B_{a} \times B_{db} \times B \times B_{db} \times B \times B_{a} \times B_{db}. \]

\[ \lbrack\text{upd}(r, X) \rbrack_{S, \zeta} = \text{true iff } \text{val}_{S, \zeta}(X) \in \Delta(r, S, \zeta). \]

\[ \lbrack\text{upm}(r, \tilde{X}) \rbrack_{S, \zeta} = \text{true iff } \{ (x_1, x_2, x_3) \mid (x_1, x_2, x_3, y) \in \text{val}_{S, \zeta}(\tilde{X}) \text{ for some } y \in B_{a} \} \in \Delta(r, S, \zeta). \]

\[ \lbrack[X] \varphi \rbrack_{S, \zeta} = \text{true iff } \Delta = \zeta(X) \text{ is inconsistent or } \lbrack\varphi \rbrack_{S + \Delta, \zeta} = \text{true for } \zeta(X) = \Delta \in \Delta(r, S, \zeta). \]

When $\Delta = \zeta(X)$ is inconsistent, successor states for the current state $S$ do not exist and thus $S + \Delta$ is undefined. In this case, $[X] \varphi$ is interpreted as $\text{true}$. With the use of modal operator $\lbrack\cdot \rbrack$ for an update set $\Delta = \zeta(X)$ (i.e., $[X]$), the logic for DB-ASMs is empowered to be a multi-modal logic. The formulae of the logic for DB-ASMs are interpreted in states represented as a Kripke frame. A Kripke frame is a pair $(U, R)$ consisting of a universe $U$ that is a non-empty set of states, and a binary accessibility relation $R$ on $U$ such that $(S, S') \in R$ for $S, S' \in U$. 
Note that the semantics of the second-order quantifiers in the logic for DB-ASMs, restricts the second-order variables to range over finite sets. This is enough for our purpose of using them to represent update sets and multisets which by Lemma 3.1 are always finite. Furthermore, the finiteness condition on the database part of a state implies that elements in the multiset
\[
\{ \text{val}_{S,\zeta}[x \mapsto a_i] | a_i \in B_{db} \text{ and } [\varphi(x, \bar{y})]_{S,\zeta}[x \mapsto a_i] = \text{true} \}
\]
used in the interpretation of a \(\rho\)-term of the form \(\rho_x(t|\varphi(x, \bar{y}))\) must be finite.

As we shall see later, using a fixed interpretation for (finite) set membership we can get rid of second-order variables completely. A detailed discussion is presented in Section 7.

5. A Proof System

In this section we develop a proof system for the logic for DB-ASMs.

**Definition 5.1.** We say that a state \(S\) is a model of a formula \(\varphi\) (denoted as \(S \models \varphi\)) iff \([\varphi]_{S,\zeta} = \text{true}\) holds for every variable assignment \(\zeta\). If \(\Psi\) is a set of formulae, we say that \(S\) models \(\Psi\) (denoted as \(S \models \Psi\)) iff \(S \models \varphi\) for each \(\varphi \in \Psi\). A formula \(\varphi\) is said to be a logical consequence of a set \(\Psi\) of formulae (denoted as \(\Psi \models \varphi\)) if for every state \(S\), if \(S \models \Psi\), then \(S \models \varphi\). A formula \(\varphi\) is said to be valid (denoted as \(\models \varphi\)) if \([\varphi]_{S,\zeta} = \text{true}\) in every state \(S\) with every variable assignments \(\zeta\). A formula \(\varphi\) is said to be derivable from a set \(\Psi\) of formulae (denoted as \(\Psi \vdash \varphi\)) if there is a deduction from formulae in \(\Psi\) to \(\varphi\) by using a set \(R\) of axioms and inference rules.

We will define such a set \(R\) of axioms and rules in Subsection 5.4. Then we simply write \(\vdash\) instead of \(\vdash_R\). We also define equivalence between two DB-ASM rules. Two equivalent rules \(r_1\) and \(r_2\) are either both defined or both undefined.

**Definition 5.2.** Let \(r_1\) and \(r_2\) be two DB-ASM rules. Then \(r_1\) and \(r_2\) are equivalent (denoted as \(r_1 \equiv r_2\)) if for every state \(S\) it holds that \(S \models \forall X(\text{upd}(r_1, X) \leftrightarrow \text{upd}(r_2, X))\).

The substitution of a term \(t\) for a variable \(x\) in a formula \(\varphi\) (denoted as \(\varphi[t/x]\)) is defined by the rule of substitution. That is, \(\varphi[t/x]\) is the result of replacing all free instances of \(x\) by \(t\) in \(\varphi\) provided that no free variable of \(t\) becomes bound after substitution.

5.1. Consistency

In [30] Nanchen and Stärk use a predicate \(\text{Con}(r)\) as an abbreviation for the statement that the rule \(r\) is consistent. As a rule \(r\) in their work is considered to be deterministic, there is no ambiguity with the reference to the update set associated with \(r\), i.e., each deterministic rule \(r\) generates exactly one (possibly empty) update set. Thus a deterministic rule \(r\) is consistent iff the update set generated by \(r\) is consistent. However, in the logic for DB-ASMs, the presence of non-determinism makes the situation less straightforward.
Let \( r \) be a DB-ASM rule and \( \Delta \) be an update set. Then the consistency of an update set \( \Delta \), denoted by the formula \( \text{conUSet}(X) \) (where \( X \) represents \( \Delta \)), can be expressed as:

\[
\text{conUSet}(X) \equiv \bigwedge_{f \in \mathcal{F}_{\text{dyn}}} \forall xyz((X(f,x,y) \land X(f,x,z)) \rightarrow y = z)
\]  

(1)

Then \( \text{con}(r, X) \) is an abbreviation of the following formula which expresses that an update set \( \Delta \) (represented by the variable \( X \)) generated by the rule \( r \) is consistent.

\[
\text{con}(r, X) \equiv \text{upd}(r, X) \land \text{conUSet}(X)
\]  

(2)

As the rule \( r \) may be non-deterministic, it is possible that \( r \) yields several update sets. Thus, we develop the consistency of DB-ASM rules in two versions:

- A rule \( r \) is \textit{weakly consistent} (denoted as \( \text{wcon}(r) \)) if at least one update set generated by \( r \) is consistent. This can be expressed as follows:

\[
\text{wcon}(r) \equiv \exists X(\text{con}(r, X))
\]  

(3)

- A rule \( r \) is \textit{strongly consistent} (denoted as \( \text{scon}(r) \)) if every update set generated by \( r \) is consistent. This can be expressed as follows:

\[
\text{scon}(r) \equiv \forall X(\text{upd}(r, X) \rightarrow \text{conUSet}(X))
\]  

(4)

In the case that a rule \( r \) is deterministic, the weak notion of consistency coincides with the strong notion of consistency, i.e., \( \text{wcon}(r) \leftrightarrow \text{scon}(r) \).

5.2. Update Sets

We present the axioms for predicate \( \text{upd}(r, X) \) in Fig. [7]. To simplify the presentation, we give the formulae only for the case in which all the function symbols in \( \mathcal{F}_{\text{dyn}} \) correspond to functions on the database part of the state. To deal with dynamic function symbols corresponding to bridge functions, we only need to slightly change the formulae by replacing some of the first-order variables in \( X_{\text{db}} \) by first-order variables in \( X_{a} \). For instance, if \( f \) is a bridge function symbol, we should write \( \forall xy(X(f,x,y) \rightarrow x = t \land y = s) \) instead of \( \forall xy(X(f,x,y) \rightarrow x = t \land y = s) \).

In the following we explain Axioms \( \text{U1-U7} \) in turn. We assume a state \( S \) of some signature \( \Upsilon \) and base set \( B = B_{\text{db}} \cup B_{a} \), and a variable assignment \( \zeta \).

As a DB-ASM rule may be non-deterministic, a straightforward extension from the formalisation of the forall and parallel rules used in the logic for ASMs [30] would not work for Axioms \( \text{U3 and U4} \). The axioms correspond to the definition of update sets in Fig. [1].

- Axiom \( \text{U1} \) says that \( X \) is an update yielded by the assignment rule \( f(t) := s \) iff it contains exactly one update which is \( (f,t,s) \).
U1. \( \text{upd}(f(t) := s, X) \leftrightarrow X(f(t, s) \land \forall x(y(X(f, x, y) \leftrightarrow x = t \land y = s) \land \bigwedge_{f' \in F_{\text{dy}} \setminus f} \forall x(y(X(f', x, y))) \) 

U2. \( \text{upd}(\text{if } \varphi \text{ then } r \text{ endif}, X) \leftrightarrow (\varphi \land \text{upd}(r, X)) \lor (\neg \varphi \land \bigwedge_{f \in F_{\text{dy}}} \forall x(y(X(f, x, y))) \) 

U3. \( \text{upd}(\text{forall } x \text{ with } \varphi \text{ do } r \text{ enddo}, X) \leftrightarrow \) 
\[ \exists X \forall x((\varphi \rightarrow \exists Y(\text{upd}(r, Y) \land \bigwedge_{f \in F_{\text{dy}}} \forall y_1 y_2 (Y(f, y_1, y_2) \leftrightarrow X(f, y_1, y_2, x)))) \land \\
(\neg \varphi \rightarrow \bigwedge_{f \in F_{\text{dy}}} \forall y_1 y_2 (-X(f, y_1, y_2, x)))) \land \\
\bigwedge_{f \in F_{\text{dy}}} \forall x_1 x_2 (X(f, x_1, x_2) \leftrightarrow \exists x_3 (X(f, x_1, x_2, x_3))) \) 

U4. \( \text{upd}(\text{par } r_1 r_2 \text{ endpar}, X) \leftrightarrow \) 
\[ \exists Y_1 Y_2(\text{upd}(r_1, Y_1) \land \text{upd}(r_2, Y_2) \land \\
\bigwedge_{f \in F_{\text{dy}}} \forall x(y(X(f, x, y) \leftrightarrow (Y_1(f, x, y) \lor Y_2(f, x, y)))) \) 

U5. \( \text{upd}(\text{choose } x \text{ with } \varphi \text{ do } r \text{ enddo}, X) \leftrightarrow \exists x(\varphi \land \text{upd}(r, X)) \) 

U6. \( \text{upd}(\text{seq } r_1 r_2 \text{ endseq}, X) \leftrightarrow (\text{upd}(r_1, X) \land \neg \text{conUSet}(X)) \lor \\
(\exists Y_1 Y_2(\text{upd}(r_1, Y_1) \land \text{conUSet}(Y_1) \land [Y_1] \text{upd}(r_2, Y_2) \land \\
\bigwedge_{f \in F_{\text{dy}}} \forall x(y(X(f, x, y) \leftrightarrow (Y_1(f, x, y) \land \forall x_2 (x_2) \land Y_2(f, x, y))))) \) 

U7. \( \text{upd}(\text{let } (f, t) \rightarrow \rho \text{ in } r \text{ endlet}, X) \leftrightarrow \\
\exists \tilde{X}(\text{upm}(r, \tilde{X}) \land \forall x(y(X(f, x, y) \leftrightarrow ((t = x \land y = \rho y'(y' \exists z(\tilde{X}(f, x, y', z)))) \lor \\
(t \neq x \land \exists z(\tilde{X}(f, x, y, z)))) \land \\
\bigwedge_{f' \in F_{\text{dy}}} \forall x(y(X(f', x, y) \leftrightarrow \exists z(\tilde{X}(f', x, y, z)))) \) 

Figure 7: Axioms for predicate upd(r,X)

- Axiom U2 asserts that, if the formula \( \varphi \) evaluates to true, then \( X \) is an update set yielded by the conditional rule if \( \varphi \) then \( r \) endif iff \( X \) is an update set yielded by the rule \( r \). Otherwise, the conditional rule yields only an empty update set.

- Axiom U3 states that \( X \) is an update set yielded by the rule forall \( x \) with \( \varphi \) do r enddo iff \( X \) coincides with \( \Delta_{a_1} \cup \cdots \cup \Delta_{a_n} \), where \( \{a_1, \ldots, a_n\} = \{a_i \in B_{db} | \)
val_{S,\zeta}[x \mapsto a_i](\varphi) = true \} and \Delta_{a_i} (for \ 1 \leq i \leq n) is an update set yielded by the rule \ r \ under the variable assignment \ \zeta[x \mapsto a_i]. \ Note \ that \ the \ update \ sets \ \Delta_{a_1}, \ldots, \Delta_{a_n} \ are \ encoded \ into \ the \ second-order \ variable \ \mathcal{X}.

- Axiom U4 states that X is an update set yielded by the parallel rule par r_1 r_2 endpar iff it corresponds to the union of an update set yielded by r_1 and an update set yielded by r_2.

- Axiom U5 asserts that X is an update set yielded by the rule choose x with \varphi \ do r enddo iff it is an update set yielded by the rule r under a variable assignment \ \zeta[x \mapsto a] \ which \ satisfies \ \varphi.

- Axiom U6 asserts that X is an update set yielded by a sequence rule seq r_1 r_2 endseq iff it corresponds either to an inconsistent update set yielded by rule r_1, or to an update set formed by the updates in an update set Y_2 yielded by rule r_2 in a successor state \ S + Y_1, \ where \ Y_1 \ encodes \ a \ consistent \ set \ of \ updates \ produced \ by \ rule \ r_1, \ plus \ the \ updates \ in \ Y_1 \ that \ correspond \ to \ locations \ other \ than \ the \ locations \ updated \ by \ Y_2.

- Axiom U7 asserts that an update multiset \ \X \ is collapsed into an update set \ \Delta \ by aggregating the update values to the location \ (f,t) \ which \ appear \ in \ \X \ using \ the \ location \ operator \ \rho, \ and \ by \ ignoring \ multiple \ identical \ updates \ to \ the \ same \ location \ if \ that \ locations \ is \ not \ (f,t).

The following lemma is an easy consequence of the axioms in Figure 7.

**Lemma 5.1.** Each formula in the DB-ASM logic can be replaced by an equivalent formula not containing any subformulae of the form upd(r, X).

### 5.3. Update Multisets

Each DB-ASM rule is associated with a set of update multisets. Axioms UM1-UM7 in Fig. 8 assert how an update multiset is yielded by a DB-ASM rule. Same as in the axioms for update sets, we simplify the presentation by giving the formulae only for the case in which all the function symbols in F_{dyn} correspond to functions on the database part of the state. The case of bridge functions can be dealt with similarly.

The axioms for the predicate upm(r, \X) are analogous to the axioms for the predicate upd(r, X), except for Axioms UM6 and UM7, and the fact that we need to deal with multisets represented as relations.

It should be noted that two second-order variables of type \X and \X' encode the same update multiset \X if there is a bijections \ F (i.e., a one-to-one correspondence) from \X to \X' that maps each tuple \ (f,t,t_0,a) \ in \X which represents an occurrence of a tuple \ (f,t,t_0) \ in \X \ to \ a \ corresponding \ tuple \ \X' of \ (f,t,t_0',s) \ in \X' which represents a corresponding occurrence of \ (f,t,t_0) \ in \X'. \ In \ particular, \ “F \ is \ a \ bijection \ from \ \X'|_f \ to \ \X'|_f\” \ in \ Axiom \ UM3 \ refers \ to \ the \ following \ formula:
∀\(x_1x_2x_3y_1y_2y_3y_4z_1z_2z_3z_4\) \begin{align*}
(F_f(f, x_1, x_2, x_3, y_1, y_2, y_3, y_4) & \rightarrow \tilde{X}(f, x_1, x_2, x_3) \land \tilde{X}(f, y_1, y_2, y_3, y_4)) \land \\
(F_f(f, x_1, x_2, x_3, y_1, y_2, y_3, y_4) & \land F_f(f, x_1, x_2, x_3, z_1, z_2, z_3, z_4) \rightarrow \ y_1 = z_1 \land y_2 = z_2 \land y_3 = z_3 \land y_4 = z_4) \land \\
(\tilde{X}(f, x_1, x_2, x_3) & \rightarrow \exists y_1y_2y_3y_4(F_f(f, x_1, x_2, x_3, y_1, y_2, y_3, y_4))) \land \\
(F_f(f, x_1, x_2, x_3, z_1, z_2, z_3, z_4) & \land F_f(f, y_1, y_2, y_3, z_1, z_2, z_3, z_4) \rightarrow \ x_1 = y_1 \land x_2 = y_2 \land x_3 = y_3) \land \\
(\tilde{X}(f, y_1, y_2, y_3, y_4) & \rightarrow \exists x_1x_2x_3(F_f(f, x_1, x_2, x_3, y_1, y_2, y_3, y_4)))
\end{align*}

Likewise, “\(G_f\) is a bijection from \(\tilde{X}|_f\) to \((\tilde{Y}_1 \uplus \tilde{Y}_2)|_f\)” in Axiom \textbf{UM4} is expressed by the following formula.

\begin{align*}
∀\(x_1x_2x_3y_1y_2y_3y_4z_1z_2z_3z_4\) \begin{align*}
(G_f(f, x_1, x_2, x_3, y_1, y_2, y_3, y_4) & \rightarrow \tilde{X}(f, x_1, x_2, x_3) \land (\tilde{Y}_1(f, y_1, y_2, y_3) \lor \tilde{Y}_2(f, y_1, y_2, y_3))) \land \\
(G_f(f, x_1, x_2, x_3, y_1, y_2, y_3, y_4) & \land G_f(f, x_1, x_2, x_3, z_1, z_2, z_3, z_4) \rightarrow \ y_1 = z_1 \land y_2 = z_2 \land y_3 = z_3 \land y_4 = z_4) \land \\
(\tilde{X}(f, x_1, x_2, x_3) & \rightarrow \exists y_1y_2y_3y_4(G_f(f, x_1, x_2, x_3, y_1, y_2, y_3, y_4))) \land \\
(G_f(f, x_1, x_2, x_3, z_1, z_2, z_3, z_4) & \land G_f(f, y_1, y_2, y_3, z_1, z_2, z_3, z_4) \rightarrow \ x_1 = y_1 \land x_2 = y_2 \land x_3 = y_3) \land \\
((\tilde{Y}_1(f, y_1, y_2, y_3) \lor \tilde{Y}_2(f, y_1, y_2, y_3)) & \rightarrow \exists x_1x_2x_3y_4(G_f(f, x_1, x_2, x_3, y_1, y_2, y_3, y_4))) \land \\
((\tilde{Y}_1(f, y_1, y_2, y_3) & \lor \tilde{Y}_2(f, y_1, y_2, y_3)) \rightarrow \exists x_1x_2x_3y_4x_1'x_2'x_3'y_4'(x_1 \neq x_1' \lor x_2 \neq x_2' \lor x_3 \neq x_3') \land y_4 \neq y_4') \land \\
G_f(f, x_1, x_2, x_3, y_1, y_2, y_3, y_4) & \land G_f(f, x_1, x_2, x_3, y_1, y_2, y_3, y_4')
\end{align*}
\end{align*}

In turn, Axioms \textbf{UM6} and \textbf{UM7} can be described as follows:

- Let a multiset of updates encoded into a second-order variable \(\tilde{X}\) be considered consistent if its corresponding update set \(X\) obtained by ignoring the multiplicities in \(\tilde{X}\) is consistent, and let it be considered inconsistent otherwise. Axiom \textbf{UM6} asserts that \(\tilde{X}\) is a multiset of updates yielded by a sequence rule \textbf{seq} \(r_1\) \(r_2\) \textbf{endseq} iff it corresponds either to an inconsistent multiset of updates yielded by rule \(r_1\), or to a multiset of updates formed by the updates in a multiset of updates \(\tilde{Y}_2\) yielded by rule \(r_2\) in a successor state \(S + Y_1\), where \(\tilde{Y}_1\) encodes a consistent multiset of updates produced by rule \(r_1\) and \(Y_1\) its corresponding update set, plus the updates in \(\tilde{Y}_1\) that correspond to locations other than the locations in the multiset of updates \(\tilde{Y}_2\).
• Axiom UM7 asserts that $\mathcal{X}$ is an update multiset yielded by a let rule $\text{let } (f, t) \rightarrow \rho \text{ in } r \text{ endlet}$ iff it corresponds to an update multiset $\mathcal{Y}$ yielded by the rule $r$ except for the updates to the location $(f, t)$ which are collapsed to a unique update in $\mathcal{X}$ by aggregating their values using the operator $\rho$.

Analogous to Lemma 5.1, the following result is a straightforward consequence of the axioms in Figures 7 and 8.

**Lemma 5.2.** Each formula in the DB-ASM logic can be replaced by an equivalent formula not containing any subformulae of the form $\text{upd}(r, X)$ or $\text{upm}(r, \mathcal{X})$.

To obtain the aggregated value $s$ of an update $(f, t, s)$ in an update set which comes from multiple updates $(f, t, s_1), \ldots, (f, t, s_n)$ in an update multiset encoded into a variable of type $\mathcal{X}$, we need $\rho\gamma(y|\exists z(\mathcal{X}(f, t, y, z)))$ such that

$$\text{val}_{S,\zeta}(\rho\gamma(y|\exists z(\mathcal{X}(f, t, y, z)))) = \rho(\{s_1, \ldots, s_n\}) = s.$$ 

Therefore, $\rho$-terms play a vital role in formalising operations over a multiset.

**Remark 1.** The inclusion of the parameters $X$ and $\mathcal{X}$ in the predicates $\text{upd}(r, X)$ and $\text{upm}(r, \mathcal{X})$ is important because a DB-ASM rule $r$ may be associated with multiple update sets or multisets and we need a way to specify which update set or multiset yielded by rule $r$ is meant. Furthermore, the use of predicates $\text{upd}(r, X)$ and $\text{upm}(r, \mathcal{X})$ provides the capability to specify the interrelationship among update sets or multisets associated with possibly different rules, e.g., Axiom U7.

### 5.4. Axioms and Inference Rules

Now we can present a set of axioms and inference rules which constitute a proof system for the logic for DB-ASMs. To avoid unnecessary repetitions of almost identical axioms and rules, we describe them only considering first-order variables in $X_{db}$ which range over the database part of the state, but the exact same axioms and inference rules are implicitly assumed for the case of first-order variables in $X_a$.

Formally, the set $\mathcal{R}$ of axioms and inference rules is formed by:

• The axioms U1-U7 in Fig. 7 which assert the properties of $\text{upd}(r, X)$.

• The axioms UM1-UM7 in Fig. 8 which assert the properties of $\text{upm}(r, \mathcal{X})$.

• Axiom M1 and Rules M2-M3 from the axiom system K of modal logic, which is the weakest normal modal logic system [19]. Axiom M1 is called *Distribution Axiom* of K, Rule M2 is called *Necessitation Rule* of K and Rule M3 is the inference rule called *Modus Ponens* in the classical logic. By using these axiom and rules together, we are able to derive all modal properties that are valid in Kripke frames.

**M1** \([X](\varphi \rightarrow \psi) \rightarrow ([X]\varphi \rightarrow [X]\psi)\)
UM1. \(\text{upm}(f(t) := s, \bar{X}) \leftrightarrow\)
\[
\exists z_1(\bar{X}(f, t, s, z_1) \land \forall x y z_2(\bar{X}(f, x, y, z_2) \rightarrow x = t \land y = s \land z_1 = z_2)) \land
\forall_{f \neq f^1} \in \mathcal{F}_{\text{dyn}} \forall x y z(\neg \bar{X}(f', x, y, z))
\]

UM2. \(\text{upm}(\text{if } \varphi \text{ then } r \text{ endif}, \bar{X}) \leftrightarrow (\varphi \land \text{upm}(r, \bar{X})) \lor
(\neg \varphi \land \bigwedge_{f \in \mathcal{F}_{\text{dyn}}} \forall x y z(\neg \bar{X}(f, x, y, z)))
\]

UM3. \(\text{upm}(\forall x \text{ with } \varphi \text{ do } r \text{ enddo}, \bar{X}) \leftrightarrow\)
\[
\exists \bar{X}(\forall x ((\varphi \rightarrow \exists \bar{Y}(\text{upm}(r, \bar{Y})) \land
\bigwedge_{f \in \mathcal{F}_{\text{dyn}}} \forall y_1 y_2 y_3(\bar{Y}(f, y_1, y_2, y_3) \leftrightarrow \bar{X}(f, y_1, y_2, y_3, x))) \land
(\neg \varphi \rightarrow \bigwedge_{f \in \mathcal{F}_{\text{dyn}}} \forall y_1 y_2 y_3(\neg \bar{X}(f, y_1, y_2, y_3, x))) \land
\bigwedge_{f \in \mathcal{F}_{\text{dyn}}} \exists \bar{F}_f ("F_f is a bijection from \bar{X}|_f to \bar{X}|_f") \land
\forall x_1 x_2 x_3 (\bar{X}(f, x_1, x_2, x_3) \leftrightarrow \exists y_3 y_4 (F_f(f, x_1, x_2, x_3, x_1, x_2, y_3, y_4))))
\]

UM4. \(\text{upm}(\text{par } r_1 r_2 \text{ endpar}, \bar{X}) \leftrightarrow\)
\[
\exists \bar{Y}_1 \bar{Y}_2(\text{upm}(r_1, \bar{Y}_1) \land \text{upm}(r_2, \bar{Y}_2) \land
\bigwedge_{f \in \mathcal{F}_{\text{dyn}}} \exists \bar{G}_f ("G_f is a bijection from \bar{X}|_f to (\bar{Y}_1 \uplus \bar{Y}_2)|_f") \land
\forall x_1 x_2 x_3 (\bar{X}(f, x_1, x_2, x_3) \leftrightarrow \exists y_3 y_4 (G_f(f, x_1, x_2, x_3, x_1, x_2, y_3, y_4))))
\]

UM5. \(\text{upm}(\text{choose } x \text{ with } \varphi \text{ do } r \text{ enddo}, \bar{X}) \leftrightarrow \exists x(\varphi \land \text{upm}(r, \bar{X}))\)

UM6. \(\text{upm}(\text{seq } r_1 r_2 \text{ endseq}, \bar{X}) \leftrightarrow\)
\[
(\text{upm}(r_1, \bar{X}) \land \forall X (\bigwedge_{f \in \mathcal{F}_{\text{dyn}}} \forall x_1 x_2 (X(f, x_1, x_2) \leftrightarrow \exists x_3 (\bar{X}(f, x_1, x_2, x_3)) \land
\neg \text{conUSet}(X))) \lor
(\exists \bar{Y}_1 \bar{Y}_2 (\text{upm}(r_1, \bar{Y}_1) \land \forall Y_1 (\bigwedge_{f \in \mathcal{F}_{\text{dyn}}} \forall x_1 x_2 (Y_1(f, x_1, x_2) \leftrightarrow \exists x_3 (\bar{Y}_1(f, x_1, x_2, x_3))) \land
\text{conUSet}(Y_1) \land [Y_1]\text{upm}(r_2, \bar{Y}_2)) \land
\bigwedge_{f \in \mathcal{F}_{\text{dyn}}} \forall x_1 x_2 x_3 (\bar{X}(f, x_1, x_2, x_3) \leftrightarrow (\bar{Y}_2(f, x_1, x_2, x_3) \lor (\bar{Y}_1(f, x_1, x_2, x_3) \land \forall y_2 y_3 (\neg \bar{Y}_2(f, x_1, y_2, y_3))))))
\]

UM7. \(\text{upm}(\text{let } (f, t) \rightarrow \rho \text{ in } r \text{ endlet}, \bar{X}) \leftrightarrow\)
\[
\exists \bar{Y} y (\text{upm}(r, \bar{Y}) \land \forall x_1 x_2 x_3 (\bar{X}(f, x_1, x_2, x_3) \leftrightarrow ((t \neq x_1 \land \bar{Y}(f, x_1, x_2, x_3)) \lor
(t = x_1 \land x_2 = \rho_y (y | \exists z (\bar{Y}(f, x_1, y, z)) \land x_3 = y))) \land
\bigwedge_{f \neq f^1 \in \mathcal{F}_{\text{dyn}}} \forall x_1 x_2 x_3 (\bar{X}(f', x_1, x_2, x_3) \leftrightarrow \bar{Y}(f', x_1, x_2, x_3)))
\]

Figure 8: Axioms for predicate upm(r, \bar{X})
\( M2 \) \( \varphi \vdash [X]\varphi \)
\( M3 \) \( \varphi, \varphi \rightarrow \psi \vdash \psi \)

- Axiom \( M4 \) asserts that, if an update set \( \Delta \) is not consistent, then there is no successor state obtained after applying \( \Delta \) over the current state and thus \( [X]\varphi \) (for \( X \) interpreted by \( \Delta \)) is interpreted as true for any formula \( \varphi \). As applying a consistent update set \( \Delta \) over the current state is deterministic, Axiom \( M5 \) describes the deterministic accessibility relation in terms of \( [X] \).

\( M4 \) \( \neg \text{conUSet}(X) \rightarrow [X]\varphi \)
\( M5 \) \( \neg [X]\varphi \rightarrow [X]\neg \varphi \)

- Axiom \( M6 \) is called Barcan Axiom. It originates from the fact that all states in a run of a DB-ASM have the same base set, and thus the quantifiers in all states always range over the same set of elements. This axiom also applies to the second-order quantifiers in the logic of DB-ASMs.

\( M6 \) \( \forall x([X]\varphi) \rightarrow [X]\forall x(\varphi) \)

- Axioms \( M7 \) and \( M8 \) assert that the interpretation of static and pure formulae is the same in all states of a DB-ASM, which is not affected by the execution of any DB-ASM rule \( r \). Note that, depending on the logic that is parameterised into the logic of meta-finite states, static and pure formulae might not be first-order formulae.

\( M7 \) \( \text{con}(r, X) \land \varphi \rightarrow [X]\varphi \) for static and pure \( \varphi \)
\( M8 \) \( \text{con}(r, X) \land [X]\varphi \rightarrow \varphi \) for static and pure \( \varphi \)

- Axiom \( A1 \) asserts that, if a consistent update set \( \Delta \) does not contain any update to the location \((f, x)\), then the content of \((f, x)\) in a successor state obtained after applying \( \Delta \) is the same as its content in the current state. Axiom \( A2 \) asserts that, if a consistent update set \( \Delta \) does contain an update which changes the content of the location \((f, x)\) to \( y \), then the content of \((f, x)\) in the successor state obtained after applying \( \Delta \) is equal to \( y \). Axiom \( A3 \) says that, if a DB-ASM rule \( r \) yields an update multiset, then the rule \( r \) also yields an update set.

\( A1 \) \( \text{conUSet}(X) \land \forall z(\neg X(f, x, z)) \land f(x) = y \rightarrow [X]f(x) = y \)
\( A2 \) \( \text{conUSet}(X) \land X(f, x, y) \rightarrow [X]f(x) = y \)
\( A3 \) \( \text{upm}(r, \tilde{X}) \rightarrow \exists X(\text{upd}(r, X)) \)

- The following are axiom schemes from classical logic.
\(P1\) \(\varphi \rightarrow (\psi \rightarrow \varphi)\)

\(P2\) \((\varphi \rightarrow (\psi \rightarrow \chi)) \rightarrow ((\varphi \rightarrow \psi) \rightarrow (\varphi \rightarrow \chi))\)

\(P3\) \((-\varphi \rightarrow -\psi) \rightarrow (\psi \rightarrow \varphi)\)

- The following four inference rules describe when the universal and existential quantifiers can be added to or deleted from a statement. Rule UI, called Universal Instantiation, says that \(\forall x(\varphi)\) implies \(\varphi[t/x]\) for any term \(t\) which evaluates to an element in the (appropriate) base set. Rule EG, called Existential Generalisation, says that if some element in the (appropriate) base set which is represented by the term \(t\) has the property \(\varphi\), then \(\exists x(\varphi)\) must hold. Rule UG, called Universal Generalisation, says that if every element from the (appropriate) base set has the property \(\varphi\) then we may infer that \(\forall (\varphi)\) holds. Rule EI, called Existential Instantiation, says that, if \(\exists x(\varphi)\) holds, then we may infer \(\varphi[t/x]\) for any term \(t\) which represents a valuation for \(x\) which satisfies \(\varphi\).

UI \(\forall x(\varphi) \vdash \varphi[t/x]\) if \(\varphi\) is pure or \(t\) is static.

EG \(\varphi[t/x] \vdash \exists x(\varphi)\) if \(\varphi\) is pure or \(t\) is static.

UG \(\varphi[t_i/x] \vdash \forall x(\varphi)\) if \(\varphi[t_i/x]\) holds for every element \(a_i\) in the domain of \(x\) and corresponding term \(t_i\) representing \(a_i\), and further \(\varphi\) is pure or every \(t_i\) is static.

EI \(\exists x(\varphi) \vdash \varphi[t/x]\) if \(t\) represents a valuation for \(x\) which satisfies \(\varphi\), and further \(\varphi\) is pure or \(t\) is static.

- The following are the equality axioms from first-order logic with equality. Axiom EQ1 asserts the reflexivity property, Axiom EQ2 asserts the substitutions for functions, and Axiom EQ3 asserts the substitutions for \(\rho\)-terms. Again, terms occurring in the axioms are restricted to be static, which do not contain any dynamic function symbols.

\(EQ1\) \(t = t\) for static term \(t\)

\(EQ2\) \(t_1 = t_{n+1} \land \ldots \land t_n = t_{2n} \rightarrow f(t_1, \ldots, t_n) = f(t_{n+1}, \ldots, t_{2n})\) for any function \(f\) and static terms \(t_i\) (\(i = 1, \ldots, 2n\))

\(EQ3\) \(t_1 = t_2 \land (\varphi_1(x, \bar{y}) \leftrightarrow \varphi_2(x, \bar{y})) \rightarrow \rho_x(t_1|\varphi_1(x, \bar{y})) = \rho_x(t_2|\varphi_2(x, \bar{y}))\) for pure formulae \(\varphi_1\) and \(\varphi_2\), and static terms \(t_1\) and \(t_2\).

- The following axiom is taken from dynamic logic, asserting that executing a sequence rule equals to executing rules sequentially.

\(DY1\) \(\exists X(\text{ upd(seq } r_1 r_2 \text{ endseq}, X) \land [X] \varphi) \leftrightarrow \exists X_1(\text{ upd}(r_1, X_1) \land [X_1] \exists X_2(\text{ upd}(r_2, X_2) \land [X_2] \varphi))\)
• Axiom \( E \) is the extensionality axiom.

\[
E \quad r_1 \equiv r_2 \rightarrow \exists X_1 X_2((\text{upd}(r_1, X_1) \land [X_1]\varphi) \leftrightarrow (\text{upd}(r_2, X_2) \land [X_2]\varphi))
\]

The following soundness theorem for the proof system is relatively straightforward, since the non-standard axioms and rules are just a formalisation of the definitions of the semantics of rules, update sets and update multisets.

**Theorem 5.3.** Let \( \varphi \) be a formula and let \( \Phi \) be a set of formulae in the logic for DB-ASMs. If \( \Phi \vdash \varphi \), then \( \Phi \models \varphi \).

### 6. Derivation

In this section we present some properties of the logic for DB-ASMs which are implied by the axioms and rules from the previous section. This includes some properties known for the logic for ASMs [30]. In particular, the logic for ASMs uses the modal expressions \( [r]\varphi \) and \( \langle r \rangle \varphi \) with the following semantics:

- \( [[r]\varphi]_{S,\zeta} = \begin{cases} 
  \text{true} & \text{if } [\varphi]_{S+\Delta,\zeta} = \text{true} \text{ for all consistent } \Delta \in \Delta(r, S, \zeta), \\
  \text{false} & \text{otherwise}
\end{cases} \)

- \( [\langle r \rangle \varphi]_{S,\zeta} = \begin{cases} 
  \text{true} & \text{if } [\varphi]_{S+\Delta,\zeta} = \text{true} \text{ for at least one consistent } \\
  \Delta \in \Delta(r, S, \zeta), \\
  \text{false} & \text{otherwise}
\end{cases} \)

Instead of introducing modal operators \([\ ]\) and \(\langle \rangle\) for a DB-ASM rule \( r \), we use the modal expression \( [X]\varphi \) for an update set yielded by a possibly non-deterministic rule. The modal expressions \( [r]\varphi \) and \( \langle r \rangle \varphi \) in the logic for ASMs can be treated as the shortcuts for the following formulae in our logic.

\[
[r]\varphi \equiv \forall X(\text{upd}(r, X) \rightarrow [X]\varphi). \quad (5)
\]

\[
\langle r \rangle \varphi \equiv \exists X(\text{upd}(r, X) \land [X]\varphi). \quad (6)
\]

**Lemma 6.1.** The following axioms and rules used in the logic for ASMs are derivable in the logic for DB-ASMs, where the rule \( r \) in Axioms (c) and (d) is assumed to be defined and deterministic.

(a) \( ([r](\varphi \rightarrow \psi) \rightarrow [r]\varphi) \rightarrow [r]\psi \)

(b) \( \varphi \rightarrow [r]\varphi \)

(c) \( \neg \text{wcon}(r) \rightarrow [r]\varphi \)
(d) $[r]\varphi \leftrightarrow \neg[r] \neg \varphi$

Proof. We can prove them as follows:

- (a): By Equation $5$ we have that $[r](\varphi \rightarrow \psi) \land [r] \varphi \equiv \forall X (\text{upd}(r, X) \rightarrow [X](\varphi \rightarrow \psi)) \land \forall X (\text{upd}(r, X) \rightarrow [X] \varphi)$. By the axioms from classical logic, this is in turn equivalent to $\forall X (\text{upd}(r, X) \rightarrow ([X](\varphi \rightarrow \psi) \land [X] \varphi))$. Then by Axiom $M1$ and axioms from the classical logic, we get $\forall X (\text{upd}(r, X) \rightarrow ([X](\varphi \rightarrow \psi) \land [X] \varphi)) \rightarrow \forall X (\text{upd}(r, X) \rightarrow [X] \psi)$. Therefore, $([r](\varphi \rightarrow \psi) \rightarrow [r] \varphi) \rightarrow [r] \psi$ is derivable.

- (b): By Rule $M2$, we have that $\varphi \rightarrow [X_i] \varphi$. Since $X$ is free in $\varphi \rightarrow [X] \varphi$, this holds for every possible valuation of $X$. Thus using Rule $UG$ (applied to the second-order variable $X$) and the axioms from classical logic, we can clearly derive $\varphi \rightarrow \forall X (\text{upd}(r, X) \rightarrow [X] \varphi)$.

- (c): By Equation $3$ we have $\neg \text{con}(r) \leftrightarrow \exists X (\text{con}(r, X))$. In turn, by Equation $2$ we get $\neg \text{con}(r) \leftrightarrow \exists X (\text{upd}(r, X) \land \text{conSet}(X))$. Since a rule $r$ in the logic for ASMs is deterministic, we get $\neg \text{con}(r) \leftrightarrow \neg \text{conSet}(X)$. By Axiom $M4$, we get $\neg \text{con}(r) \rightarrow [r] \varphi$.

- (d): By Equation $5$ we have $\neg[r] \neg \varphi \equiv \exists X (\text{upd}(r, X) \land \neg[X] \neg \varphi)$. By applying Axiom $M5$ to $\neg[X] \neg \varphi$, we get $\neg[r] \neg \varphi \equiv \exists X (\text{upd}(r, X) \land [X] \varphi)$. When the rule $r$ is deterministic, the interpretation of $\forall X (\text{upd}(r, X) \rightarrow [X] \varphi)$ coincides the interpretation of $\exists X (\text{upd}(r, X) \land [X] \varphi)$ and therefore $[r] \varphi \leftrightarrow \neg[r] \neg \varphi$.

The logic for ASMs introduced in [30] is deterministic, i.e., it excludes nondeterministic choice rules. In contrast, our logic for DB-ASMs includes a nondeterministic choice rule. Note that the formula $\text{Con}(R)$ in Axiom $5$ in [30] (i.e., in $\neg \text{Con}(R) \rightarrow [R] \varphi$) corresponds to the weak version of consistency (i.e., $\text{wcon}(r)$) in the context of our logic for DB-ASMs.

Lemma 6.2. The following properties are derivable in the logic for DB-ASMs.

(e) $\text{con}(r, X) \land [X] f(x) = y \rightarrow X(f, x, y) \lor (\forall z (\neg X(f, x, z)) \land f(x) = y)$

(f) $\text{con}(r, X) \land [X] \varphi \rightarrow \neg[X] \neg \varphi$

(g) $[X] \exists x(\varphi) \rightarrow \exists x([X] \varphi)$

(h) $[X] \varphi_1 \land [X] \varphi_2 \rightarrow [X](\varphi_1 \land \varphi_2)$

Proof. (e) is derivable by applying Axioms $A1$ and $A2$. (f) is a straightforward result of Axiom $M5$. (g) can be derived by applying Axioms $M5$ and $M6$. Regarding (h), it is derivable by using Axioms $M1$-$M3$. □
Lemma 6.3. For arbitrary term \( t \) and \( s \), the following properties in \([16]\) are derivable in the logic for DB-ASMs.

\[
\begin{align*}
\bullet & \quad x = t \rightarrow (y = s \leftrightarrow [f(t) := s] f(x) = y) \\
\bullet & \quad x \neq t \rightarrow (y = f(x) \leftrightarrow [f(t) := s] f(x) = y)
\end{align*}
\]

In DB-ASMs, two parallel computations may produce an update multiset, in which there are identical updates to a location assigned with a location operator. Without an outer \texttt{let} rule \texttt{par r r endpar} could be simplified to \texttt{r}. This however is no longer the case if we consider update multisets.

Example 6.1. In the DB-ASM rule below, \texttt{sum} is a location operator assigned to the location \( \texttt{(tnum,())} \). Two identical updates (i.e., \( \texttt{(tnum,(),1)} \) and \( \texttt{(tnum,(),1)} \)) are first generated in an update multiset, and then aggregated into one update \( \texttt{(tnum,(),2)} \) in an update set.

\[
\begin{align*}
\texttt{let} \quad \texttt{(tnum,())} & \rightarrow \texttt{sum in} \texttt{par} \\
\texttt{tnum} & := 1 \\
\texttt{tnum} & := 1 \\
\texttt{endpar} \\
\texttt{endlet}
\end{align*}
\]

The update multiset is collapsed into the update set \( \{ \texttt{(tnum,(),2)} \} \), whereas without the \texttt{let} rule we would obtain \( \{ \texttt{(tnum,(),1)} \} \).

Following the approach of defining the predicate joinable in \([30]\), we define the predicate joinable over two DB-ASM rules. As DB-ASM rules are allowed to be nondeterministic, the predicate joinable\( (r_1, r_2) \) means that there exists a pair of update sets without conflicting updates, which are yielded by rules \( r_1 \) and \( r_2 \), respectively. Then, based on the use of predicate joinable, the properties in Lemma 6.4 are all derivable.

\[
\text{joinable}(r_1, r_2) \equiv \exists X_1 X_2 (\text{upd}(r_1, X_1) \land \text{upd}(r_2, X_2) \land \bigwedge_{f \in F_{\text{dyn}}} \forall x y z (X_1(f, x, y) \land X_2(f, x, z) \rightarrow y = z)) \quad (7)
\]

Lemma 6.4. The following properties for weak consistency are derivable in the logic of DB-ASMs.

(i) \( \wcon(f(t) := s) \)

(j) \( \wcon(\text{if} \ \varphi \ \text{then} \ r \ \text{endif}) \leftrightarrow \neg \varphi \lor (\varphi \land \wcon(r)) \)

(k) \( \wcon(\text{forall} \ x \ \text{with} \ \varphi \ \text{do} \ r \ \text{endo}) \leftrightarrow \\
\forall x (\varphi \rightarrow \wcon(r) \land \forall y (\varphi[y/x] \rightarrow \text{joinable}(r, r[y/x]))) \)

32
We omit the proof of the previous lemma as well as the proof of the remaining lemmas in this section, since they are lengthy but relatively easy exercises.

**Lemma 6.5.** The following properties for the formula $[r]\varphi$ are derivable in the logic for DB-ASMs.

\[(p)\] $[if, \varphi, then, r, endif] \psi \leftrightarrow (\varphi \land [r] \psi) \lor (\neg \varphi \land \psi)$

\[(q)\] $[choose x with \varphi do r enddo] \psi \leftrightarrow \forall x(\varphi \rightarrow [r] \psi)$

**Lemma 6.6.** The following properties for parallel and sequential compositions are derivable in the logic for DB-ASMs.

\[(r)\] $par r_1 r_2 endpar \equiv par r_2 r_1 endpar$

\[(s)\] $par (par r_1 r_2 endpar) r_3 endpar \equiv par r_1 (par r_2 r_3 endpar) endpar$

\[(t)\] $seq (seq r_1 r_2 endseq) r_3 endseq \equiv seq r_1 (seq r_2 r_3 endseq) endseq$

**Lemma 6.7.** The extensionality axiom for transition rules in the logic for ASMs [30] is derivable in the logic for DB-ASMs.

\[(u)\] $r_1 \equiv r_2 \rightarrow ([r_1] \varphi \leftrightarrow [r_2] \varphi)$

7. Completeness

In this section we investigate the completeness of the proof system. As DB-ASMs are a variation of hierarchical ASMs, which do not have recursive rule declarations, we adopt G. Renardel de Lavalette’s approach to prove the completeness of the logic for DB-ASMs, i.e. we will show that the logic for DB-ASMs is a definitional extension of a complete logic. This approach was also used in [30] to prove the completeness of the logic for hierarchical ASMs. However, the logic for hierarchical ASMs in [30] is a definitional extension of
the first-order logic. In the case of the logic for DB-ASMs, things are more complicated since we have to deal with second-order formulae and $\rho$-terms. The key idea is to show instead that the logic for DB-ASMs is a \textit{definitional extension} of first-order logic extended with a finite number of membership predicates with respect to finite sets, which in turn constitutes itself a complete logic.

7.1. $L^\in\subseteq$: A logic with set membership predicates

In the remaining of this section, we will use $\mathcal{L}$ to denote the logic for DB-ASM as introduced in Section 4 and $\mathcal{L}^\in\subseteq$ to denote the following first-order logic extended with set membership predicates.

**Definition 7.1.** The logic $\mathcal{L}^\in\subseteq$ is defined over many sorted first-order structures which have:

- a \textit{finite individual sort} with variables $x_1, x_2, \ldots$ which range over a finite domain $D_1$,
- an \textit{individual sort} with variables $x_1, x_2, \ldots$, which range over a (possibly infinite) domain $D_2$, and
- seven \textit{predicate sorts}, where for every $n = 1, \ldots, 7$, the predicate sort $n$ has variables $x^n_1, x^n_2, \ldots$, which range over the a domain $P_n$ formed by all finite subsets (relations) on
  
  \[ D_1 \times D_2 \]
  \[ \mathcal{F}_{dyn} \times D_1 \times (D_1 \cup D_2) \]
  \[ \mathcal{F}_{dyn} \times D_1 \times (D_1 \cup D_2) \times D_1 \]
  \[ \mathcal{F}_{dyn} \times D_1 \times (D_1 \cup D_2) \times D_2 \]
  \[ \mathcal{F}_{dyn} \times D_1 \times (D_1 \cup D_2) \times D_1 \times D_1 \]
  \[ \mathcal{F}_{dyn} \times D_1 \times (D_1 \cup D_2) \times D_2 \times D_1 \times D_1 \]
  \[ \mathcal{F}_{dyn} \times D_1 \times (D_1 \cup D_2) \times D_2 \times D_1 \times (D_1 \cup D_2) \times D_2 \times D_2 \]

depending on whether $n$ is 1, 2, \ldots or 7, respectively.

A signature $\Sigma$ of the logic $\mathcal{L}^\in\subseteq$ comprises a finite set $F_1$ of names for functions on $D_1$, a finite set $F_2$ of names for functions on $D_2$, a finite set $F_b$ of names for functions which take arguments from $D_1$ and return values on $D_2$, and a finite set $F_\rho$ of names for unary functions from the first predicate sort $P_1 = D_1 \times D_2$ to $D_2$.

We define terms of $\mathcal{L}^\in\subseteq$ by induction. Variables $x_1, x_2, \ldots$ are terms of the first individual sort. Variables $x_1, x_2, \ldots$ are terms of the second individual sort. Every variable of the $i$-th predicate sort is a term of the $i$-th predicate sort. If $f$ is an $n$-ary function name in $F_1$ and $t_1, \ldots, t_n$ are terms of the first individual sort, then $f(t_1, \ldots, t_n)$ is a term of the first individual sort. If $f$ is an $n$-ary function name in $F_2$ and $t_1, \ldots, t_n$ are terms of the second individual sort, then $f(t_1, \ldots, t_n)$ is a term of the second individual sort. If $f$ is an $n$-ary
function of sound and complete axioms and rules for first-order logic and the set of sound and complete axioms and rules in the axiomatisation of the properties of finite sets introduced in [5]. It is not difficult to see that such theory of \( L \) is a conservative extension of the first-order theory, in the sense that if \( \Phi \) is a set of pure first-order formulae and \( \varphi \) is a pure first-order formula (not containing subformulae of the form \( \varepsilon^n(x^n, t_1, \ldots, t_m) \)) and \( \Phi \vdash \varphi \) holds in the theory of \( \mathcal{L}^\varepsilon \), then there already exists a derivation using the sound and complete axioms and rules for first-order logic. Indeed, due to the soundness of the axioms and rules in the theory of \( \mathcal{L}^\varepsilon \), we obtain \( \Phi \models \varphi \), which is a pure statement about models for first-order logic. Thus the known completeness for first-order logic gives \( \Phi \models \varphi \) in an axiomatisation for first-order logic, hence the claimed conservativism of the extension. Since then the theory of \( \mathcal{L}^\varepsilon \) proves no new theorems about first-order logic and all the new theorems belong to the theory of properties of finite sets and thus can be derived by using the sound and complete axioms and rules from [5] (which also form part of the set of axioms and rules in the theory of \( \mathcal{L}^\varepsilon \)), we get the following important result.

**Theorem 7.1.** Let \( \varphi \) be a formula and \( \Phi \) be a set of formulae in the language of \( \mathcal{L}^\varepsilon \). If \( \Phi \models \varphi \), then \( \Phi \vdash \varphi \).
7.2. Completeness of the logic $\mathcal{L}$ for DB-ASMs

In order to show that the logic $\mathcal{L}$ for DB-ASMs is actually a definitional extension of the complete first-order logic extended with set membership predicates $\mathcal{L}^e$, i.e., that we can translate formulae $\varphi$ of $\mathcal{L}$ into formulae $\varphi^*$ of $\mathcal{L}^e$ such that: (a) $\varphi \leftrightarrow \varphi^*$ is derivable in $\mathcal{L}$ and (b) $\varphi^*$ is derivable in $\mathcal{L}^e$ whenever $\varphi$ is derivable in $\mathcal{L}$, we need first to redefine the syntax, semantics and models (states) of $\mathcal{L}$ in a way which is compatible with the syntax, semantics and models of $\mathcal{L}^e$.

Since meta-finite states are just a special kind of two sorted first-order structures in which one of the sorts is finite, we can identify every meta-finite state $S$ of $\mathcal{L}$ with a corresponding many sorted first-order structure $S'$ of the class used in Definition 7.1. This can be done by taking the domains $D_1$ and $D_2$ of the individual sorts of $S'$ to be the base sets $B_{db}$ and $B_a$ of $S$, respectively, the sets $F_1$, $F_2$ and $F_b$ of function names of the signature $\Sigma$ of $S'$ to be the sets $\Upsilon_{db}$, $\Upsilon_a$ and $\mathcal{F}_b$ of the signature $\Upsilon$ of $S$, respectively, and the interpretation in $S'$ of the function names in $\Sigma$ to coincide with the interpretation in $S$ of the corresponding function symbols in $\Upsilon$.

In addition, we treat location (multiset) operators as standard non-axiomatized functions since clearly we cannot axiomatize an arbitrary set $\Lambda$ of location operators. Note that even if we just take a simple location operator such as PRODUCT and axiomatize it, that leads us outside linear arithmetic and we would end up with an incomplete theory. Thus, we identify every location operators $\rho$ in $\Lambda$ with a function name $f_\rho$ in $F_\rho$ and define its interpretation in $S'$ as $f_\rho(A) = \rho_b(\{b | (a, b) \in A\})$ where $A$ is an element in the domain of the first predicate sort $P_1$ of $S'$.

Remark 2. Recall that the finiteness condition on the database part of every state $S$, and thus also on $D_1$, implies that the cardinality of every multiset

$$M = \{val_{S,\varsigma}[x \to a_i](t) | a_i \in B_{db} \text{ and } [\varphi(x, \bar{y})]_{S,\varsigma}[x \to a] = \text{true}\}$$

used in the interpretation of a $\rho$-term of the form $\rho_x(t)[\varphi(x, \bar{y})]$ is always finite. Since in addition $t \in T_\rho$ and $x \in X_{db}$, we can clearly represent every finite multiset $M$ of this type by a corresponding element $A_M$ of the first predicate sort $P_1$ such that $(a, b) \in A_M$ iff $[\varphi(x, \bar{y})]_{S,\varsigma}[x \to a] = \text{true}$ and $val_{S,\varsigma}[x \to a](t) = b$.

We can now redefine the logic $\mathcal{L}$ for DB-ASM by simply replacing in Definition 7.2 the second-order quantifications of the form $\forall M, \forall X, \forall \bar{X}, \forall \bar{\bar{X}}, \forall F$ and $\forall G$ by first-order quantification of the form $\forall x^1, \forall x^2, \forall x^3, \forall x^4, \forall x^5, \forall x^6$ and $\forall x^7$, respectively, and by replacing formulae of the form $\text{upd}(r, X)$, $\text{upm}(r, \bar{X}), [X] \varphi$, $M(s, t_0)$, $X(f, t, t_0)$, $\bar{X}(f, t, t_0, t_a, s)$, $X(f, t, t_0, t_{a}, s)$, $\bar{X}(f, t, t_0, t_1, t')$, and $G(f, t, t_0, t_{a}', t_{a}^\prime, s_0)$ by formulae of the form $\text{upd}(r, x^2)$, $\text{upm}(r, x^4)$, $[x^2] \varphi$, $\in^1(x^1, s, t_a)$, $\in^2(x^2, f, t, t_0)$, $\in^3(x^3, f, t, t_0, s)$, $\in^4(x^4, f, t, t_0, t_a)$, $\in^5(x^5, f, t, t_0, t_{a}, s)$, $\in^6(x^6, f, t, t_0, t_{a}, t', t_{a}' , s)$ and $\in^7(x^7, f, t, t_0, t_{a}', t_{a}^\prime, s)$, respectively.

Regarding $\rho$-terms $\rho$ of the form $\rho_x(t)[\varphi(x, \bar{y})]$ which can appear in an atomic (sub)formulae $\psi(\rho)$, we replace them by a (sub)formulae of the form

$$\forall x^1(\forall x \bar{x}_1(\in^1(x^1, x, x) \leftrightarrow \varphi(x, \bar{y}) \land t = x) \rightarrow \psi(f_\rho(x^1)))$$

36
It is quite clear that this redefinition of the logic $\mathcal{L}$ for DB-ASMs has at least the same expressive power than the original one. Notice that the identification between meta-finite states and many sorted first-order structures, plus the proposed redefinition of $\mathcal{L}$, implies that there is a total function $g_1$ from the set of meta-finite states of $\mathcal{L}$ to the class of many sorted first-order structures used in Definition [7.1] as well a total function $g_2$ from the formulae in the original definition of $\mathcal{L}$ to the formulae in the redefinition of $\mathcal{L}$, respectively, such that for every meta-finite state $S$ and formula $\varphi$ in the original definition of $\mathcal{L}$, $S \models \varphi$ iff $g_1(S) \models g_2(\varphi)$.

Finally, we need to show that all the formulae in the (redefined) logic $\mathcal{L}$ for DB-ASMs which are not formulae of $\mathcal{L}^\mathcal{E}$ can be translated into formulae of $\mathcal{L}^\mathcal{E}$ based on derivable equivalences in the theory of the (redefined) logic $\mathcal{L}$ for DB-ASMs.

First, we reduce the general atomic formulae in the (redefined) logic $\mathcal{L}$ for DB-ASMs to atomic formulae of the form

\[
x = y, x = y, f(x) = y, f(x) = x, f(x) = y, \epsilon^1(x, y), \epsilon^2(x, f, x, y), \epsilon^3(x, f, x, y, z),
\]

where $y, y_1$ and $y_2$ denote variables from either the first or the second individual sort depending on whether $f$ is a function name in $F_1$ or $F_0$, respectively. This can be done by using the following equivalences.

\[
s = t \leftrightarrow \exists x (s = x \land x = t)
\]
\[
s_a = t_a \leftrightarrow \exists x(s_a = x \land x = t_a)
\]
\[
f(s) = y \leftrightarrow \exists x(s = x \land f(x) = y)
\]
\[
f(s) = y \leftrightarrow \exists x(s = x \land f(x) = y)
\]
\[
f(s_a) = y \leftrightarrow \exists x(s_a = x \land f(x) = y)
\]
\[
\epsilon^1(x, s, t_a) \leftrightarrow \exists y(s \land t_a = y \land \epsilon^1(x, y))
\]
\[
\epsilon^2(x, f, t, s) \leftrightarrow \exists y(t = x \land s = y \land \epsilon^2(x, f, y))
\]
\[
\epsilon^2(x, f, t, s_a) \leftrightarrow \exists y(t = x \land s_a = y \land \epsilon^2(x, f, y))
\]
\[
\vdots
\]
\[
\epsilon^7(x, f, t, s_a, t_a, x, y, x', x', x', x', x') \leftrightarrow \exists y_1 y_1 x_1 x_2 y_2 x_2 z (x_1 = t \land y_1 = s_a \land x_1 = t_a \land x_2 = t' \land y_2 = s_a \land x_2 = t_a \land z = s_a \land \epsilon^7(x, f, x_1, y_1, x_1, x_2, y_2, x_2, z)
\]

The translation of modal formulae into $\mathcal{L}^\mathcal{E}$ distributes over negation, Boolean connectives and quantifiers. For eliminating the atomic formulae of the form $\text{upd}(r, x^2)$ and $\text{upm}(r, x^4)$ we use Axioms U1-U7 and Axioms UM1-UM7, respectively. For eliminating the modal operator in formulae of the form $[x^2] \varphi$, where $\varphi$ is a formula that was already translated to $\mathcal{L}^\mathcal{E}$, we use the following equivalences which are derivable in the (redefined) logic $\mathcal{L}$ for
DB-ASMs.

\[ [x^2]x = y \leftrightarrow (\text{conUSet}(x^2) \rightarrow x = y) \]

\[ [x^2]x = y \leftrightarrow (\text{conUSet}(x^2) \rightarrow x = y) \]

\[ [x^2]f(x) = y \leftrightarrow (\text{conUSet}(x^2) \rightarrow \in^2(x^2, f, x, y) \vee (\forall z(\neg \in^2(x^2, f, x, z)) \land f(x) = y)) \]

\[ [x^2]f(x) = y \leftrightarrow (\text{conUSet}(x^2) \rightarrow \in^2(x^2, f, x, y) \vee (\forall z(\neg \in^2(x^2, f, x, z)) \land f(x) = y)) \]

\[ [x^2]f(x) = y \leftrightarrow (\text{conUSet}(x^2) \rightarrow \in^2(x^2, f, x, y) \vee (\forall z(\neg \in^2(x^2, f, x, z)) \land f(x) = y)) \]

\[ [x^2] \in^1(x^1, x, y) \leftrightarrow (\text{conUSet}(x^2) \rightarrow \in^1(x^1, x, y)) \]

\[ \vdots \]

\[ [x^2] \in^7(x^7, f, x_1, y_1, x_1, x_2, y_2, x_2, z) \leftrightarrow (\text{conUSet}(x^2) \rightarrow \in^7(x^7, f, x_1, y_1, x_1, x_2, y_2, x_2, z)) \]

\[ [x^2] \neg \varphi \leftrightarrow (\text{conUSet}(x^2) \rightarrow \neg [x^2] \varphi) \]

\[ [x^2](\varphi \land \psi) \leftrightarrow ([x^2] \varphi \land [x^2] \psi) \]

\[ [x^2] \forall x(\varphi) \leftrightarrow \forall x([x^2] \varphi) \]

\[ \vdots \]

\[ [x^2] \forall x^7(\varphi) \leftrightarrow \forall x^7([x^2] \varphi) \]

It is not difficult to prove that the described translation from formulae of the (redefined) logic \( L \) for DB-ASMs to formulae of \( L^\in \) satisfies the properties required for \( L \) to be a definitional extension of \( L^\in \). This fact together with Theorem 7.1 implies our main completeness result.

**Theorem 7.2.** Let \( \varphi \) be a formula and \( \Phi \) a set of formulae in the logic \( L \) for DB-ASMs. If \( \Phi \models \varphi \), then \( \Phi \vdash \varphi \).

8. Conclusions

This article presents a logic for DB-ASMs. In accordance with the result that DB-ASMs and database transformations are behaviourally equivalent, it thus represents a logical characterisation for database transformations in general.

The logic for DB-ASMs is built upon the logic of meta-finite structures. The formalisation of multiset operations is captured by the notion of \( \rho \)-term. The use of \( \rho \)-terms greatly enhances the expressive power of the logic for DB-ASMs since aggregate computing in database applications can be easily expressed by using \( \rho \)-terms. On the other hand, \( \rho \)-terms can easily lead to incompleteness if we try to axiomatize them in the proof system. We avoid this problem by considering them as non-interpreted functions. In this way, we cannot reason about properties of \( \rho \)-terms themselves, but we can still use them in the formulae of our complete proof system to express meaningful properties of DB-ASMs.
As discussed in the paper, the non-determinism accompanied with the use of choice rules poses a further challenging problem. Fortunately, DB-ASMs are restricted to have quantifiers only over the database part of a state which is represented as a finite structure. This implies that any update set or multiset yielded by a DB-ASM rule must be finite. Based on the finiteness of update sets, we can use the modal operator $[X]$ where $X$ is a second-order variable that represents an update set $\Delta$ generated by a DB-ASM rule $r$. By introducing $[X]$ into the logic for DB-ASMs, it is shown that nondeterministic database transformations can also be captured.

Another important result of this article is the establishment of a sound and complete proof system for the logic for DB-ASMs, which can be turned into a tool for reasoning about database transformations. However, this is restricted to reasoning about steps, not full runs, but no complete logic for reasoning about runs can be expected. In the future we will continue to investigate how the logic for DB-ASMs can be tailored towards different classes of database transformations such as XML data transformations and used for verifying the properties of database transformations in practice.

References


