Hamiltonian Weights and Unique 3-Edge-Colorings of Cubic Graphs

Cun-Quan Zhang
DEPARTMENT OF MATHEMATICS
WEST VIRGINIA UNIVERSITY
MORGANTOWN, WEST VIRGINIA

ABSTRACT

A (1,2)-eulerian weight \( w \) of a graph is hamiltonian if every faithful cover of \( w \) is a set of two Hamilton circuits. Let \( G \) be a 3-connected cubic graph containing no subdivision of the Petersen graph. We prove that if \( G \) admits a hamiltonian weight then \( G \) is uniquely 3-edge-colorable. © 1995 John Wiley & Sons, Inc.

1. INTRODUCTION

Most standard graph-theoretic terms that are used in this paper can be found for instance in [3]. All graphs we considered in this paper may have multiple edges but no loops. A \((1, 2)\)-eulerian weight \( w \) of a 2-connected graph \( G \) is a weight \( w: E(G) \rightarrow \{1, 2\} \) such that the total weight of each edge-cut is even. A faithful cover of \( w \) is a family \( C \) of circuits such that each edge \( e \) is contained in precisely \( w(e) \) circuits of \( C \). The topic of faithful coverings of eulerian weights has attracted many mathematicians (see survey papers [6–9,19], etc.). Some well-known conjectures on this topic (such as the circuit double cover conjecture due to Szekeres [12] and, Seymour [11]) still remain open. A cubic graph is uniquely 3-edge-colorable if \( G \) has precisely one I-factorization. The topic of uniquely 3-edge-colorable cubic graphs is also a very interesting topic in graph theory (see [13,14,17,4, and 10], etc.). Some well-known conjectures (such as every planar, uniquely 3-edge-colorable, cubic graph has a triangle due to Fiorini and Wilson [4]) still remain open. In this paper, we will study the relations between the faithful coverings and the uniquely 3-edge-colorability of cubic graphs.

© 1995 John Wiley & Sons, Inc. CCC 0364-9024/95/010091-09
Let $w$ be a (1,2)-eulerian weight of a cubic graph $G$. A faithful cover $C$ of $w$ is **hamiltonian** if $C$ is a set of two Hamilton circuits. A (1,2)-eulerian weight $w$ of $G$ is **hamiltonian** if every faithful cover of $w$ is hamiltonian. Let $G$ be a uniquely 3-edge-colorable cubic graph with the 1-factorization $\mathcal{F} = \{F_1, F_2, F_3\}$. Let $w: E(G) \to \{1, 2\}$ such that

$$w(e) = \begin{cases} 2 & \text{if } e \in F_3, \\ 1 & \text{if } e \in F_1 \cup F_2. \end{cases}$$

It is obvious that $w$ has a hamiltonian cover $\{F_1 \cup F_3, F_2 \cup F_3\}$. It is natural to ask the following question: Let $G$ and $w$ be defined as above. Is every faithful cover of $w$ hamiltonian (i.e., is $w$ a hamiltonian weight)? The answer is no. A uniquely 3-edge-colorable graph $P(9, 2)$ (see Figure 1) constructed by Tutte ([17]) does not admit a hamiltonian weight. (A graph $P(n, k)$, called a **generalized Pterersen graph**, is defined as follows: $P(n, k)$ has $2n$ vertices, namely $v_0, v_{n-1}, u_0, ..., u_{n-1}$, the vertex $v_i$ is joined to $v_{i+1}, v_{i-1}$ and $u_i, u_i$ is further joined to $u_{i+k}$ and $u_{i-k}$ (where addition is mod $n$).)

The 1-factorization $\mathcal{F} = \{F_0, F_1, F_2\}$ of $P(9, 2)$ is

$$F_1 = \{u_2 v_2, v_5 u_5, v_8 u_8, v_3 u_4, v_6 v_7, v_0 v_1, u_3 u_7, u_4 u_0, u_6 u_1\},$$

$$F_2 = \{v_0 u_0, v_3 u_3, v_6 u_6, v_1 v_2, v_4 v_5, v_7 v_8, u_1 u_5, u_2 u_7, u_4 u_8\},$$

and

$$F_3 = \{v_1 u_1, v_4 u_4, v_7 u_7, v_2 v_3, v_5 v_6, v_8 v_0, u_2 u_6, u_3 u_8, u_5 u_0\}.$$

**FIGURE 1.** Tutte's graph $P(9, 2)$. 

For the eulerian weight
\[ w(e) = \begin{cases} 2 & \text{if } e \in F_3, \\ 1 & \text{if } e \in F_1 \cup F_2, \end{cases} \]
the graph $P(9.2)$ has a non-Hamiltonian faithful cover $C = \{C_1, C_2, C_3, C_4\}$, where
\begin{align*}
C_1 &= v_2v_3v_4u_0u_5v_5v_6u_6u_2v_2, \\
C_2 &= u_0u_5u_1v_1v_2v_3u_8v_8v_0, \\
C_3 &= v_0u_1u_6u_2u_7v_7v_8v_0,
\end{align*}
and
\[ C_4 = v_7u_7u_3u_8v_4v_5v_6v_7. \]

The study of hamiltonian weights and uniquely 3-edge-colorable cubic graphs are motivated by the circuit double cover conjecture that every 2-edge-connected graph has a family of circuits that covers each edge precisely twice. Let a cubic graph $G$ be a minimal counterexample to the circuit double cover conjecture. For each $e_0 \in E(G)$, the graph $G \setminus e_0$ has a circuit double cover. Choose a circuit double cover $C$ such that $C$ contains the maximum number of circuits. Let $C_1, C_2 \in C$ such that $E(C_1) \cap E(C_2) \neq \emptyset$. Let $H$ be the underlying (cubic) graph of the induced subgraph $G[E(C_1) \cup E(C_2)]$. Let $w$ be the $(1,2)$-eulerian weight on $E(H)$ such that $w(e)$ is the number of circuits of $\{C_1, C_2\}$ containing the edge $e$, for each edge $e \in E(H)$. It is not hard to see that $\{C_1, C_2\}$ is a hamiltonian cover of $w$ and $w$ is a hamiltonian weight of $H$. Hopefully, some results about cubic graphs admitting hamiltonian weight will provide some new tools to attack the circuit double cover conjecture.

**2. LEMMAS AND THEOREMS**

A subgraph $H$ of a graph $G$ is even if the degree of each vertex of $H$ is even. Let $w$ be a $(1,2)$-eulerian weight of $G$. Denote $E_i = \{e \in E(G): w(e) = i\}$. Since $w$ is an eulerian weight, it is obvious that the subgraph of $G$ induced by $E_1$ is an even subgraph of $G$. The following lemma is straightforward.

**Lemma 2.1.** Let $G$ be a cubic graph with a 1-factorization $\mathcal{F} = \{F_1, F_2, F_3\}$ and $w$ be a $(1,2)$-eulerian weight of $G$. Then

(i) The set of even subgraphs $\{L_{ij} : \{i, j\} \subset \{1, 2, 3\} \text{ and } i \neq j\}$ covers each edge $e \in E(G)$ precisely $w(e)$ times, where $L_{ij} = E_1 \Delta (F_i \cup F_j)$.

(ii) One of $\{L_{12}, L_{13}, L_{23}\}$ is empty if and only if $E_2 \in \mathcal{F}$. 
Theorem 2.2. Let $G$ be a cubic graph admitting a hamiltonian weight $w$. Then the following statements are equivalent:

(i) $G$ is uniquely 3-edge-colorable.
(ii) $G$ has precisely three Hamilton circuits.
(iii) the hamiltonian weight $w$ has precisely one faithful cover.
(iv) $E_1$ is a Hamilton circuit of $G$.

Proof: (i) $\Rightarrow$ (ii). It is well known and easy to prove (even without the assumption of admitting a hamiltonian weight).

(ii) $\Rightarrow$ (iii). If the hamiltonian weight $w$ has at least two distinct hamiltonian covers, then $G$ has at least four distinct Hamilton circuits, which is a contradiction.

(iii) $\Rightarrow$ (iv). Let $\{H_1, H_2\}$ be the unique hamiltonian cover of $w$. Let $C$ be a component of $E_1$. If $C$ is not a Hamilton circuit of $G$, then $\{H_1 \Delta C, H_2 \Delta C\}$ is also a faithful cover of $w$ and distinct from $\{H_1, H_2\}$. This contradicts that $w$ is a hamiltonian weight with only one faithful cover.

(iv) $\Rightarrow$ (i). Since $E_1$ is a Hamilton circuit and $E_2$ is a 1-factor, $G$ has a 1-factorization $\mathcal{F} = \{F_1, F_2, F_3\}$, where $F_1 \cup F_2 = E_1$ and $F_3 = E_2$. Assume that $G$ is not uniquely 3-edge-colorable. Let $\mathcal{F}_2 = \{F_4, F_5, F_6\}$ be a 1-factorization distinct from $\mathcal{F}_1$. By (i) of Lemma 2.1, the set of even subgraphs

$$ C = \{E_1 \Delta (F_i \cup F_j); \{i, j\} \subset \{4, 5, 6\} \text{ and } i \neq j\} $$

covers each edge $e \in E(G)$ precisely $w(e)$ times. Thus, the set of circuits of circuit decompositions of the members of $C$ is a faithful circuit cover of $w$. Since $w$ is a hamiltonian weight, one member of $C$ is empty. By (ii) of Lemma 2.1, $E_2 \in \mathcal{F}_2$. Without loss of generality, let $E_2 = F_6$. Therefore, $F_6 = F_3$ and $F_4 \cup F_5 = E_1 = F_1 \cup F_2$. Since $E_1$ is a Hamilton circuit, we have that $\{F_1, F_3\} = \{F_4, F_5\}$ and this is a contradiction since $\mathcal{F}_2$ is a 1-factorization distinct from $\mathcal{F}_1$. Then $G$ is uniquely 3-edge-colorable.

Note that (i) and (ii) are not always equivalent (see [13]) without the assumption that $G$ admits a hamiltonian weight. It was conjectured by Greenwell and Kronk ([5], also see [13]) that if a cubic graph $G$ has exactly three Hamilton circuits, then $G$ is uniquely 3-edge-colorable. This conjecture was disproved by A. Thomason ([13]), who found a family of counterexamples: $P(6k + 3, 2)$ for $k \geq 2$, each of which has exactly three Hamilton circuits, but is not uniquely 3-edge-colorable. By Theorem 2.2, those graphs constructed by Thomason do not admit hamiltonian weights.

Lemma 2.3 (Alspach and Zhang [1] or see [2]). Let $G$ be a 2-connected cubic graph containing no subdivision of the Petersen graph. Then $G$ has a faithful cover for every $(1, 2)$-eulerian weight of $G$.
Theorem 2.4. If a 3-connected cubic graph $G$ admits a hamiltonian weight and contains no subdivision of the Petersen graph, then $G$ is uniquely 3-edge-colorable.

Proof. Let $w$ be a hamiltonian weight of $G$ and $\{H_1, H_2\}$ be a hamiltonian cover of $w$.

1. Each component of the 2-factor $E_1 = H_1 \Delta H_2$ is a circuit of even length since $\{H_1, H_2\}$ induces a 3-edge-coloring $\{H_1 \setminus H_2, H_2 \setminus H_1, H_1 \cap H_2\}$ of $G$.

2. For each weight two edge $e_0 = xy$, by Lemma 2.3, $G' = G \setminus \{e_0\}$ has a faithful cover $C$ with respect to the restriction of $w$ to $E(G')$. Define an auxiliary graph $A(C)$ with the vertex set $C$ and two vertices $C_i$ and $C_j$ are adjacent in $A(C)$ if and only if the corresponding circuits $C_i$ and $C_j$ have a nonempty intersection. A circuit chain $P = C_1 \cdots C_r$ joining the vertices $x$ and $y$ of $G$ is a shortest path in $A(C)$ joining a circuit $C_i$ containing $x$ and a circuit $C_r$ containing $y$.

Let $H$ be the graph induced by edges covered by circuits of $P$ and the edge $e_0$, and let $w'$ be a $(1,2)$-eulerian weight on $E(H)$ such that $w'(e)$ is the number of circuits of $P$ containing the edge $e$, and $w'(e_0) = 2$. By Lemma 2.3, $H$ has a faithful cover $C'$. Then $C' \cup [C \setminus P]$ is a faithful cover of $w$. If $C \neq P$, then the faithful cover $C' \cup [C \setminus P]$ of $w$ is not hamiltonian. This contradicts that $w$ is hamiltonian and therefore $C = P$.

Color the edges of $[E(C_1) \cup E(C_2) \cdots \cup E(C_r)]$ red, the edges of $[E(C_2) \cup E(C_4) \cdots \cup E(C_1) \cup E(C_3) \cdots]$ blue and the edges contained in the intersections of two circuits of $C$ yellow. The graph $H$ is 3-edge-colored. Here each component of $E_1$ is colored red and blue. Since $x$ and $y$ are the only degree two vertices of $H$, each component of $E_1$ containing neither $x$ nor $y$ is a circuit of even length. If the vertices $x$ and $y$ are contained in distinct circuits of $E_1$, then $E_1$ has two components of odd lengths. But by 1, every component of $E_1$ is of even length. Therefore, $x$ and $y$ must be in the same component of $E_1$.

Since $e_0 = xy$ is an arbitrary weight two edge of $G$, we have that every weight two edge of $G$ must join two vertices on the same component of $E_1$. Thus $E_1$ has only one component, which is, therefore, a Hamilton circuit of $G$. By (iv) of Theorem 2.2, $G$ is uniquely 3-edge-colorable.

Theorem 2.5. Let $G$ be cubic graph. If $G$ admits at least two hamiltonian weights, then $G$ is uniquely 3-edge-colorable.

Proof. Let $w$ and $w'$ be two distinct hamiltonian weights of $G$ and $\{H_1, H_2\}$, $\{H'_1, H'_2\}$ be hamiltonian covers of $w$ and $w'$, respectively. Both hamiltonian covers induce 1-factorizations $F = \{H_1 \setminus H_2, H_2 \setminus H_1, H_1 \cap H_2\}$ and $F' = \{H'_1 \setminus H'_2, H'_2 \setminus H'_1, H'_1 \cap H'_2\}$. Denote $E_{w-1}$ (and $E_{w'-1}$), the set of edges $e$ of $G$ with weight $w(e) = i$ (and $w'(e) = i$, respectively), for $i = 1, 2$. Since $w$ and $w'$ are two distinct eulerian weights, $E_{w-2} \neq E_{w'-2}$. By
(ii) of Lemma 2.1, $H_1 \cap H_2 = E_{w=2} \in \mathcal{F}$ and $H'_1 \cap H'_2 = E_{w'=2} \in \mathcal{F}$.
Without loss of generality, we assume that $H_1 \cap H_2 = E_{w=2} = H'_1 \setminus H'_2$. Thus $E_{w=1} = H'_2$ is a Hamilton circuit of $G$. By (iv) of Theorem 2.2, $G$ is uniquely 3-edge-colorable.

3. $\Delta-Y$ OPERATION AND EQUIVALENCE CLASSES OF CUBIC GRAPHS

Definition. The $\Delta-Y$ operation of a cubic graph is either (i) contracting the edges of a triangle, or (ii) replacing a vertex of the graph by a triangle (see Figure 2).

Let $\mathcal{T}$ be a graph with the vertex set $V(\mathcal{T})$ being the collection of all connected cubic graphs. Two vertices $G_1$ and $G_2 \in V(\mathcal{T})$ are adjacent in $\mathcal{T}$ if and only if $G_1$ can be obtained from $G_2$ by a $\Delta-Y$ operation. $V(\mathcal{T})$ has a partition into equivalence classes: each class is a component of the graph $\mathcal{T}$. That is, $G_1 \equiv G_2$ if and only if $G_1$ can be obtained from $G_2$ by a series of $\Delta-Y$ operations. In each class, there is only one triangle-free element that is the element with the least number of vertices in the class. Denote by $T_1$ the class containing $K^3_3$, which is the graph with two vertices and three parallel edges, and by $T_2$ the class containing $P(9,2)$.

Remarks. The $\Delta-Y$ operation preserves each of the following properties:

1. planarity,
2. 3-edge-colorability,
3. number of 1-factorizations,
4. number of Hamilton circuits,
5. number of hamiltonian weights,
6. number of hamiltonian covers of a $(1,2)$-eulerian weight.

4. CONJECTURES AND REMARKS

In this section, all graphs we consider are 3-connected cubic graphs. Let $S_1$ be the collection of all uniquely 3-edge-colorable graphs, $S_2$ be the

![FIGURE 2. $\Delta-Y$ operation.](image-url)
collections of all graphs admitting hamiltonian weights, \( PL \) be the collection of all planar graphs, and \( \overline{P}_{10} \) be the collection of all graphs containing no subdivision of the Petersen graph. (Here, \( PL \subseteq \overline{P}_{10} \). It was conjectured by Greenwell and Kronk ([5], also see [13]) that \( S_1 = T_1 \). (That is, every uniquely 3-edge-colorable cubic graph is planar and has a triangle.) This conjecture was disproved by Tutte, who found the counterexample \( P(9,2) \) (see Figure 1). Since the generalized Petersen graph \( P(9,2) \) is not planar, the conjecture was later modified as follows,

**Conjecture 4.1** (Fiorini and Wilson [4]). Let \( G \) be a 3-connected planar cubic graph with at least 4 vertices. If \( G \) is uniquely 3-edge-colorable cubic graph, then \( G \) has a triangle. That is,

\[
PL \cap S_1 = T_1.
\]

The author believes that in the construction of a triangle-free, uniquely 3-edge-colorable cubic graph other than \( K_2^3 \), a non-3-edge-colorable cubic graph (snark) must be somehow involved. Based on the famous 4-flow conjecture of Tutte ([16]) that no snark belongs to \( \overline{P}_{10} \), the author proposes the following conjecture.

**Conjecture 4.2.** Let \( G \) be a 3-connected cubic graph containing no subdivision of the Petersen graph. If \( G \) is uniquely 3-edge-colorable, then \( G \) must contain a triangle. That is,

\[
\overline{P}_{10} \cap S_1 = T_1.
\]

Recall Theorem 2.4: \( \overline{P}_{10} \cap S_2 \subset S_1 \). Note that \( T_2 \), which contains \( P(9,2) \) is a subset of neither \( \overline{P}_{10} \) nor \( S_2 \). We propose the following conjecture:

**Conjecture 4.3.** Let \( G \) be a 3-connected cubic graph containing no subdivision of the Petersen graph. If \( G \) is uniquely 3-edge-colorable, then \( G \) must admit a hamiltonian weight. That is,

\[
\overline{P}_{10} \cap S_1 \subset S_2.
\]

By Theorem 2.4, we have the following equivalent version of Conjecture 4.3.

**Conjecture 4.4.** If \( G \) is a 3-connected cubic graph containing no subdivision of the Petersen graph, then \( G \) admits a hamiltonian weight if and only if \( G \) is uniquely 3-edge-colorable. That is,

\[
\overline{P}_{10} \cap S_1 = \overline{P}_{10} \cap S_2.
\]
Since $T_1 \subseteq S_2$, Conjecture 4.3, as well as Conjecture 4.4, is implied by Conjecture 4.2. Similar to Conjecture 4.1, we propose

**Conjecture 4.5.** Every 3-connected cubic graph admitting a hamiltonian weight contains a triangle. That is,

$$S_2 = T_1.$$

The following conjecture is a generalization of Theorem 2.4,

**Conjecture 4.6.** Every 3-connected cubic graph admitting a hamiltonian weight is uniquely 3-edge-colorable. That is,

$$S_2 \subseteq S_1.$$

Note that the 3-connectivity in most conjectures of this section cannot be relaxed, since the 2-connected cubic graph $H$ with four vertices $\{a, b, c, d\}$ and the six edges $\{ab, ab, ac, bd, cd, cd\}$ admits a hamiltonian weight $w$ with $E_{w-2} = \{ac, bd\}$ but contains no triangle, and all cubic graphs obtained from $H$ by $\Delta - Y$ operations admit hamiltonian weights but are not uniquely 3-edge-colorable.

**ACKNOWLEDGMENT**

Partial support was provided by the National Science Foundation under Grant DMS-9306379.

**References**


Received January 31, 1994