

A note on the Lorentz force, magnetic charges and the Casimir effect

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Abstract

We show that in order to account for the repulsive Casimir effect in the parallel plate geometry in terms of the quantum version of the Lorentz force, virtual surface densities of magnetic charge and currents must be introduced. The quantum version of the Lorentz force expressed in terms of the correlators of the electric and the magnetic fields for planar geometries yields then correctly the Casimir pressure.

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1 Introduction

Since its prediction by Casimir in 1948, the Casimir effect [1] has been the object of an increasing theoretical and experimental investigation. This is due to its growing recognition as a fundamental feature of quantum field theory and also to its importance in elementary particle physics, cosmology and condensed matter physics as well as its practical and decisive role in nanotechnology. For an introduction to this remarkable effect, see for example [2]; for an updated review of the recent research and applications see [3] and references therein.

Even in simple examples the Casimir interaction can exhibit surprising features. Some time ago Gonzales [4], among other things, correctly pointed out that an alternative computation of the Casimir force between two perfectly conducting plates can be carried out starting from the consideration of the Lorentz force acting on the plates. The reason is that in the context this is the only force that could act on a metallic plate and therefore one should be able to

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obtain Casimir's result from this physical fact. Here we will develop further this point of view and consider its consequences when applied to Boyer's variant of the standard Casimir effect in which one of the conducting plates is replaced by a magnetically permeable one [5]. We will show that when applied to this particular case, Gonzales' conception of the Casimir interaction leads to the introduction of virtual magnetic charges and currents.

2 The standard Casimir effect

In order to state clearly our point of view we begin by considering the standard experimental setup proposed by Casimir which consists of two infinite perfectly conducting parallel plates kept at a fixed distance a from each other. We will choose the coordinates axis in such a way that the OZ direction is perpendicular to the plates. One of the plates will be placed at $z = 0$ and the other one at $z = a$. Classically, the Lorentz force per unit area on, say, the conducting plate at $z = a$ is given by

$$\vec{f}_e = \frac{1}{2}\sigma_e\vec{E} + \frac{1}{2c}\vec{K}_e \times \vec{B}, \quad (1)$$

where σ_e is the electric charge density and \vec{K}_e is the electric current surface density. The boundary conditions on the electric and the magnetic fields on the plate are: the tangential components E_x and E_y of the electric field and the normal component B_z of the magnetic field must be zero on the plate. Under the boundary conditions imposed on the field at $z = a$, however, it is easily seen that the resultant classical Lorentz force is perpendicular to the conducting plate. We expect the quantum version of Eq. (1) to show the same feature and ultimately to be the source of the Casimir pressure between the two conducting plates. The electric charge and current densities on the conducting plate are related to the fields through

$$\hat{n} \cdot \vec{E} = 4\pi\sigma_e, \quad (2)$$

$$\hat{n} \times \vec{B} = \frac{4\pi}{c}\vec{K}_e, \quad (3)$$

where \hat{n} is the normal to the plate under consideration. The quantum version of Eq. (1) reads

$$\langle \vec{f}_e \rangle_0 = \frac{1}{8\pi} \langle \vec{E}^2 - \vec{B}^2 \rangle_0 \hat{n}, \quad (4)$$

and can be obtained by combining the vacuum expectation value of Eqs. (1), (2), and (3). In order to proceed – from now on we depart from Ref. [4] – we need to evaluate the vacuum expectation value of the quantum operators $E_i(\vec{r}, t) E_j(\vec{r}, t)$, $B_i(\vec{r}, t) B_j(\vec{r}, t)$, and $E_i(\vec{r}, t) B_j(\vec{r}, t)$. The evaluation of these correlators depends on the specific choice of the boundary conditions. A regularisation recipe is also necessary, for these objects are mathematically ill-defined. Regularisation recipes vary from the relatively simple cutoff method employed by Casimir himself [1] to the sophisticated and mathematical elegant generalised zeta function techniques, see Refs. [6] for an introduction to these techniques. Here we will make use of the results and send the

reader to the relevant references. The electric field correlators for a pair of perfectly conducting plates are given by [9, 10, 11, 13]

$$\langle E_i(\vec{r}, t) E_j(\vec{r}, t) \rangle_0 = \left(\frac{\pi}{a} \right)^4 \frac{2}{3\pi} \left[\frac{(-\delta^{\parallel} + \delta^{\perp})_{ij}}{120} + \delta_{ij} F(\xi) \right] \quad (5)$$

The function $F(\xi)$ with $\xi := \pi z/a$ is defined by

$$F(\xi) := -\frac{1}{8} \frac{d^3}{d\xi^3} \frac{1}{2} \cot(\xi), \quad (6)$$

and its expansion about $\xi = \xi_0$ is given by

$$F(\xi) \approx \frac{3}{8} (\xi - \xi_0)^{-4} + \frac{1}{120} + O[(\xi - \xi_0)]^2. \quad (7)$$

Notice that due to the behavior of $F(\xi)$ near $\xi_0 = 0, \pi$, strong divergences control the behavior of the correlators near the plates. The corresponding magnetic field correlators are

$$\langle B_i(\vec{r}, t) B_j(\vec{r}, t) \rangle_0 = \left(\frac{\pi}{a} \right)^4 \frac{2}{3\pi} \left[\frac{(-\delta^{\parallel} + \delta^{\perp})_{ij}}{120} - \delta_{ij} F(\xi) \right]. \quad (8)$$

A direct evaluation also shows that the correlators $\langle E_i(\vec{r}, t) B_j(\vec{r}, t) \rangle_0$ are zero. For calculational purposes it is convenient to consider a third conducting plate placed perpendicularly to the OZ axis at $z = \ell$. Consider the plate at $z = a$. The Lorentz force per unit area on its left side ($\hat{n} = -\hat{z}$) reads

$$\langle f_z^L \rangle_0 = -\frac{1}{8\pi} \langle \vec{E}^2 - \vec{B}^2 \rangle_0 \approx -\frac{3}{16\pi^2 (z-a)^4} - \frac{\pi^2}{240 a^4}, \quad (9)$$

where we have also made use of Eq. (7). On the other hand, after simple modifications in Eqs. (5), (8) and (7) the Lorentz force on the right side of the plate ($\hat{n} = \hat{z}$) reads

$$\langle f_z^R \rangle_0 = \frac{1}{8\pi} \langle \vec{E}^2 - \vec{B}^2 \rangle_0 \approx \frac{3}{16\pi^2 (z-a)^4} + \frac{\pi^2}{240 (\ell-a)^4}. \quad (10)$$

Adding the forces on both sides of the plate and setting $\ell \rightarrow \infty$ we obtain the well known result

$$\langle f_z \rangle_0 = \langle f_z^R \rangle_0 + \langle f_z^L \rangle_0 = -\frac{\pi^2}{240 a^4}. \quad (11)$$

The minus sign means that the probe plate at $z = a$ is attracted towards the other one at the origin.

3 The repulsive version of the standard Casimir effect

Let us now consider an alternative setup to the standard one in which a perfectly conducting plate is placed at $z = 0$ and perfectly permeable one is placed at $z = a$. This setup was

analysed for the first time by Boyer in the context of stochastic electrodynamics [5], a kind of classical electrodynamics that includes the zero-point electromagnetic radiation, and leads to the simplest example of a repulsive Casimir interaction. For alternative evaluations see Refs. [7, 8]. The boundary conditions now are: (a) the tangential components E_x and E_y of the electric field as well as the normal component B_z of the magnetic field must vanish on the surface of the plate at $z = 0$; (b) the tangential components of B_x e B_y of the magnetic field as well as normal component E_z of the electric field must vanish on the surface of the plate at $z = a$. From the classical point of view the perfectly permeable plate at $z = a$ poses a problem when we apply Eq. (1) to it. This is so because due to the boundary conditions this time the Lorentz force on either side of the plate is *parallel* to the permeable plate and therefore the resultant Lorentz force will be also parallel to the plate. This is a puzzling feature if we wish to describe the Casimir interaction between the plates through the Lorentz force. Things can be mended, however, if we allow for a virtual surface magnetic charge density σ_m and a virtual magnetic charge current surface density \vec{K}_m . In this case the modified Lorentz force per unit area on the permeable plate reads

$$\vec{f}_m = \frac{1}{2}\sigma_m\vec{B} - \frac{1}{2c}\vec{K}_m \times \vec{E}. \quad (12)$$

The charge and current surface densities are related to the fields through

$$\hat{n} \cdot \vec{B} = 4\pi\sigma_m, \quad (13)$$

$$\hat{n} \times \vec{E} = -\frac{4\pi}{c}\vec{K}_m. \quad (14)$$

It is easily seen that the modified Lorentz force given by Eq. (12) combined with the boundary conditions on the permeable plate yields on either side of the plate a force perpendicular to the plate as it must be. Proceeding as above we now have

$$\langle \vec{f}_m \rangle_0 = \frac{1}{8\pi} \langle \vec{B}^2 - \vec{E}^2 \rangle_0 \hat{n} \quad (15)$$

For Boyer's setup the relevant correlators were evaluated in Refs. [12, 13]. The results are

$$\langle E_i(\vec{r}, t) E_j(\vec{r}, t) \rangle_0 = \left(\frac{\pi}{a}\right)^4 \frac{2}{3\pi} \left[\left(-\frac{7}{8}\right) \frac{(-\delta^{\parallel} + \delta^{\perp})_{ij}}{120} + \delta_{ij} G(\xi) \right], \quad (16)$$

$$\langle B_i(\vec{r}, t) B_j(\vec{r}, t) \rangle_0 = \left(\frac{\pi}{a}\right)^4 \frac{2}{3\pi} \left[\left(-\frac{7}{8}\right) \frac{(-\delta^{\parallel} + \delta^{\perp})_{ij}}{120} - \delta_{ij} G(\xi) \right], \quad (17)$$

where

$$G(\xi) = -\frac{1}{8} \frac{d^3}{d\xi^3} \frac{1}{2 \sin(\xi)}. \quad (18)$$

Near $\xi = 0$ the function $G(\xi)$ behaves as

$$G(\xi) = \frac{3}{8}\xi^{-4} - \frac{7}{8}\frac{1}{120} + O(\xi^2), \quad (19)$$

but near $\xi = \pi$ its behavior is slightly different

$$G(\xi) = -\frac{3}{8}(\xi - \pi)^{-4} + \frac{7}{8}\frac{1}{120} + O[(\xi - \pi)^2]. \quad (20)$$

Again, a direct calculation shows that $\langle E_i(\vec{r}, t) B_j(\vec{r}, t) \rangle_0 = 0$ for this case also.

To obtain the Casimir force it is convenient to replace the third plate at $z = \ell$ for a permeable one. It is not hard to see that if we do so we can use Eqs. (5) and (8) in the region between the two permeable plates with small modifications. The force on the left side ($\hat{n} = -\hat{z}$) of the permeable plate at $z = a$ then reads

$$\langle f_{m,z}^L \rangle_0 = -\frac{3}{16\pi^2(z-a)^4} + \frac{7}{8} \times \frac{\pi^2}{240a^4}, \quad (21)$$

and the force on the right side is

$$\langle f_{m,z}^R \rangle_0 \approx \frac{3}{16\pi^2(z-a)^4} + \frac{\pi^2}{240(\ell-a)^4}. \quad (22)$$

As before we add the forces on each side and set $\ell \rightarrow \infty$ to obtain the repulsive Casimir force per unit area

$$\langle f_{m,z} \rangle_0 = \langle f_{m,z}^L \rangle_0 + \langle f_{m,z}^R \rangle_0 = \frac{7}{8} \times \frac{\pi^2}{240a^4}, \quad (23)$$

in agreement with Boyer [5]. Notice that this time the force per unit area is repulsive. Notice also that in both cases the divergent pieces cancel out; these cancellations yield finite Casimir energies [9, 11, 12, 13].

4 Conclusions

As we can see the introduction of virtual magnetic charges and currents can account for Boyer's variant of the Casimir effect in terms of the quantum version of the Lorentz force. Of course, for other geometries and boundary conditions, for example a perfectly conducting cube, the Casimir force can be repulsive and accountable for by the usual virtual electric charges and currents. On the other hand, it is plausible to state that whenever we have an ideal magnetically permeable wall as one of the confining surfaces a qualitative analysis of the interaction between the zero point electromagnetic fields and this confining surface, which is modelled by appropriate boundary conditions, will show the need of introducing virtual magnetic charges and currents. The introduction of these charges and currents avoids the need of constructing a model of the Casimir interaction based on an appropriate distribution of amperian currents, a considerably harder task.

As a final remark we observe that in the framework of the usual cavity QED, the result given by Eq. (23) is obtained by first evaluating the confined vacuum renormalized energy and followed by a variation of the volume of the confining region. It can be easily shown, however, that in terms of partition functions and free energies, Boyer's setup is mathematically equivalent to the difference between two standard setups, one with the distance between the plates equal to $2a$ and the other one with the distance between the plates equal to a , see Ref. [7]. This can be also easily proved for the Casimir pressure at zero and finite temperature.

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