GRAPHICAL CONDENSATION, OVERLAPPING PFaffIANS AND SUPERPOSITIONS OF MATCHINGS

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Abstract. The purpose of this note is to exhibit clearly how the “graphical condensation” identities of Kuo, Yan, Yeh and Zhang follow from classical Pfaffian identities by the Kasteleyn–Percus method for the enumeration of matchings. Knuth termed the relevant identities “overlapping Pfaffian” identities and the key concept of proof “superpositions of matchings”. In our uniform presentation of the material, we also give an apparently unpublished general “overlapping Pfaffian” identity of Krattenthaler.

1. Introduction

In the last 5 years, several authors [10, 11, 15, 21, 22] came up with identities related to the enumeration of matchings in planar graphs, together with a beautiful method of proof, which they termed graphical condensation.

In this paper, we show that these identities are special cases of certain Pfaffian identities (in the simplest case Tanner’s identity [18]), by simply applying the Kasteleyn–Percus method [6, 14]. These identities involve products of Pfaffians, for which Knuth [8] coined the term overlapping Pfaffians. Overlapping Pfaffians were further investigated by Hamel [5].

Knuth gave a very clear and concise exposition not only of the results, but also of the main idea of proof, which he termed superposition of matchings.

Tanner’s identity dates back to the 19th century — and so does the basic idea of superposition of matchings, which was used for a proof of Cayley’s Theorem [1] by Veltmann in 1871 [19] and independently by Mertens in 1877 [12] (as was already pointed out by Knuth [8]). Basically the same proof of Cayley’s Theorem was presented by Stembridge [17], who gave a very elegant “graphical” description of Pfaffians.

The purpose of this note is to exhibit clearly how “graphical condensation” is connected to “overlapping Pfaffian” identities. This is achieved by

• using Stembridge’s description of Pfaffians to give a simple, uniform presentation of the underlying idea of “superposition of matchings”, accompanied by many graphical illustrations (which should demonstrate ad oculos the beauty of this idea),

Date: February 7, 2014.

Research supported by the National Research Network “Analytic Combinatorics and Probabilistic Number Theory”, funded by the Austrian Science Foundation.
• using this idea to give uniform proofs for several known “overlapping Pfaffian” identities and a general “overlapping Pfaffian” identity, which to the best of our knowledge is due to Krattenthaler [9] and was not published before,
• and (last but not least) making clear how the “graphical condensation” identities of Kuo [10] Theorem 2.1 and Theorem 2.3], Yan, Yeh and Zhang [22, Theorem 2.2 and Theorem 3.2] and Yan and Zhang [21, Theorem 2.2] follow immediately from the “overlapping Pfaffian” identities via the classical Kasteleyn–Percus method for the enumeration of (perfect) matchings.

This note is organized as follows:

• Section 2 presents the basic definitions and notations used in this note,
• Section 3 presents the concept of “superposition of matchings”,
• Section 4 presents Stembridge’s description of Pfaffians together with the “superposition of matchings”–proof of Cayley’s Theorem,
• Section 5 recalls (quite briefly) the Kasteleyn–Percus method,
• Section 6 presents Tanner’s classical identity and more general “overlapping Pfaffian” identities, together with “superposition of matchings”–proofs, and deduces the “graphical condensation” identities [10, Theorem 2.1 and Theorem 2.3], [22, Theorem 2.2 and Theorem 3.2] and [21, Theorem 2.2].

2. Basic notation: Ordered sets (words), graphs and Matchings

The sets we shall consider in this paper will always be finite and ordered, whence we may view them as words of distinct letters

$$\alpha = \{\alpha_1, \alpha_2, \ldots, \alpha_n\} \simeq (\alpha_1, \alpha_2, \ldots, \alpha_n).$$

When considering a subset $\gamma \subseteq \alpha$, we shall always assume that the elements (letters) of $\gamma$ appear in the same order as in $\alpha$, i.e.,

$$\gamma = \{\alpha_{i_1}, \alpha_{i_2}, \ldots, \alpha_{i_k}\} \simeq (\alpha_{i_1}, \alpha_{i_2}, \ldots, \alpha_{i_k}) \text{ with } i_1 < i_2 < \cdots < i_k.$$

We choose this somewhat indecisive notation because the order of the elements (letters) is not always relevant. For instance, for graphs $G$ we shall employ the usual (set–theoretic) terminology: $G = G(V, E)$ with vertex set $V(G) = V$ and edge set $E(G) = E$, and the ordering of $V$ is irrelevant for typical graph–theoretic questions like “is $G$ a planar graph?”.

The graphs we shall consider in this paper will always be finite and loopless (they may, however, have multiple edges). Moreover, the graphs will always be weighted, i.e., we assume a weight function $\omega : E(G) \rightarrow R$, where $R$ is some integral domain. (If we are interested in mere enumeration, we may simply choose $\omega \equiv 1$.)

The weight $\omega(U)$ of some subset of edges $U \subseteq E(G)$ is defined as

$$\omega(U) := \prod_{e \in U} \omega(e).$$
The total weight (or generating function) of some family $F$ of subsets of $E(G)$ is defined as

$$\omega(F) := \sum_{U \in F} \omega(U).$$

For some subset $S \subseteq V(G)$, we denote by $[G - S]$ the subgraph of $G$ induced by the vertex set $(V(G) \setminus S)$.

A matching in $G$ is a subset $\mu \subseteq E(G)$ of edges such that

- no two edges in $\mu$ have a vertex in common,
- and every vertex in $V(G)$ is incident with precisely one edge in $\mu$.

(This concept often is called a perfect matching). Note that a matching $\mu$ may be viewed as a partition of $V(G)$ into blocks of cardinality 2.

3. Kuo’s Proposition and superposition of matchings

Denote the family of all matchings of $G$ by $M_G$, and denote the total weight of all matchings of $G$ by $M_G := \omega(M_G)$. According to Kuo [10], the following proposition is a generalization of results of Propp [15, section 6] and Kuo [11], and was first proved combinatorially by Yan, Yeh and Zhang [22]:

**Proposition 1.** Let $G$ be a planar graph with four vertices $a$, $b$, $c$ and $d$ that appear in that cyclic order on the boundary of a face of $G$. Then

$$M_G M_{[G - \{a,b,c,d\}]} + M_{[G - \{a,c\}]} M_{[G - \{b,d\}]} = M_{[G - \{a,b\}]} M_{[G - \{c,d\}]} + M_{[G - \{a,d\}]} M_{[G - \{b,c\}]}.$$  \hfill (1)

We present a simple proof for this statement which arises from a straightforward combinatorial interpretation of the products involved in (1), for which Knuth [8] coined the term superposition of matchings.

Consider a simple graph $G$ and two disjoint (but otherwise arbitrary) subsets of vertices $b \subseteq V(G)$ and $r \subseteq V(G)$. Call $b$ the blue vertices, $r$ the red vertices, $c := r \cup b$ the coloured vertices and the remaining $w := V(G) \setminus c$ the white vertices. Now consider the bicoloured graph $B = G_{r|b}$

- with vertex set $V(B) := V(G)$,
- and with edge set $E(B)$ equal to the disjoint union of
  - the edges of $E([G - b])$, which are coloured red,
  - and the edges of $E([G - r])$, which are coloured blue.

Here, “disjoint union” should be understood in the sense that $E([G - b])$ and $E([G - r])$ are subsets of two different “copies” of $E(G)$, respectively. This concept will appear frequently in the following: assume that we have two copies of some set $M$. We may imagine these copies to have different colours, red and blue, and denote them accordingly by $M_r$ and $M_b$, respectively. Then “by definition” subsets $A \subseteq M_r$ and $B \subseteq M_b$ are disjoint: every element in $A \cap B$ (in the ordinary sense, as subsets of $M$) appears twice
Figure 1. Illustration: A graph $G$ with two disjoint subsets of vertices $r$ and $b$ (shown in the left picture) gives rise to the bicoloured graph $G_{r|b}$ (shown in the right picture; blue edges are shown as dashed lines).

(a) as red copy and as blue copy) in $A \cup B$. Introducing the notation $X \cup Y$ as a shortcut for disjoint union, i.e., for “$X \cup Y$, where $X \cap Y = \emptyset$”, we can write:

$$E(B) = E([G - b]) \cup E([G - r]).$$

Note that in $B = G_{r|b}$

- all edges incident with blue vertices (i.e., with vertices in $b$) are blue,
- all edges incident with red vertices (i.e., with vertices in $r$) are red,
- and all edges in $E([G - c])$ appear as double edges in $E(B)$; one coloured red and the other coloured blue.

See Figure 1 for an illustration.

The weight function $\omega$ on the edges of graph $B = G_{r|b}$ is assumed to be inherited from graph $G$: $\omega(e)$ in $B$ equals $\omega(e)$ in $G$ (irrespective of the colour of $e$ in $B$).

**Observation 1 (superposition of matchings).** Define the weight $\omega(X, Y)$ of a pair $(X, Y)$ of subsets of edges as

$$\omega(X, Y) := \omega(X) \cdot \omega(Y).$$

Then $M_{[G - b]} M_{[G - r]}$ clearly equals the total weight of $M_{[G - b]} \times M_{[G - r]}$; and the typical summand in $M_{[G - b]} M_{[G - r]}$ is the product of the weights $\omega(\mu_r) \cdot \omega(\mu_b)$ of some pair of matchings $(\mu_r, \mu_b) \in M_{[G - b]} \times M_{[G - r]}$. Such pair of matchings can be viewed as the disjoint union $\mu_r \cup \mu_b \subseteq E(B)$ in the bicoloured graph $B$, where $\mu_r$ is a subset of the red edges, and $\mu_b$ is a subset of the blue edges. We call any subset in $E(B)$ which arises from a pair of matchings in this way a superposition of matchings, and we denote by $S_B$ the family of superpositions of matchings of $B$. So there is a weight preserving bijection

$$M_{[G - b]} \times M_{[G - r]} \leftrightarrow S_B.$$

**Observation 2 (nonintersecting bicoloured paths/cycles).** It is obvious that some subset $S \subseteq E(B)$ of edges of the bicoloured graph $B$ is a superposition of matchings if and only if

- every blue vertex $v$ (i.e., $v \in b$) is incident with precisely one blue edge from $S$,
- every red vertex $v$ (i.e., $v \in r$) is incident with precisely one red edge from $S$,
- every white vertex $v$ (i.e., $v \in w$) is incident with precisely one blue edge and precisely one red edge from $S$. 

Figure 2. Illustration: Take graph $G$ of Figure 1 and consider a matching in $[G - r]$ ($r = \{x, t\}$), whose edges are colored blue (shown as dashed lines), and a matching in $[G - b]$ ($b = \{y, z\}$), whose edges are colored red. This superposition of matchings determines a unique path $p$ connecting $x$ and $y$ in the bicoloured graph $G_{r|b}$. Swapping the colours of the edges of $p$ determines uniquely a matching in $[G - r']$ ($r' = \{y, t\}$) and a matching in $[G - b']$ ($b' = \{x, z\}$).

Stated otherwise: A superposition of matchings in $B$ may be viewed as a family of paths and cycles,

- such that every vertex of $B$ belongs to precisely one path or cycle (i.e., the paths/cycles are nonintersecting: no two different cycles/paths have a vertex in common),
- such that edges of each cycle/path alternate in colour along the cycle/path (therefore, we call them bicoloured: Note that a bicoloured cycle must have even length),
- such that precisely the end vertices of paths are coloured (i.e., red or blue), and all other vertices are white.

Observation 3 (colour–swap along paths). For an arbitrary coloured vertex $x$ in some superposition of matchings $S$ of $E(B)$, we may swap colours for all the edges in the unique path $p$ in $S$ with end vertex $x$ (see Figure 2). Without loss of generality, assume that $x$ is red. Depending on the colour of the other end vertex $y$ of $p$, this colour–swap results in a set of coloured edges $\mathcal{S}$, which is a superposition of matchings in

- $B' = G_{r'|b'}$, where $r' := (r \setminus \{x\}) \cup \{y\}$ and $b' := (b \setminus \{y\}) \cup \{x\}$, if $y$ is blue (i.e., of the opposite colour as $x$; the length of the path $p$ is even in this case — this case is illustrated in Figure 2),
- $B'' = G_{r''|b''}$, where $r'' := (r \setminus \{x, y\})$ and $b'' := b \cup \{x, y\}$, if $y$ is red (i.e., of the same colour as $x$; the length of path $p$ is odd in this case).

Clearly, this operation of swapping colours defines a weight preserving injection

$$\chi_x : \mathcal{S}_B \to (\mathcal{S}_{B'} \cup \mathcal{S}_{B''})$$

(which, viewed as mapping onto its image, is an involution: $\chi_x = \chi_x^{-1}$). So $\chi_x$ together with the bijection (2) gives a weight preserving injection

$$\mathcal{M}_{[G - b]} \times \mathcal{M}_{[G - r]} \to \mathcal{M}_{[G - b']} \times \mathcal{M}_{[G - r']} \cup \mathcal{M}_{[G - b'']} \times \mathcal{M}_{[G - r'']}.$$
3.1. The “graphical condensation method”. Now we apply the reasoning outlined in Observations 1, 2 and 3 for the proof of Proposition 1 (basically the same proof is presented in [10]):

Proof of Proposition 1. Clearly, for all the superpositions of matchings (see Observation 1) involved in (1), the set of coloured vertices in the associated bicoloured graphs is \( \{a, b, c, d\} \). In any superposition of matchings, there are two nonintersecting paths (see Observation 2) with end vertices in \( \{a, b, c, d\} \). Since \( G \) is planar and the vertices \( a, b, c \) and \( d \) appear in this cyclic order in the boundary of a face \( F \) of \( G \), the path starting in vertex \( a \) cannot end in vertex \( c \) (otherwise it would intersect the path connecting \( b \) and \( d \); see Figure 3 for an illustration).

So consider the bicoloured graphs

- \( B_1 := G_{r_1|b_1} \) with \( r_1 := \{a, b, c, d\} \), \( b_1 = \emptyset \),
- and \( B_2 := G_{r_2|b_2} \) with \( r_2 := \{a, c\} \), \( b_2 = \{b, d\} \).

Observe that

\[
M_G M_{[G-r_1]} + M_{[G-b_2]} M_{[G-r_2]} = \omega \left( \left( M_{[G-b_2]} \times M_{[G-r_2]} \right) \cup \left( M_{[G-b_2]} \times M_{[G-r_1]} \right) \right) = \omega(S_{B_1} \cup S_{B_2}).
\]

Note that for any superposition of matchings, the other end–vertex of the bicoloured path starting at \( a \) necessarily has

- the same colour as \( a \) in \( B_1 \) (i.e., red),
- the other colour as \( a \) in \( B_2 \) (i.e., blue).

(See Figure 4.) So consider the bicoloured graphs

- \( B'_1 := G_{r'_1|b'_1} \) with \( r'_1 := \{b, c\} \), \( b'_1 = \{a, d\} \),
- and \( B'_2 := G_{r'_2|b'_2} \) with \( r'_2 := \{c, d\} \), \( b'_2 = \{a, b\} \).

It is easy to see that the operation \( \chi_a \) of swapping colours of edges along the path starting at vertex \( a \) (see Observation 3) defines a weight preserving involution

\[
\chi_a : S_{B_1} \cup S_{B_2} \leftrightarrow S_{B'_1} \cup S_{B'_2},
\]

and thus gives a weight preserving involution

\[
\left( M_G \times M_{[G-r_1]} \right) \cup \left( M_{[G-b_2]} \times M_{[G-r_2]} \right) \leftrightarrow \left( M_{[G-b'_2]} \times M_{[G-r'_2]} \right) \cup \left( M_{[G-b'_1]} \times M_{[G-r'_1]} \right).
\]

(See Figure 4 for an illustration.)

This bijective proof certainly is very satisfactory. But since there is a well–known powerful method for enumerating perfect matchings in planar graphs, namely the Kasteleyn–Percus method (see [6, 7, 14]) which involves Pfaffians, the question arises whether Proposition 1 (or the bijective method of proof) gives additional insight or provides a new perspective.
Figure 3. A simple planar graph $G$ with vertices $a$, $b$, $c$ and $d$ appearing in this order in the boundary of face $F$.

Figure 4. Take the planar graph $G$ from Figure 3 and consider the bicoloured graphs $B_1$, $B_2$, $B'_1$ and $B'_2$ from the proof of Proposition 1. The pictures show certain superpositions of matchings in these bicoloured graphs (the edges belonging to the matchings are drawn as thick lines, the blue edges appear as dashed lines). The arrows indicate the possible effects of the operation $\chi_a$, i.e., of swapping colours of edges in the unique path with end vertex $a$.

4. Pfaffians

The name Pfaffian was introduced by Cayley [2] (see [8] page 10f) for a concise historical survey. Here, we follow closely Stembridge’s exposition [17].
Definition 1. Consider the complete graph $K_V$ on the (ordered) set of vertices $V = (v_1, \ldots, v_n)$, with weight function $\omega : E(K_V) \to \mathbb{R}$. Draw this graph in the upper halfplane in the following specific way:

- Vertex $v_i$ is represented by the point $(i, 0)$,
- edge $\{v_i, v_j\}$ is represented by the half-circle with center $(\frac{i+j}{2}, 0)$ and radius $\frac{|j-i|}{2}$.

(See the left picture in Figure 5).

Consider some matching $\mu = \{\{v_{i_1}, v_{j_1}\}, \ldots, \{v_{i_m}, v_{j_m}\}\}$ in $K_V$. Clearly, if such $\mu$ exists, then $n = 2m$ must be even. By convention, we assume $i_k < j_k$ for $k = 1, \ldots, m$.

A crossing of $\mu$ corresponds to a crossing of edges in the specific drawing just described, or more formally: A crossing of $\mu$ is a pair of edges $(\{v_{i_k}, v_{j_k}\}, \{v_{i_l}, v_{j_l}\})$ of $\mu$ such that $i_k < i_l < j_k < j_l$.

Then the sign of $\mu$ is defined as 

$$\text{sgn} (\mu) := (-1)^{\#(\text{crossings of } \mu)}.$$ 

(See the right picture in Figure 5).

Now arrange the weights $a_{i,j} := \omega(\{v_i, v_j\})$ in an upper triangular array $A = (a_{i,j})_{1 \leq i < j \leq n}$:

The Pfaffian of $A$ is defined as 

$$\text{Pf} (A) := \sum_{\mu \in \mathcal{M}_{K_V}} \text{sgn} (\mu) \omega(\mu),$$

where the sum runs over all matchings of $K_V$.

Since we always view $K_V$ as weighted graph (with some weight function $\omega$), we also write Pf($K_V$), or even simpler Pf($V$), instead of Pf($A$). Moreover, since an upper triangular matrix $A$ determines uniquely a skew symmetric matrix $A'$ (by letting $A'_{i,j} = A_{i,j}$ if $j > i$ and $A'_{j,i} = -A_{j,i}$ if $j < i$), we also write Pf($A'$) instead of Pf($A$). We set $\text{Pf}(\emptyset) := 1$ by definition.

With regard to the identities for matchings we are interested in, an edge not present in graph $G$ may safely be added if it is given weight zero. Hence every simple finite weighted graph $G$ graph may be viewed as a subgraph (in general not an induced subgraph!) of $K_V$ with $V(K_V) = V(G)$, where the weight of edge $e$ in $K_V$ is defined to be 

- $\omega(e)$ in $G$, if $e \in E(G)$,
- zero, if $e \not\in E(G)$.

Keeping that in mind, we also write Pf($G$) (or Pf($V$), again) instead of Pf($K_V$).

The following simple observation is central for many of the following arguments.

Observation 4 (contribution of a single edge to the sign of some matching). Let $V = (v_1, \ldots, v_{2n})$. Removing an edge $e = \{v_i, v_j\}$, $i < j$, together with the vertices $v_i$ and $v_j$, from some matching $\mu$ of $K_V$ gives a matching $\overline{\mu}$ of the complete graph on the remaining vertices $(v_1, v_2, \ldots, v_{2n}) \setminus \{v_i, v_j\}$, and the change in sign from $\mu$ to $\overline{\mu}$ is determined by
**Figure 5.** Pfaffians according to Definition 1. The left picture shows $K_4$, drawn in the specific way described in Definition 1. The right picture shows the matching $\mu = \{\{v_1, v_3\}, \{v_2, v_4\}\}$. Since there is precisely one crossing of edges in the picture, $\text{sgn}(\mu) = (-1)^1 = -1$.

**Figure 6.** The contribution of edge $e = \{v_i, v_j\}$ to the sign of the perfect matching $\pi$ amounts to $(-1)^{(\text{#}(\text{vertices between } v_i \text{ and } v_j))}$ (which is the same as $(-1)^{j-i-1}$ if the vertices $\{v_1, v_2, \ldots, v_{2n}\}$ appear in ascending order).

the number of vertices lying between $v_i$ and $v_j$ (see Figure 6). By the ordering of the vertices, $\text{#}(\text{vertices between } v_i \text{ and } v_j) = j - i - 1$, whence we have:

$$\text{sgn}(\mu) = \text{sgn}(\pi) \cdot (-1)^{j-i-1}.$$  

4.1. Cayley’s Theorem and the long history of superposition of matchings. The following Theorem of Cayley is well-known. Stembridge presented a beautiful proof (see [17, Proposition 2.2]) which was based on superposition of matchings. Basically the same proof was already found in the 19th century. We cite from [8]:

> An elegant graph-theoretic proof of Cayley’s theorem... was found by Veltmann in 1871 [19] and independently by Mertens in 1877 [12]. Their proof anticipated 20th-century studies on the superposition of two matchings, and the ideas have frequently been rediscovered.

**Theorem 1** (Cayley). Given an upper triangular array $A = (a_{i,j})_{1 \leq i \leq j \leq n}$, extend it to a skew symmetric matrix $A' = (a'_{i,j})_{1 \leq i,j \leq n}$ by setting

$$a'_{i,j} = \begin{cases} 
    a_{i,j} & \text{if } i < j, \\
    -a_{i,j} & \text{if } i > j, \\
    0 & \text{if } i = j.
\end{cases}$$

Then we have:

$$(\text{Pf}(A))^2 = \det (A').$$  \hfill (6)
Proof. By the definition of the determinant, we may view det \((A')\) as the generating function of the symmetric group \(S_n\)
\[
\det (A') = \sum_{\pi \in S_n} \text{sgn}(\pi) \omega(\pi),
\]
where the weight of a permutation \(\pi \in S_n\) is given as
\[
\omega(\pi) := \prod_{i=1}^{n} a'_{i,\pi(i)},
\]
(7)

The proof proceeds in two steps:

Step 1: Denote by \(S_0^n\) the set of permutations \(\pi \in S_n\) where the cycle decomposition of \(\pi\) does not contain a cycle of odd length. Then we claim:
\[
\det (A') = \sum_{\pi \in S_0^n} \text{sgn}(\pi) \omega(\pi).
\]
(8)

To prove this, we define a weight–preserving and sign–reversing involution on the set of all permutations \(\pi \in (S_n \setminus S_0^n)\) which do contain a cycle of odd length: of all odd–length cycles in \(\pi\), choose the one which contains the smallest element \(i\),
\[
\kappa_1 = (i, \pi(i), \pi^2(i), \ldots, \pi^{2m}(i)),
\]
and replace it by its inverse
\[
\kappa_1^{-1} = (\pi^{2m}(i), \pi^{2m-1}(i), \ldots, \pi(i), i).
\]
This operation obviously is an involution:
\[
\pi = (\kappa_1, \kappa_2, \ldots) \leftrightarrow \pi' = (\kappa_1^{-1}, \kappa_2, \ldots).
\]
Since \(\omega(\pi') = -\omega(\pi)\) and \(\text{sgn}(\pi') = \text{sgn}(\pi)\), this involution is weight–preserving and sign–reversing. So the terms corresponding to \((S_n \setminus S_0^n)\) in the right–hand side of (7) sum up to zero, which proves (8).

Step 2: We shall construct a weight– and sign–preserving bijection between the terms
- \(\text{sgn}(\pi) \omega(\pi)\) corresponding to the determinant as given in (8) (i.e., \(\pi \in S_0^n\))
- and \(\text{sgn}(\mu) \omega(\mu) \text{sgn}(\nu) \omega(\nu)\) corresponding to the square of the Pfaffian in (6).

To this end, consider the unique cycle decomposition of \(\pi\)
\[
\pi = (i_1, \pi(i_1), \pi^2(i_1), \ldots) (i_2, \pi(i_2), \pi^2(i_2), \ldots) \cdots (i_m, \pi(i_m), \pi^2(i_m), \ldots),
\]
(9)
where
- \(i_k\) is the smallest element in its cycle,
- and \(i_1 < i_2 < \cdots < i_m\).

Visualize \(\pi\) as directed graph as follows: represent
\[
i_1, \pi(i_1), \ldots, i_2, \pi(i_2), \ldots
\]
\textit{in this order} (i.e., in the order in which the elements appear in (9)) by vertices
\[
(1, 0), (2, 0), \ldots, (n, 0)
\]
in the plane. Call \((1, 0), (3, 0), \ldots\) the odd vertices, and call \((2, 0), (4, 0), \ldots\) the even vertices. Note that \(\pi\) maps elements corresponding to even vertices to elements corresponding to odd vertices, and vice versa. If some element \(i\) corresponds to an odd vertex \(v\), then draw a blue semicircle arc in the upper halfplane from \(v\) to the even vertex \(w\) which corresponds to \(\pi(i)\). If some element \(j\) corresponds to an even vertex \(s\), then draw a red semicircle arc in the lower halfplane from \(s\) to the odd vertex \(t\) which corresponds to \(\pi(j)\). (See the left picture in Figure 7 for an illustration.)

Note that if we forget the orientation of the arcs, we simply have a superposition \((\mu, \nu)\) of a blue and a red matching. Some of the arcs are co–oriented (i.e., they point from left to right), and some are contra–oriented (i.e., they point from right to left). Define the sign of any such oriented superposition of matchings by

\[
\text{sgn}(\mu, \nu) := \text{sgn}(\mu) \cdot \text{sgn}(\nu) \cdot (-1)^{\#\text{contra–oriented arcs in } \mu \cup \nu} 
\]  

and observe that for the particular oriented superposition of matchings obtained by visualizing permutation \(\pi\) as above, this definition gives precisely \(\text{sgn}(\pi)\). (Again, see the left picture in Figure 7 for an illustration.)

Furthermore, observe that with notation

\[
d(\pi) := |\{i : 1 \leq i \leq n, \pi(i) > i\}|
\]

we can rewrite the weight of \(\pi\) as

\[
\omega(\pi) = \omega(\mu) \cdot \omega(\nu) \cdot (-1)^{d(\pi)}.
\]  

However, the vertices do not appear in their original order. Clearly, we can obtain the original ordering by interchanging neighbouring vertices \((i, 0)\) and \((i + 1, 0)\) whose corresponding elements appear in the wrong order, one after another, together with the arcs being attached to them: see the right picture in Figure 7 and observe that this interchanging of vertices does not change the sign as defined in (10). Note that after finishing this “sorting procedure”, the number of contra–oriented arcs equals \(d(\pi)\), so we have altogether

\[
\text{sgn}(\pi) = \text{sgn}(\mu) \cdot \text{sgn}(\nu) \cdot (-1)^{d(\pi)}. 
\]

Together with (11), this amounts to

\[
\text{sgn}(\pi) \cdot \omega(\pi) = (\text{sgn}(\mu) \cdot \omega(\mu)) \cdot (\text{sgn}(\nu) \cdot \omega(\nu)),
\]

the right–hand side of which obviously corresponds to a term in \((\text{Pf}(A))^2\).

On the other hand, every term in \((\text{Pf}(A))^2\) corresponds to some superposition of matchings \(S = (\mu, \nu)\). For a bicoloured cycle \(C\) in \(S\), identify the smallest vertex \(v \in C\) and consider the unique blue edge \(\{v, w\}\) in \(C\). Orienting all bicoloured cycles \(C\) such that this blue edge points “from \(v\) to \(w\)” gives an oriented superposition of matchings, from which we obtain a permutation without odd–length cycles and with the same weight and the same sign (by simply reversing the above “sorting procedure”). \(\square\)
Figure 7. Illustration for Cayley’s Theorem. The left picture shows a cycle $c$ of length 8, whose smallest element is $i$, i.e.,
\[ c = (i, \pi(i), \pi^2(i), \ldots, \pi^7(i)) , \]
drawn as superposition of two directed matchings. Note that there is no crossing and precisely one contra–oriented arc, whence, according to (10), $c$ contributes $(-1)$ to the sign of $\pi$, as it should be for an even–length cycle. The right picture shows the effect of changing the position of two neighbouring vertices $a$ and $b$. For both matchings (red and blue; blue arcs appear as dashed lines), we have:
- the number of crossings changes by $\pm 1$ if $a$ and $b$ belong to different arcs,
- and if $a$ and $b$ belong to the same arc $e$, the orientation of $e$ is changed.
Since this amounts to a change in sign for the red matching and for the blue matching, the total effect is that the sign does not change.

4.2. A corollary to Cayley’s Theorem. The following Corollary is an immediate consequence of Cayley’s theorem. However, we shall provide a direct “graphical” proof.
Assume that the set of vertices $V$ is partitioned in two disjoint sets $A = (a_1, \ldots, a_m)$ and $B = (b_1, \ldots, b_n)$ such that the ordered set $V$ appears as $(a_1, \ldots, a_m, b_1, \ldots, b_n)$. Denote the complete bipartite graph on $V$ (with set of edges $\{\{a_i, b_j\} : 1 \leq i \leq m, 1 \leq j \leq n\}$) by $K_{A:B}$. (For our purposes, we may view $K_{A:B}$ as the complete graph $K_{A\cup B}$, where $\omega(\{a_i, a_j\}) = \omega(\{b_k, b_l\}) = 0$ for all $1 \leq i < j \leq m$ and for all $1 \leq k < l \leq n$.) We introduce the notation
\[ \text{Pf}(A, B) := \text{Pf}(K_{A:B}) . \]

Corollary 1. Let $A = (a_1, \ldots, a_m)$ and $B = (b_1, \ldots, b_n)$ be two disjoint ordered sets. Then we have
\[ \text{Pf}(A, B) = \begin{cases} (-1)^{\binom{m}{2}} \det(\omega(a_i, b_j))_{i,j=1}^n & \text{if } m = n, \\ 0 & \text{otherwise.} \end{cases} \]

Proof. If $m \neq n$, then there is no matching in $K_{A:B}$, and thus the Pfaffian clearly is 0.
If $m = n$, consider the $n \times n$–matrix $M := (\omega(\{a_i, b_{n-j+1}\}))_{i,j=1}^n$. Note that for every permutation $\pi \in \mathfrak{S}_n$, the corresponding term in the expansion of $\det(M)$ may be
Consider the $4 \times 4$–matrix $M := (\omega(\{a_i, b_{5-j}\}))_{i,j=1}^4$ and the permutation $\pi = (2, 3, 4, 1)$ in $S_4$. The left pictures shows $\pi$ as (bijective) function mapping the set $\{1, 2, 3, 4\}$ onto itself: It is obvious that the “arrows” indicating the function constitute a matching $\mu$. The right picture shows the same matching $\mu$ drawn in the specific way of Definition 1. Inversions of $\pi$ are in one–to–one correspondence with crossings of $\mu$, whence we see:

$$\text{sgn}(\pi) = \text{sgn}(\mu).$$

4.3. Another Definition for Pfaffians. There is another (less “graphical”) approach to Pfaffians, see [8] or [5]:

**Definition 2.** Let $X$ be some finite ordered alphabet (we may assume $X = \{1, \ldots, n\}$ and interpret $X$ as row and column indices of some $n \times n$–matrix). Consider quantities $f[x, y]$ defined on ordered pairs of elements of $X$ which satisfy

$$f[x, y] = -f[y, x].$$

This notation is extended to $f[\alpha]$ for arbitrary words $\alpha = (x_1, x_2, \ldots, x_{2n})$ of even length over $X$ by defining

$$f[x_1, \ldots, x_{2n}] := \sum s(x_1, \ldots, x_{2n}; y_1, \ldots, y_{2n}) f[y_1, y_2] \cdots f[y_{2n-1}, y_{2n}], \quad (12)$$

where the sum is over all $(2n-1)(2n-3)\cdots3\cdot1$ ways to write $\{x_1, \ldots, x_{2n}\}$ as a union of pairs $\{y_1, y_2\} \cup \cdots \cup \{y_{2n-1}, y_{2n}\}$, and where $s(x_1, \ldots, x_{2n}; y_1, \ldots, y_{2n})$ is the sign of the permutation that takes $(x_1, \ldots, x_{2n})$ into $(y_1, \ldots, y_{2n})$.

Note that $f$ is well defined, even though there are $2^n n!$ different permutations that yield the same partition into pairs. In particular, we may assume $y_{2k-1} < y_{2k}$ for $k = 1, \ldots, n$ and $y_{2k-1} < y_{2k+1}$ for $1 \leq k \leq n - 1$: This determines the permutation $\pi =$...
Figure 9. Pfaffians according to Definition 2. Assume $X = \{1, \ldots, 2n\}$, then the canonical representation $(y_1, y_2, \ldots, y_{2n})$ of some perfect matching $\mu$ is a permutation $\pi_1, \ldots, \pi_{2n}$, where $\pi_{2k-1} < \pi_{2k}$ for $1 \leq k \leq n$, and $\pi_{2k} - 1 < \pi_{2k+1}$ for $1 \leq k \leq n - 1$. We may depict this as follows:

\begin{center}
\begin{tikzpicture}
\node[draw,circle,inner sep=0pt,minimum size=10pt] (n1) at (0,0) {$\pi_1$};
\node[draw,circle,inner sep=0pt,minimum size=10pt] (n2) at (1,0) {$\pi_2$};
\node[draw,circle,inner sep=0pt,minimum size=10pt] (n3) at (2,0) {$\pi_3$};
\node[draw,circle,inner sep=0pt,minimum size=10pt] (n4) at (3,0) {$\pi_4$};
\node[draw,circle,inner sep=0pt,minimum size=10pt] (n5) at (4,0) {$\pi_5$};
\node[draw,circle,inner sep=0pt,minimum size=10pt] (n6) at (5,0) {$\pi_6$};
\node[draw,circle,inner sep=0pt,minimum size=10pt] (n7) at (6,0) {$\pi_7$};
\node[draw,circle,inner sep=0pt,minimum size=10pt] (n8) at (7,0) {$\pi_8$};
\node[draw,circle,inner sep=0pt,minimum size=10pt] (n9) at (8,0) {$\pi_9$};
\end{tikzpicture}
\end{center}

Figure 10. The 2 perfect matchings $\mu_1 = \{(1, 3), (2, 7), (4, 8), (5, 6)\}$ (shown in the left pictures) and $\mu_2 = \{(1, 4), (2, 5), (3, 6), (7, 8)\}$ (shown in the right pictures) are presented according to Definition 1 (in the upper pictures) and according to Definition 2 (in the lower pictures):

\begin{center}
\begin{tikzpicture}
\node[draw,circle,inner sep=0pt,minimum size=10pt] (n1) at (0,0) {$1$};
\node[draw,circle,inner sep=0pt,minimum size=10pt] (n2) at (1,0) {$2$};
\node[draw,circle,inner sep=0pt,minimum size=10pt] (n3) at (2,0) {$3$};
\node[draw,circle,inner sep=0pt,minimum size=10pt] (n4) at (3,0) {$4$};
\node[draw,circle,inner sep=0pt,minimum size=10pt] (n5) at (4,0) {$5$};
\node[draw,circle,inner sep=0pt,minimum size=10pt] (n6) at (5,0) {$6$};
\node[draw,circle,inner sep=0pt,minimum size=10pt] (n7) at (6,0) {$7$};
\node[draw,circle,inner sep=0pt,minimum size=10pt] (n8) at (7,0) {$8$};
\end{tikzpicture}
\end{center}

\begin{center}
\begin{tikzpicture}
\node[draw,circle,inner sep=0pt,minimum size=10pt] (n1) at (0,0) {$1$};
\node[draw,circle,inner sep=0pt,minimum size=10pt] (n2) at (1,0) {$2$};
\node[draw,circle,inner sep=0pt,minimum size=10pt] (n3) at (2,0) {$3$};
\node[draw,circle,inner sep=0pt,minimum size=10pt] (n4) at (3,0) {$4$};
\node[draw,circle,inner sep=0pt,minimum size=10pt] (n5) at (4,0) {$5$};
\node[draw,circle,inner sep=0pt,minimum size=10pt] (n6) at (5,0) {$6$};
\node[draw,circle,inner sep=0pt,minimum size=10pt] (n7) at (6,0) {$7$};
\node[draw,circle,inner sep=0pt,minimum size=10pt] (n8) at (7,0) {$8$};
\end{tikzpicture}
\end{center}

\begin{center}
\begin{tikzpicture}
\node[draw,circle,inner sep=0pt,minimum size=10pt] (n1) at (0,0) {$1$};
\node[draw,circle,inner sep=0pt,minimum size=10pt] (n2) at (1,0) {$2$};
\node[draw,circle,inner sep=0pt,minimum size=10pt] (n3) at (2,0) {$3$};
\node[draw,circle,inner sep=0pt,minimum size=10pt] (n4) at (3,0) {$4$};
\node[draw,circle,inner sep=0pt,minimum size=10pt] (n5) at (4,0) {$5$};
\node[draw,circle,inner sep=0pt,minimum size=10pt] (n6) at (5,0) {$6$};
\node[draw,circle,inner sep=0pt,minimum size=10pt] (n7) at (6,0) {$7$};
\node[draw,circle,inner sep=0pt,minimum size=10pt] (n8) at (7,0) {$8$};
\end{tikzpicture}
\end{center}

\begin{center}
\begin{tikzpicture}
\node[draw,circle,inner sep=0pt,minimum size=10pt] (n1) at (0,0) {$1$};
\node[draw,circle,inner sep=0pt,minimum size=10pt] (n2) at (1,0) {$2$};
\node[draw,circle,inner sep=0pt,minimum size=10pt] (n3) at (2,0) {$3$};
\node[draw,circle,inner sep=0pt,minimum size=10pt] (n4) at (3,0) {$4$};
\node[draw,circle,inner sep=0pt,minimum size=10pt] (n5) at (4,0) {$5$};
\node[draw,circle,inner sep=0pt,minimum size=10pt] (n6) at (5,0) {$6$};
\node[draw,circle,inner sep=0pt,minimum size=10pt] (n7) at (6,0) {$7$};
\node[draw,circle,inner sep=0pt,minimum size=10pt] (n8) at (7,0) {$8$};
\end{tikzpicture}
\end{center}

$(y_1, y_2, \ldots, y_{2n})$ uniquely; we call this unique permutation the canonical representation of the partition into pairs; see Figure 9.

4.3.1. Equivalence of the two definitions. Clearly, a partition into pairs of $\alpha = (x_1, \ldots, x_{2n})$ may be viewed as a perfect matching of the complete graph $K_{2n}$ on the set of vertices $\{x_1, \ldots, x_{2n}\}$. Therefore, we may conveniently abbreviate (12) in the form

$$f[\alpha] = \sum_{\mu \in \mathcal{M}_{K_{2n}}} s(\alpha; \mu) \prod_{i=1}^{n} f[y_{2i-1}, y_{2i}], \quad (13)$$

where $\mathcal{M}_{K_{2n}}$ is the family of perfect matchings of the complete graph $K_{2n}$, and where $(y_1, \ldots, y_{2n})$ is the canonical representation of the perfect matching $\mu$. Now define the weight $\omega(\{x, y\})$ of some edge of $K_{2n}$ with $x < y$ as

$$\omega(\{x, y\}) := f[x, y].$$

This implies

$$\prod_{i=1}^{n} f[y_{2i-1}, y_{2i}] = \omega(\mu).$$

for $\mu = (y_1, \ldots, y_{2n})$. 
Figure 11. From Definition 2 to Definition 1: For $i = 1, 2, \ldots$, shift the vertex corresponding to $\pi_{2i}$ (together with the arc incident to it) to its “correct position”.

If we can also show that
\[ s(x_1, \ldots, x_{2n}; y_1, \ldots, y_{2n}) = \text{sgn}(\mu) \]  
under this interpretation, then Definitions 1 and 2 are equivalent. (Note that the sign is encoded in the “crossings of the edges” of a perfect matching $\mu$ according to Definition 1, while it is encoded in the canonical representation of $\mu$ according to Definition 2, see Figure 10.) But this is an immediate consequence of the Observation 4.

From the pictures in Figure 10 it is obvious how the presentation according to Definition 2 can be transformed to the one according to Definition 1: In the canonical representation $\pi$ of some perfect matching $\mu$, we have to rearrange the vertices (together with the edges incident to them) in their natural order. Recall that by definition of the canonical representation, the “odd–labeled vertices” $\pi_{2k} - 1$ already appear in their natural order:
\[ \pi_1 < \pi_3 < \cdots < \pi_{2n-1}. \]
So whenever some “even–labeled vertex” $\pi_{2m}$ does not appear in the proper position (i.e., immediately after $(\pi_{2m} - 1)$), we move it there (together with the edge incident to it; see Figure 11 where “odd–labeled vertices” are indicated by black circles). This operation can be viewed as removing the old edge (with $\pi_{2m}$ in the wrong position) and inserting a new edge (with $\pi_{2m}$ in the correct position): According to Observation 4 the change of sign for the perfect matching amounts to
\[ (-1)^{\# \text{(vertices between old (wrong) and new (correct) position of $\pi_{2m}$)}}, \]
which obviously coincides with the change in sign for the permutation
\[ (-1)^{\# \text{(transpositions of neighbours needed to move $\pi_{2m}$ from old (wrong) to new (correct) position)}}. \]
This proves (14).

4.4. A generalization of Corollary 1. We may use Observation 4 to prove another identity involving Pfaffians. To state it conveniently, we need some further notation.

Assume that the ordered set of vertices $V$ appears as $(a_1, \ldots, a_m, b_1, \ldots, b_n)$ for disjoint sets $A = (a_1, \ldots, a_m)$ and $B = (b_1, \ldots, b_n)$. Consider the complete graph $K_V$ and delete (or assign weight zero to) all edges joining two vertices from $A$. Call the resulting graph
the semi-bipartite graph $S_{A,B}$. (Note that every matching $\mu$ in $S_{A,B}$ constitutes an injective mapping $A \to B$.)

Let $M = (m_1, m_2, \ldots, m_n)$ be some ordered set. For some arbitrary subset $X = (m_{i_1}, \ldots, m_{i_k}) \subseteq M$, denote by $\Sigma(X \subseteq M)$ the sum of the indices $i_j$ of $X$:

$$\Sigma(X \subseteq M) := i_1 + i_2 + \cdots + i_k.$$ 

(Recall that subsets “inherit” the ordering in our presentation, i.e., $i_1 < i_2 < \cdots < i_k$: we might also call $X$ a subword of $M$.)

**Remark 1.** Note that this concept is related to the permutation $\pi \in S_M$ which moves $m_{i_j}$ to position $j$, $j = 1, \ldots, k$, while leaving the order of all remaining elements in $M \setminus X$ unchanged:

$$\text{sgn}(\pi) = (-1)^{i_1-1+i_2-2+\cdots+i_k-k} = (-1)^{\Sigma(X \subseteq M)-(k+1)/2}.$$ 

(This observation provides the translation to the presentation of Pfaffians given in [8] and [5], it is not needed for our presentation.)

**Corollary 2.** Let $V = (a_1, \ldots, a_m, b_1, \ldots, b_n)$, $A = (a_1, \ldots, a_m)$ and $B = (b_1, \ldots, b_n)$. For every subset $Y = \{b_{k_1}, b_{k_2}, \ldots, b_{k_m}\} \subseteq B$, denote by $M_Y$ the $m \times m$-matrix

$$M_Y := (\omega(\{a_i, b_{k_j}\}))_{i,j=1}^m.$$ 

Then we have

$$\text{Pf}(S_{A,B}) = (-1)^m \sum_{\substack{Y \subseteq B, \cr |Y| = m}} (-1)^{\Sigma(Y \subseteq B)} \cdot \text{Pf}(B \setminus Y) \cdot \det(M_Y). \quad (15)$$

**Proof.** For every matching $\rho$ in $S_{A,B}$, let $Y \subseteq B$ be the set of vertices which are joined with a vertex in $A$ by some edge in $\rho$. Note that $|Y| = |A| = m$ (if such matching exists), and observe that $\rho$ may be viewed as a superposition of matchings, namely

- a red matching $\mu$ in the complete bipartite graph $K_{A,Y}$
- and a blue matching $\nu$ in the complete graph $K_{B \setminus Y}$,

where

$$\omega(\rho) = \omega(\mu) \cdot \omega(\nu).$$

For an illustration, see Figure [12]. Note that the crossings of $\rho$ are partitioned in

- crossings of two edges from $\mu$,
- crossings of two edges from $\nu$
- and crossings of an edge from $\mu$ with an edge from $\nu$,

whence we have

$$\text{sgn}(\rho) = (-1)^{\#(\text{crossings of an edge from } \mu \text{ and an edge from } \nu)} \cdot \text{sgn}(\mu) \cdot \text{sgn}(\nu).$$
Figure 12. Illustration for Corollary 2. Consider the ordered set of vertices
\[ V = \{ a_1, a_2, a_3; b_1, b_2, \ldots, b_7 \}. \]

The picture shows the matching \( \rho = \{ \{ a_1, b_5 \}, \{ a_2, b_2 \}, \{ a_3, b_7 \}, \{ b_1, b_4 \}, \{ b_3, b_6 \} \} \) in \( S_{A,B} \), where \( A = \{ a_1, a_2, a_3 \} \) and \( B = \{ b_1, b_2, \ldots, b_7 \} \). Let \( Y = \{ b_2, b_5, b_7 \} \) and observe that \( \rho \) may be viewed as superposition of the red matching \( \mu = \{ \{ a_1, b_5 \}, \{ a_2, b_2 \}, \{ a_3, b_7 \} \} \) in \( K_{A,Y} \) and the blue matching \( \nu = \{ \{ b_1, b_4 \}, \{ b_3, b_6 \} \} \) in \( K_{B \setminus Y} \) (blue edges are drawn as dashed lines). All crossings in \( \rho \) are indicated by circles; the crossings which are not present in \( \mu \) or in \( \nu \) are indicated by two concentric circles.

Assume that \( Y = (b_{k_1}, b_{k_2}, \ldots, b_{k_m}) \) and observe that modulo 2 the number of crossings

- of the edge from \( \mu \) which ends in \( b_{k_j} \)
- with edges from \( \nu \)

equals the number of vertices of \( B \setminus Y \) which lie to the left of \( b_{k_j} \), which is \( k_j - j \). Hence we have

\[
\text{sgn}(\rho) \cdot \text{sgn}(\mu) \cdot \text{sgn}(\nu) = (-1)^{(k_1-1)+(k_2-2)+\ldots+(k_m-m)} = (-1)^{\sum_{Y \subseteq B} - \binom{m+1}{2}}.
\]

From this we obtain

\[
Pf(S_{A,B}) = (-1)^m \sum_{Y \subseteq B, \ |Y|=m} (-1)^{\text{sgn}(Y \subseteq B) \cdot \text{sgn}(Y \subseteq B)} \cdot Pf(B \setminus Y) \cdot (-1)^{\binom{m}{2}} \cdot Pf(A, Y), \quad (16)
\]

which by Corollary 1 equals (15).

5. Matchings and Pfaffians: The Kasteleyn–Percus Method

Let \( G \) be some loopless graph with weight function \( \omega \) and assume some (arbitrary) orientation \( \xi \) on the pairs of vertices of \( G \):

\[
\xi : V(G) \times V(G) \rightarrow \{1, -1\} \text{ such that } \xi(v, u) = -\xi(u, v).
\]
Consider the skew-symmetric square matrix $D(G, \xi)$ with row and column indices corresponding to the vertices $\{v_1, \ldots, v_n\}$ of $G$ (in some arbitrary order) and entries
\[
d_{i,i} = 0,
\]
\[
d_{i,j} = \xi(v_i, v_j) \times \sum_{e \in E(G), e = \{v_i, v_j\}} \omega(e) \quad \text{for } i \neq j.
\]

Recall that the superposition of two arbitrary matchings of $G$ yields a covering of the bicoloured graph $B = G_{r|b}$, $r = b = \emptyset$ (i.e., there are no coloured vertices) with even-length cycles. Consider the “inherited” orientation $\xi$ for $B$ (i.e., the same orientation as in $G$). The orientation $\xi$ is called admissible if in every even-length cycle $C$ in $B$ which arises from the superposition of two matchings of $G$, there is an odd number of edges co-oriented with $\xi$ (and an odd number of edges contra-oriented with $\xi$; here, co- and contra-orientation refer to some arbitrary but fixed orientation of the even-length cycle $C$).

Then we have \cite{6} Theorem 1 on page 92:

**Theorem 2** (Kasteleyn). Let $G$ be a graph with weight function $\omega$. Assume $G$ has an admissible orientation $\xi$. Then the total weight of all matchings of $G$ equals the Pfaffian of $D(G, \xi)$:
\[
M_G = \Pf(D(G, \xi)). \tag{18}
\]

While the existence of admissible orientations is not guaranteed in general, for planar graphs we have the following result \cite{6} Theorem 2 on page 94:

**Theorem 3** (Kasteleyn). For every planar graph $G$ there exists an orientation, such that for every even-length cycle in $G$ the number of co-oriented edges is odd. (In particular, the orientation is admissible).

Observe that the property in Theorem 3 is preserved by taking induced subgraphs:

**Observation 5.** For an oriented graph $G$, where in every even-length cycle the number of co-oriented edges is odd, the same property holds for any induced subgraph of $G$ with the “inherited” orientation.

So it seems that Proposition 1 gives an identity for special Pfaffians which correspond to planar graphs $G$: Let $\xi$ be an admissible orientation for $G$, and assume the same (inherited) orientation for all of $G$’s subgraphs, then (1) translates to
\[
\Pf(D(G, \xi)) \cdot \Pf(D([G - \{a, b, c, d\}], \xi)) + \\
\Pf(D([G - \{a, b\}], \xi)) \cdot \Pf(D([G - \{b, d\}], \xi)) = \\
\Pf(D([G - \{a, c\}], \xi)) \cdot \Pf(D([G - \{b, d\}], \xi)) + \\
\Pf(D([G - \{a, b\}], \xi)) \cdot \Pf(D([G - \{c, d\}], \xi)) + \\
\Pf(D([G - \{a, d\}], \xi)) \cdot \Pf(D([G - \{b, c\}], \xi)). \tag{19}
\]

But it turns out that (19) is in fact an identity for Pfaffians in general, namely the special case \cite{8} Equation (1.1) of an identity \cite{8} Equation (1.0) due to Tanner 18.
The following theorem is due to H.W.L. Tanner [18] (see [8, Equation (1.0)].

**Theorem 4.** Let \( \alpha = (\alpha_1, \ldots, \alpha_n) \) and \( \beta = (\beta_1, \ldots, \beta_m) \) be subsets of the ordered set \( \gamma = (\alpha_1, \ldots, \alpha_n, \beta_1, \ldots, \beta_m) \). Let \( 1 \leq k \leq m \), be arbitrary but fixed. Then there holds

\[
\Pf(\alpha) \Pf(\alpha \cup \beta) = (-1)^k \cdot \sum_{j=1, j\neq k}^{m} (-1)^{j-1} \Pf(\alpha \cup \{\beta_k, \beta_j\}) \Pf((\alpha \cup \beta) \setminus \{\beta_k, \beta_j\}). \tag{20}
\]

(Recall that all subsets inherit the order of \( \gamma \).)

Observe that for the special case \( \beta = (a, b, c, d) \), (20) reads

\[
\Pf(\alpha) \cdot \Pf(\alpha \cup (a, b, c, d)) + \Pf(\alpha \cup (a, c)) \cdot \Pf(\alpha \cup (b, d)) + \Pf(\alpha \cup (a, d)) \cdot \Pf(\alpha \cup (b, c)),
\]

which implies (19) (simply set \( \alpha := V(G) \setminus \beta \)).

Tanner’s identity has the following generalization which Hamel [5, Theorem 2.1] attributes to Ohta [13]; it was also found by Wenzel (see [20, Proposition 2.3] and [4, Theorem 1]).

For arbitrary sets \( A \) and \( B \), denote the symmetric difference of \( A \) and \( B \) by

\[ A \triangle B := (A \cup B) \setminus (A \cap B). \]

**Theorem 5** (Ohta). Assume that the ordered set \( \gamma = (v_1, v_2, \ldots, v_n) \) appears as \( \gamma = \alpha \cup \beta \) with \( \alpha \triangle \beta = (v_{i_1}, \ldots, v_{i_t}) \). Then we have:

\[
\sum_{\tau=1}^{t} (-1)^{\tau} \cdot \Pf(\alpha \triangle \{v_i\}) \cdot \Pf(\beta \triangle \{v_{i_t}\}) = 0. \tag{21}
\]

(Recall again that all subsets inherit the order of \( \gamma \).)

The following proof given by Krattenthaler [9] uses superposition of matchings once more, together with proper accounting for the sign–changes associated to swapping colours in bicoloured paths.

**Lemma 1.** Consider the complete graph \( K_V \) on the ordered set of vertices \( V \), given as disjoint union of red, blue and white vertices

\[ V = b \cup r \cup w = (v_1, v_2, \ldots, v_n), \]

where \( |b| \equiv |r| \pmod{2} \). Draw the edges of \( K_V \) in the specific way described in Definition [7]. Let \( B \) be the corresponding bicoloured graph, let \( S = \mu \cup \nu \) be a superposition of matchings in \( B \) (recall Observation [7], and let \( p \) be a bicoloured path in \( S \) (recall Observation [2] with end vertices \( v_i \) and \( v_j \). Swapping colours in \( p \) (recall Observation [2]) gives a superposition of matchings \( S = \mu' \cup \nu' \) in a bicoloured graph \( B' \), and we have

\[
\text{sgn}(\mu) \cdot \text{sgn}(\nu) = (-1)^{\# \text{vertices of } (b \cup r) \text{ between } v_i \text{ and } v_j} \cdot \text{sgn}(\mu') \cdot \text{sgn}(\nu').
\]

If \( v_i \) and \( v_j \) appear in the ordered set of coloured vertices

\[ c = b \cup r = (w_1, w_2, \ldots, w_{|c|}) \]
as \( v_i = w_x \) and \( v_j = w_y \), then this amounts to

\[
\text{sgn}(\mu) \cdot \text{sgn}(\nu) = (-1)^{y-x+1} \text{sgn}(\mu') \cdot \text{sgn}(\nu').
\]  

(22)

\textbf{Proof.} According to Observation 4, the change in sign corresponding to removing some single arc \( \{v_k, v_l\} \) from \( \mu \) and adding it to \( \nu \) amounts to

\[
(-1)^{\text{# (vertices of (b \cup w) between } v_k \text{ and } v_l)} - \text{# (vertices of (r \cup w) between } v_k \text{ and } v_l)} =
\]

\[
(-1)^{\text{# (vertices of (b \cup r) between } v_k \text{ and } v_l)}.
\]

Recolouring all the arcs in path \( p \) with end vertices \( v_i \) and \( v_j \) thus gives a change in sign equal to the product of all these single sign–changes, which clearly amounts to

\[
(-1)^{\text{# (vertices of (b \cup r) between } v_i \text{ and } v_j)}.
\]

\[\square\]

\textbf{Proof of Theorem 5.} For the combinatorial interpretation of the left–hand side of (21), simply combine Observation 1 (superposition of matchings) with the definition of Pfaffians as given in Definition 1: after expansion of the products of Pfaffians, the typical summand is of the form

\[
(-1)^\tau \cdot \text{sgn}(\mu) \cdot \text{sgn}(\nu) \cdot \omega(\mu) \cdot \omega(\nu),
\]

where \((\mu, \nu)\) can be interpreted as superposition of matchings in the bicoloured graph \( G_{r|b} \) derived from the complete graph \( G \) on vertices \( \alpha \cup \beta \), with

- \( b = (\alpha \setminus \beta) \Delta \{v_{i,\tau}\} \),
- \( r = (\beta \setminus \alpha) \Delta \{v_{i,\tau}\} \).

According to Observation 1 the operation \( \chi_v \) of swapping colours in the unique path \( p \) starting in vertex \( v = v_{i,\tau} \) is an involution which preserves the (absolute) weight \( \omega(\mu) \cdot \omega(\nu) \).

So the proof is complete if we can show that \( \chi_v \) is in fact sign–reversing in the following sense. Note that \( \chi_v \) yields a superposition of matchings \((\mu', \nu')\) in the bicoloured graph \( G_{r'|b'} \) with

- \( b = (\alpha \setminus \beta) \Delta \{v_{i',\tau}\} \),
- \( r = (\beta \setminus \alpha) \Delta \{v_{i',\tau}\} \),

where \( v_{i',\tau} \) is the other endpoint of the unique path \( p \). Thus we have to show

\[
(-1)^\tau \cdot \text{sgn}(\mu) \cdot \text{sgn}(\nu) = -(-1)^\rho \cdot \text{sgn}(\mu') \cdot \text{sgn}(\nu').
\]

But this follows immediately from Lemma 1. \[\square\]

The same idea of proof applies to the following generalization, which to the best of my knowledge is due to Krattenthaler [9]:

\textbf{Theorem 6 (Krattenthaler).} Assume that the ordered set \( \gamma = (v_1, v_2, \ldots, v_n) \) appears as \( \gamma = \alpha \cup \beta \). Let \( M = \alpha \Delta \beta \).
If \(|\alpha|\) is odd, then we have for all \(s \geq 0\):
\[
\sum_{\substack{Y \subseteq M \mid |Y| = 2s+1}} (-1)^{\Sigma Y \subseteq M} \cdot \Pf(\alpha \triangle Y) \cdot \Pf(\beta \triangle Y) = 0.
\] (23)

If \(|\alpha|\) is even, then we only have the weaker statement
\[
\sum_{s=0}^{[t/2]} \sum_{\substack{Y \subseteq M \mid |Y| = 2s}} (-1)^{\Sigma Y \subseteq M} \cdot \Pf(\alpha \triangle Y) \cdot \Pf(\beta \triangle Y) = 0.
\] (24)

**Proof.** For every superposition of matchings \((\mu, \nu)\) involved in (23) (in the sense of the proof of Theorem 5) consider the subset \(S \subseteq Y\) of vertices \(v\) with the property that the other endpoint of the unique path starting in \(v\) does not belong to \(Y\). Note that \(S\) is of odd cardinality, so in particular \(S \neq \emptyset\).

By the same reasoning as in the proof of Theorem 5, simultaneously swapping colours in all paths starting in vertices from \(S\) yields a sign-reversing and weight-preserving involution, which proves (23).

Now consider the family of superpositions of matchings corresponding to (24). Let \(v\) be some arbitrary, but fixed element in \(M\), and note that the operation \(\chi_v\) of swapping colours in the unique path with starting point \(v\) yields weight-preserving involution, which is sign-reversing according to Lemma 1. □

Since Pf\((X) \equiv 0\) if the cardinality of \(X\) is odd, we may restate the above theorem in a uniform way.

**Corollary 3.** Assume that the ordered set \(\gamma = (v_1, v_2, \ldots, v_n)\) appears as \(\gamma = \alpha \cup \beta\). Let \(M = \alpha \Delta \beta\). Then we have:
\[
\sum_{Y \subseteq M} (-1)^{\Sigma Y \subseteq M} \cdot \Pf(\alpha \triangle Y) \cdot \Pf(\beta \triangle Y) = 0.
\] (25)

6.1. **Further applications.** For the rest of this paper, consider the complete graph \(G = K_V\) with (ordered) vertex set \(V = (v_1, v_2, \ldots, v_n)\), where \(V\) appears as the disjoint union of coloured vertices and white vertices, \(V = c \cup w\), and where the set of coloured vertices is partitioned into two sets of equal size \(c = R \cup B, |R| = |B|\). For the following assertions, imagine that all vertices in \(B\) are “initially” blue and that all vertices in \(R\) are “initially” red, and bicoloured graphs are derived therefrom by changing this “initial” colouring.

6.1.1. Srinivasan’s result. For the next Lemma, let \(X\) be some fixed subset \(X \subseteq B\), and let \(b := B \setminus X\) and \(r := R \cup X\). Consider the set of all superpositions of matchings \(\mu \cup \nu\) in \(G_{r\|b}\) with the additional property, that every bicoloured path in \(\mu \cup \nu\) has at most one end vertex in \(B\): denote this set by \(\mathcal{F}(G\|X)\). To every object \(\mu \cup \nu \in \mathcal{F}(G\|X)\) assign signed weight
\[
\omega(\mu \cup \nu) := (-1)^{\Sigma B \setminus c} \sgn(\mu) \cdot \sgn(\nu) \cdot \omega(\mu) \cdot \omega(\nu).
\]
Now consider the generating function $F$ of $\bigcup_{X \subseteq B} \mathcal{F}(G \| X)$, i.e.,

$$F := \sum_{X \subseteq B} \sum_{f \in \mathcal{F}(G \| X)} \omega(f).$$

**Lemma 2.** With the above definitions, we have for the generating function $F$:

$$F = (-1)^{\Sigma(B \subseteq c)} \sum_{X \subseteq B} (-1)^{\Sigma(X \subseteq c)} \cdot \text{Pf}(R \cup w \cup X) \cdot \text{Pf}(B \cup w \setminus X). \quad (26)$$

**Proof.** Clearly, all the signed weights of objects from $\bigcup_{X \subseteq B} \mathcal{F}(G \| X)$ do appear in the sum on the right–hand side of (26). So we have to show that all the “superfluous” terms in (26) cancel.

These “superfluous” objects are precisely superpositions of matchings, where there exists a bicoloured path connecting two vertices of $B$: of all such paths choose the one, $p$, with the smallest end vertex, and swap colours in $p$. This gives a weight–preserving involution, which is sign–reversing according to Lemma 1. \qed

From this, we easily deduce (a slight generalization of) a result of Srinivasan ([16, Corollary 3.2], see also [5, Theorem 3.3]):

**Corollary 4 (Srinivasan).** Let $m + n$ be an even integer and consider the ordered set $V = (v_1, \ldots, v_{m+n})$, partitioned into the disjoint subsets $A = (v_1, \ldots, v_m)$ and $B = (v_{m+1}, \ldots, v_{m+n})$. Then we have the following expansions:

If $m < n$, then

$$\text{Pf}(V) = - \sum_{X \subseteq B, X \neq B} (-1)^{\Sigma(B \setminus X \subseteq V)} \cdot \text{Pf}(A \cup X) \cdot \text{Pf}(B \setminus X). \quad (27)$$

If $m = n$, then

$$\text{Pf}(V) = \text{Pf}(A, B) - \sum_{X \subseteq B, X \neq B} (-1)^{\Sigma(B \setminus X \subseteq V)} \cdot \text{Pf}(A \cup X) \cdot \text{Pf}(B \setminus X). \quad (28)$$

If $m > n$, then

$$\text{Pf}(V) = (-1)^\left(\frac{n+1}{2}\right) \sum_{Y \subseteq A, |Y| = n} (-1)^{\Sigma(Y \subseteq A)} \text{Pf}(A \setminus Y) \cdot \text{Pf}(B, Y)$$

$$- \sum_{X \subseteq B, X \neq B} (-1)^{\Sigma(B \setminus X \subseteq V)} \cdot \text{Pf}(A \cup X) \cdot \text{Pf}(B \setminus X). \quad (29)$$

**Proof.** We apply Lemma 2 with $R = A$, $B = B$ and $w = \emptyset$: since there are no white vertices, bicoloured paths in any superposition of matchings $(\mu \cup \nu) \in \bigcup_{X \subseteq B} \mathcal{F}(G \| X)$ are simply edges, whose end vertices are of the same colour: hence we must have $X = B$ (because there is no matching on $B \setminus X \neq \emptyset$ where every edge has only one end vertex.
in \( B \). So \( \mu \cup \nu \) corresponds to a unique matching in the semi–bipartite graph \( S_{B,A} \), and vice versa, whence we have

\[
F = \text{Pf}(S_{B,A}).
\]

Expanding the trivial identity

\[
\text{Pf}(V) = \text{Pf}(S_{B,A}) - (F - \text{Pf}(A \cup B) \cdot \text{Pf}(\emptyset))
\]

according to (16) and (26), respectively, proves all three assertions: note that \( \text{Pf}(S_{B,A}) = 0 \) if \( m < n \), and \( \text{Pf}(S_{B,A}) = \text{Pf}(A, B) \) if \( m = n \). \( \square \)

6.1.2. **Graphical condensation.** For the rest of this paper, assume that \( B = (b_1, b_2, \ldots, b_k) \) and \( R = (r_1, r_2, \ldots, r_k) \) and that the ordered set of coloured vertices appears as \( c = (r_1, b_1, r_2, b_2, \ldots, r_k, b_k) \).

**Definition 3 (Planar weight function).** We call a weight function \( \omega \) on \( E(G) \) a planar weight function if it assigns weight zero to every superposition of matchings \( \mu \cup \nu \) in every \( G_{r\cup b} \) (where \( r \cup b = R \cup B \) is an arbitrary partition of the coloured vertices in two subsets of blue and red vertices) that contains a bicoloured path \( p \) connecting two vertices from \( B \) or two vertices from \( R \).

**Lemma 3.** In the setting described above, assume that the weight function is planar. Then we have for all (fixed) subsets \( X \subseteq R \):

\[
\sum_{W \subseteq B} \text{Pf}((R \cup w) \cup W) \cdot \text{Pf}((B \cup w) \setminus W) = \sum_{V \subseteq B} \text{Pf}(((R \cup w) \cup V) \setminus X) \cdot \text{Pf}(((B \cup w) \setminus V) \cup X). \quad (30)
\]

**Proof.** The left–hand side of (30) corresponds to superpositions of matchings \( \mu \cup \nu \) of non–vanishing weight with red vertices \( R \cup W \) and blue vertices \( B \setminus W \), for some subset \( W \subseteq B \). The assumption implies that every bicoloured path \( p \), which starts in some vertex \( v \in R \), must have its other end in \( B \). So according to (22) (and the specific “alternating ordering” of \( c = B \cup R \)), the operation \( \chi_v \) of swapping colours in \( p \) gives a weight– and sign–preserving bijection. Thus, swapping colours in all bicoloured paths with end vertex in the fixed subset \( X = \{v_1, \ldots, v_m\} \subseteq R \) gives a weight– and sign–preserving bijection \( \chi := \chi_{v_1} \circ \cdots \circ \chi_{v_m} \). Denote by \( Y \) the set of the other end vertices of these paths. Note that \( Y \subseteq B \) by assumption and consider the set \( V := W \Delta Y \): it is obvious that the image of \( \chi \) is precisely the set of superpositions of matchings corresponding to the right hand side of (30). \( \square \)

This yields immediately the following generalization of a result by Yan, Yeh and Zhang [22, Theorem 2.2] (also given in Kuo [10, Theorem 2.1 and Theorem 2.3]):

**Corollary 5.** Let \( G \) be a planar graph with vertices \( r_1, b_1, \ldots, r_k, b_k \) appearing in that cyclic order on the boundary of a face of \( G \). Let \( R = \{r_1, \ldots, r_k\} \) and \( B = \{b_1, \ldots, b_k\} \).
Then we have for every fixed subset $X \subseteq \mathbb{R}$

$$
\sum_{W \subseteq B} M_{G-(B\setminus W)} M_{G-(R\cup W)} = \sum_{V \subseteq B} M_{G-(V\cup X)} M_{G-(R\setminus V\setminus X)}.
$$

**Proof.** Recall that no two different paths arising from some superposition of matchings can have a vertex in common. So the ordering of $R \cup B$ along the boundary of a face of the planar graph $G$ implies that no bicoloured path can have both end vertices in $R$ or in $B$, hence the weight function is planar and we may apply Lemma 3. By the Kasteleyn–Percus method, the assertion follows from (30). \qed

6.1.3. **Ciucu’s matching factorization theorem.** In the same manner, we may prove Ciucu’s matching factorization Theorem [3, Theorem 1.2], in the (equivalent) formulation given in [21, Theorem 2.2].

**Lemma 4.** In the setting described above, assume that the weight function is planar. Moreover, assume that the set of coloured vertices is partitioned into two subsets of equal size, $c = U \cup V$, $|U| = |V|$, such

- that there is no even–length path of non–vanishing weight connecting any two vertices $(x, y)$ with $x \in U$ and $y \in V$,
- and that there is no odd–length path of non–vanishing weight connecting any two vertices $(x, y)$ with $x, y \in U$ or $x, y \in V$.

Let $B_U := B \cap U$, $B_V := B \cap V$, $R_U := R \cap U$ and $R_V := R \cap U$. (Note that $|B_U| = |R_V|$ and $|B_V| = |R_U|$.) Then we have

$$
2^k \cdot \text{Pf}(w \cup B_U \cup R_V) \cdot \text{Pf}(w \cup B_V \cup R_U) =
\sum_{X \subseteq V, Y \subseteq U} \text{Pf}(w \cup X \cup Y) \cdot \text{Pf}(w \cup \overline{X} \cup \overline{Y}). \quad (31)
$$

**Proof.** First, consider an arbitrary superposition of matchings in the bicoloured graph $G_{b|r}$, where with $b \cup r$ is an arbitrary partition of the set $R \cup B$. Since

- bicoloured paths of even length have end vertices of different colour,
- and bicoloured paths of odd length have end vertices of the same colour,

the assumption implies that if $\omega(\mu \cup \nu) \neq 0$, then every bicoloured path in $\mu \cup \nu$ must connect (vertices from) $r \cap U$ with $b \cap U$, $b \cap U$ with $b \cap V$, $b \cap V$ with $r \cap V$, or $r \cap V$ with $r \cap U$. See the left picture in Figure 13, where the possible connections by bicoloured paths are indicated by arrows.

Now consider the product of Pfaffians in the left hand side of (31), i.e., choose $b = B_U \cup R_V$ and $r = B_V \cup R_U$. Note that in this case we have $r \cap U = R_U$, $r \cap V = B_V$, $b \cap U = B_U$, and $b \cap V = R_V$, and the assertion about the possible connections with bicoloured paths (as depicted in the left picture of Figure 13) would hold also without the additional assumption, since the weight function $\omega$ is planar.

Consider some arbitrary superposition of matchings $\mu \cup \nu$ in $G_{b|r}$. Swapping colours in an arbitrary subset of bicoloured paths in $\mu \cup \nu$ gives a unique superposition of matchings...
Figure 13. Illustration to the proof of Lemma 4. The left picture shows the partition of the set of coloured vertices in four subsets, induced by the bipartitions \( r \cup b = U \cup V \). The arrows indicate the only possible connections by bicoloured paths: for instance, no bicoloured path can connect a vertex from \( r \cap U \) with \( b \cap V \), since such paths must have even length, which is ruled out by the additional assumption in Lemma 4. The right picture shows the partition of the set of coloured vertices in eight subsets, induced by the bipartitions \( r \cup b = U \cup V = R \cup B \). The arrows indicate the only possible connections between “wrongly coloured” vertices (these are the ones indicated by the inscribed square with the dashed boundary) by bicoloured paths: in particular, every bicoloured path must have either no “wrongly coloured” end vertex or two “wrongly coloured” end vertices.
Since the weight function is planar, it is not possible for a superposition of matchings (of non–vanishing weight) to have only one “wrongly coloured” end vertex (for instance, there is no bicoloured connection from the “wrongly coloured” set $\overline{X} \cap B$ to $\overline{Y} \cap B$ or to $X \cap B$). Thus $\chi$ is in fact a weight–preserving and sign–preserving bijection. □

**Corollary 6.** Let $G$ be a planar bipartite graph, where the bipartition of the vertex set of $G$ is given as $V(G) = A \cup B$. Assume vertices $r_1, b_1, \ldots, r_k, b_k$ appear in that cyclic order on the boundary of a face of $G$, and let $R = \{r_1, \ldots, r_k\}$ and $B = \{b_1, \ldots, b_k\}$. Let $U := A \cap (R \cup B)$ and $V := B \cap (R \cup B)$, and assume that $|U| = |V|$. Then we have

$$2^k \cdot M_{[G-(B \cup R \cup V) \setminus R]} \cdot M_{[G-(B \cup R \cup V) \setminus V]} = \sum_{X \subseteq V, Y \subseteq U, |X| = |Y|} M_{[G-(X \cup Y)]} \cdot M_{[G-(\overline{X} \cup \overline{Y})]}.$$  

**Proof.** Note that the assumption that $G$ is bipartite implies the additional assumption of Lemma 3. The assertion now follows from (31) by the Kasteleyn–Percus method. □

6.1.4. **Graphical edge–condensation.** Consider again the situation of Lemma 3. Denote the edge connecting vertex $r_i$ with vertex $b_i$ by $e_i := \{r_i, b_i\}$. From the given complete graph $G = K_V$ construct a new graph $G'$ by replacing every edge $e_i$ by a path of length three $(r_i, r_i', b_i', b_i)$, thus inserting

- new vertices $r_i'$ and $b_i'$, where $r_i'$ is the immediate successor of $r_i$ and $b_i'$ is the immediate predecessor of $b_i$ in the set of ordered vertices of $G'$,
- and new edges $e_i' := \{r_i, r_i'\}$, $e_i' := \{r_i', b_i'\}$ and $e_i'' := \{b_i', b_i\}$, where the weights of the new edges are given as $\omega(e_i') = \omega(e_i)$ and $\omega(e_i'') = \omega(e_i') = 1$.

Note that $G'$ is not a complete graph ($r_i'$ and $b_i'$ are vertices of degree 2 in $G$), but we may view it as a complete graph by introducing edges of weight zero. (Figure 14 illustrates this construction.)

Observe that there is an obvious weight–preserving bijection between matchings of $G$ and matchings of $G'$: For arbitrary $\mu \in \mathcal{M}_G$ and all $i = 1, \ldots, k$,

- replace $e_i$ by edges $e_i'$ and $e_i''$ if $e_i \notin \mu$,
- add $e_i'$ to $\mu$ if $e_i \notin \mu$,

to obtain a matching $\mu' \in \mathcal{M}_{G'}$. Note that this bijection yields a change in sign equal to

$$\prod_{i=1}^{k} (-1)^{\#(\text{vertices between } r_i \text{ and } b_i)} = (-1)^{k + \sum \epsilon \subseteq V},$$

according to Observation 4 (since the edges $e_i'$ and $e_i''$ in $G'$ can never be involved in any crossing). Thus we obtain immediately

$$(-1)^{k + \sum \epsilon \subseteq V} \Pf(G) = \Pf(G').$$  (32)

Now consider the bicoloured graph $B' = G'_{b'/r'}$, where $b' \cup r'$ is some partition of the set of coloured vertices $e' := \{r_1', b_1', \ldots, r_k', b_k'\}$ in $G'$.
If $r'_i$ and $b'_i$ are of different colours, then edges $e^r_i$ and $e^b_i$ must both belong to every superposition of matchings $\mu' \cup \nu'$ in the bicoloured graph $B' = G'_{b \cup r'}$. So we may simply remove them together with their end–vertices from the corresponding subgraph (blue or red, respectively) of $G'$, thus obtaining a superposition of matchings $\mu'' \cup \nu''$ in the bicoloured graph $[G' - \{r'_i, b'_i\}]_{b'' \cup r''}$ with

- $b'' := b' \cup \{b'_i\} \setminus \{r'_i\}$ and $r'' := r' \cup \{r_i\} \setminus \{b'_i\}$, if $r'_i \in b'$ (as in Figure 15),
- $b'' := b' \cup \{r_i\} \setminus \{b'_i\}$ and $r'' := r' \cup \{b_i\} \setminus \{r'_i\}$, if $r'_i \in r'$.

Observe that $[G' - \{r'_i, b'_i\}]$ “locally looks like” the original graph $G$. For later use, note that

$$\omega(\mu' \cup \nu') = \omega(e_i) \times \omega(\mu'' \cup \nu'')$$  \hspace{1cm} (33)

there is no sign–change here, since the edges $e^r_i$ and $e^b_i$ in $G'$ can never be involved in any crossing.

If $r'_i$ and $b'_i$ are of the same colour (say red), then for any superposition of matchings $\mu' \cup \nu'$ in $B'$

- the blue subgraph $[G' - r']$ (which contains the blue matching $\nu'$) “locally looks like” the original graph $G$ with edge $e_i$ removed (since this is “locally equivalent” to removing the two red vertices $r'_i$ and $b'_i$, see the lower left picture in Figure 16),
- and in the red subgraph $[G' - b']$ (which contains the red matching $\mu'$) we may re–replace the path of length 3 $(r_i, r'_i, b'_i, b_i)$ in $G'$ by the “original” edge $e_i = \{r_i, b_i\}$ in $G$: This operation changes the corresponding Pfaffian only by a sign–factor according to the considerations preceding (32), and the resulting red subgraph “locally looks like” the original graph $G$ (see the lower right picture in Figure 16).

Applying these simple considerations to all pairs $(r'_i, b'_i)$, $i = 1, \ldots, k$ in $G'$, we see that superpositions of matchings in $B' = G'_{b' \cup r'}$ are in bijection with superpositions of matchings in a certain bicoloured graph derived from $G$, which we now describe:
Figure 15. Illustration of the construction preceding Lemma 5: If \( r_i' \) and \( b_i' \) are of different colour, then \( e_{r_i}' \) and \( e_{b_i}' \) must necessarily both belong to every superposition of matchings \( \mu' \cup \nu' \) and thus can be removed (after accounting for their weights, of course) together with the vertices \( r_i' \) and \( b_i' \). The resulting situation “locally” looks like the original graph \( G' \), with coloured vertices \( r_i \) (of the same colour as \( b_i' \)) and \( b_i \) (of the same colour as \( r_i' \)).

\[
\begin{array}{c}
\text{\( e_{r_i}' \)} \\
\text{\( r_i \)} & \text{\( r_i' \)} & \text{\( b_i' \)} & \text{\( b_i \)} \\
\text{\( G' \)}
\end{array}
\]

\[
\begin{array}{c}
\text{\( G' - \{ r_i', b_i' \} \)} \\
\text{\( \{ G' - \{ r_i', b_i' \} \} \)}
\end{array}
\]

Figure 16. Illustration of the construction preceding Lemma 5: If \( r_i' \) and \( b_i' \) are of the same colour (say red), then the construction shown in Figure 14 can be reversed in the “red subgraph” (see the right lower picture). This operation introduces a sign factor according to the considerations preceding (32). In the “blue subgraph” the vertices \( r_i' \) and \( b_i' \) simply are missing (see the left lower picture).

\[
\begin{array}{c}
\text{\( e_{r_i}' \)} & \text{\( e_{i}' \)} & \text{\( e_{b_i}' \)} \\
\text{\( r_i \)} & \text{\( r_i' \)} & \text{\( b_i' \)} & \text{\( b_i \)} \\
\text{\( G' \)}
\end{array}
\]

\[
\begin{array}{c}
\text{\( e_i \)} \\
\text{\( G' - \{ r_i', b_i' \} \)} & \text{\( G' - \{ r_i', b_i' \} \)}
\end{array}
\]
Define the sets of blue/red vertices in $G$, whose “partner with the same subscript” has a different colour (see Figure 15):

\[
\begin{align*}
\mathbf{b}' &:= \{ r_i : r'_i \in r' \land b'_i \in b' \} \cup \{ b_i : b'_i \in r' \land r'_i \in b' \}, \\
\mathbf{r}' &:= \{ b_i : r'_i \in r' \land b'_i \in b' \} \cup \{ r_i : b'_i \in r' \land r'_i \in b' \}.
\end{align*}
\]

(34)

Define the sets of blue/red edges in $G$, where both end–vertices are of the same colour (see Figure 16):

\[
\begin{align*}
\mathbf{r}_e &:= \{ e_i : b'_i \in r' \land r'_i \in r' \}, \\
\mathbf{b}_e &:= \{ e_i : b'_i \in b' \land r'_i \in r' \}.
\end{align*}
\]

(35)

Finally, define the set of vertices

\[
Z := \{ b_i, r_i : b_i \text{ and } r_i \text{ are of the same colour} \} = \left( \bigcup_{e \in \mathbf{r}_e} e \right) \cup \left( \bigcup_{e \in \mathbf{b}_e} e \right).
\]

For an arbitrary graph $H$ and subsets $V' \subseteq V(H)$ and $E' \subseteq E(H)$ introduce the notation $[H - V' - E']$ for the graph $[H - V']$ with all edges in $E'$ removed. Then the above reasoning amounts to the following Pfaffian identity:

\[
Pf([G' - \mathbf{b}']) Pf([G' - \mathbf{r}']) = (-1)^{|Z| + \sum_{Z \subseteq V} \prod_{e \in (\mathbf{r}_e \cup \mathbf{b}_e)} \omega(e_i) Pf([G - \mathbf{b}'' - \mathbf{b}_e]) Pf([G - \mathbf{r}'' - \mathbf{r}_e]).
\]

(36)

These considerations lead to the following assertion:

Lemma 5. Assume the situation of Lemma 3 with the additional condition that there is no vertex between $r_i$ and $b_i$ in the ordered set of vertices $V(G)$ ($G = K_V$). For an arbitrary subset $I \subseteq \{1, \ldots, k\}$ denote by $T$ the subset $\{1, \ldots, k\} \setminus I$ and introduce the following “template notation”:

\[
x_I := \{ x_i : i \in I \},
\]

where $x$ may be any symbol from the set $\{r, r', b, b', e, e'\}$.

Then we have for all fixed subsets $B \subseteq \{1, \ldots, k\}$:

\[
\sum_{R \subseteq \{1, \ldots, k\}} \left( \prod_{e_i \in e} \omega(e_i) \right) \cdot Pf([G - r_B]) \cdot Pf([G - b_B - e]) = \\
\sum_{R \subseteq \{1, \ldots, k\}} \left( \prod_{e_i \in (G - \mathbf{r} \setminus e)} \omega(e_i) \right) \times \\
Pf([G - (r_B \setminus e) \cup b_B \setminus e] - e) \cdot Pf([G - (b_B \setminus e) \cup r_B \setminus e] - e) (37)
\]
Proof. Consider the graph $G'$ as defined in the considerations preceding Lemma 5. It is obvious that every bicoloured path in $G$ connecting two vertices $b_i$ and $b_j$ (or $r_i$ and $r_j$) corresponds bijectively to a bicoloured path in $G'$ connecting the vertices $b'_i$ and $b'_j$ (or $r'_i$ and $r'_j$), which thus also has weight zero. Hence the weight function of $G'$ is also planar, and we may apply Lemma 3 to $G'$.

Let $R, B \subseteq \{1, \ldots, k\}$ such that $V = b'_R \subseteq B'$ and as $X = r'_B \subseteq R'$. Then we may write

$$\text{Pf}(((R' \cup \omega) \setminus V) \setminus X) \cdot \text{Pf}(((B' \cup \omega) \setminus V) \setminus X) = \text{Pf}([G' - b']) \text{Pf}([G' - r']),$$

where $b' = r_B \cup b_{\overline{R}}$ and $r' = b_R \cup r_{\overline{R}}$.

The definitions of $b''$ and $r''$ (according to (34)) and of $b_e$ and $r_e$ (according to (35)) read:

$$b'' = r_{\overline{R}} \cup b_{R \cap B},$$
$$r'' = b_{\overline{R}} \cup r_{B \cap R},$$
$$b_e = e_{B \setminus R},$$
$$r_e = e_{R \setminus B}.$$

Then (36) translates to:

$$\text{Pf}([G' - b']) \text{Pf}([G' - r']) = \left( \prod_{e_i \in (R \cup R')} \omega(e_i) \right) \times$$

$$\text{Pf} \left( \left[ G - (r_{\overline{R}} \cup b_{R \cap B}) - e_{B \setminus R} \right] \right) \text{Pf} \left( \left[ G - (b_{\overline{R}} \cup r_{B \cap R}) - e_{R \setminus B} \right] \right).$$

(Note that there is no sign-change here, since there are no vertices between $r_i$ and $b_i$ by assumption.)

Putting together (38) and (39) gives the translation from (30) to (37).

This yields immediately the following generalization of a result by Yan, Yeh and Zhang [22, Theorem 3.2] (here, we write the summands for $R$ instead of $R$ when summing over all $R \subseteq \{1, \ldots, k\}$, to make the equivalence with [22, equation (8)] more transparent):

**Corollary 7.** Let $G$ be a planar graph with $k$ independent edges $e_i = \{a_i, e_i\}$, $i = 1, \ldots, k$, in the boundary of some face $f$ of $G$, such that the vertices $a_1, b_1, \ldots, a_k, b_k$ appear in that cyclic order in $f$.

Then we have for all fixed subsets $B \subseteq \{1, \ldots, k\}$:

$$\sum_{R \subseteq \{1, \ldots, k\}} \left( \prod_{e_i \in e_R} \omega(e_i) \right) \cdot M_{[G - r_R]} \cdot M_{[G - b_R - e_R]} =$$

$$\sum_{R \subseteq \{1, \ldots, k\}} \left( \prod_{e_i \in e_{R \cup B}} \omega(e_i) \right) \cdot M_{[G - (r_{\overline{R}} \cup b_{R \cap B}) - e_{B \setminus R}]} \cdot M_{[G - (b_{\overline{R}} \cup r_{B \cap R}) - e_{R \setminus B}]}$$

**Proof.** The statement follows from (37) by the Kasteleyn–Percus method. \qed
REFERENCES