Coordinated path following for a multi-agent system of unicycles

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Abstract— We investigate a coordinated path following problem for a multi-agent system of \(N\) unicycles. Each unicycle is required to follow its own path while coordinating its motion along its path with the other unicycles. We assume that all the unicycles can communicate with each other. We view coordinated path following as a nested set stabilization problem. Stabilization of the first set corresponds to meeting the path following specification. Stabilization of the second, nested set, corresponds to meeting the coordination specification. The first set is characterized for the multi-agent system of \(N\) unicycles and feedback linearization is proposed to stabilize it. We provide sufficient conditions under which the second set is non-empty and stabilize it using feedback linearization.

Index Terms—coordinated path following, set stabilization

I. INTRODUCTION

Control of multi-agent systems has received considerable attention in the robotics and control communities [1]. Coordinated path following has received comparatively less attention. Coordinated path following, as defined in this paper, involves getting a group of mobile robots to each follow a path, with no \emph{a priori} time parameterization of the path, in a coordinated way, e.g., each unicycle must pass through its check point at the same time. This is a suitable control objective in applications that demand each mobile robot follow its path accurately while maintaining coordination with the rest of the group. Examples include search and rescue operations involving mobile robots [2], patrolling a pre-defined region to prevent intruders or for the sampling of an area [3], and marine applications [4].

Roughly speaking, three distinct approaches have been employed to accomplish coordinated path following. The first method is called the formation reference point (FRP) method. In this method, each agent requires a reference path which are parameterized so that when all the path parameters are synchronized, the agents are in formation. In order to generate the reference paths a geometric structure is considered whose center moves along a reference path. The shape of this structure is allowed to change over the time. In other words, formation can vary over the time. The parameter of the path is as a virtual vehicle moving along the path and the path following specification is achieved by steering each agent toward this virtual vehicle. The coordination specification is accomplished through coordinating the parameters of the paths [5]. In the second method, coordinated path following is solved by decoupling the path following and coordination problems. The path following problem is achieved by assuming the stabilization of tracking errors between the state of each agent and the virtual target on the path. The desired formation is obtained by guaranteeing the consensus of the arc-length [6]. The third method is called the curve extension method. In the curve extension approach, the path of each agent is extended to the set of level curves of a smooth function. The path following specification is achieved when the value of the smooth function reaches the desired value. Formation is achieved by forcing the relative arc-length between each pair of vehicles to a constant value [7], [8]. The main shortcoming with all these methods is that once agents reach their paths they might leave it. Similarly, once the coordination is achieved the agents might leave the coordination.

In this paper the coordinated path following problem with full communication is decomposed into two, completely decoupled, sub-problems. The first subproblem involves designing a local control law for each unicycle that renders the desired path attractive. The second subproblem entails designing control laws for each unicycle to achieve coordination. Unlike existing studies on coordinated path following our approach guarantees the invariance of both path following and coordination specifications in the sense that, after the agents reach their paths and are in coordination, they maintain these properties for all future time. We have assumed all unicycles can communicate with each other. In other words, we assume that the inter-unicycle communication is modelled by a complete graph.

A. Notation

Let \(\text{col}(x_1,\ldots,x_k) := [x_1\cdots x_k]^\top\) where \(\top\) denotes transpose and, given two column vectors \(a\) and \(b\), we let \(\text{col}(a, b) := [a^\top b^\top]^\top\). The notation \(so\ h : A \to C\) represents the composition of maps \(s : B \to C\) and \(h : A \to B\). The symbols \(1_{n \times n}\) and \(0_{n \times n}\) represent, respectively, the \(n \times n\) identity matrix and matrix of zeros. Let \(\arg : \mathbb{C} \to (-\pi, \pi]\) map a complex number to its principle argument. Given \(L \in \mathbb{R}\), the reals modulo \(L\) is written \(\mathbb{R} \mod L\).

If \(f : \mathbb{R}^n \to \mathbb{R}\) is a differentiable function, we denote by \(\partial_x f\) its partial derivative with respect to \(x_i\). The Jacobian of a \(C^1\) map \(f : \mathbb{R}^n \to \mathbb{R}^m\) evaluated at \(p \in \mathbb{R}^n\) is written \(df_p\). If \(f, g : \mathbb{R}^n \to \mathbb{R}^n\) are smooth vector fields and \(\phi : \mathbb{R}^n \to \mathbb{R}^m\) is a smooth map, we use the following standard notation for iterated Lie derivatives \(L_0^\phi \phi := \phi, L_f^\phi := L_f(L_{f}^{k-1}\phi) = \langle dL_{f}^{k-1}\phi_x, f(x)\rangle\). If \(f : \mathbb{R} \to \mathbb{R}^n\) then \(f'(\lambda) := \frac{df(\lambda)}{d\lambda}\).
II. COORDINATED PATH FOLLOWING PROBLEM

In this section we formulate a coordinated path following problem for a multi-agent system of \( N \) unicycles. We assume that all unicycles can communicate with each other. Under this assumption, centralized control laws are permitted.

A. Model of unicycle

We consider a multi-agent system consisting of \( N \) dynamic unicycles. Following [9], the model of unicycle \( i \), \( i \in \{1, \ldots, N\} \), is given by

\[
\begin{bmatrix}
\dot{x}_i \\
\dot{y}_i \\
\dot{\theta}_i \\
\dot{w}_i
\end{bmatrix}
= \begin{bmatrix}
(v + w_i) \cos (\theta_i) \\
(v + w_i) \sin (\theta_i) \\
0 \\
0
\end{bmatrix}
+ \begin{bmatrix}
u_i^1 \\
u_i^2 \\
-1 \\
0
\end{bmatrix}
\]  

(1)

where \((x_i, y_i)\) denotes the position of the unicycle in the plane, \( \theta_i \) is the heading angle, and \( w_i + v \), with \( v > 0 \) a fixed constant, is the forward velocity of the unicycle. The control inputs \( u_i^1 \) and \( u_i^2 \) are, respectively, the forward acceleration and angular velocity. Let \( x_i := \text{col}(x_i, y_i, \theta_i, w_i) \in \mathbb{R}^4 \). We take the position of the unicycle \( i \) as its output \( z_i := \text{col}(x_i, y_i) \). Define the state of the multi-agent system as \( x := \text{col}(x_1, \ldots, x_N) \in \mathbb{R}^{4N} \). The information flow of the multi-agent system is modelled as an undirected graph \( \mathcal{G} \).

Assumption 1. The undirected graph \( \mathcal{G} \) that models the information flow in the multi-agent unicycle system is complete.

B. Path following

Suppose that each unicycle is assigned a path \( \gamma_i \subset \mathbb{R}^2 \) in its output space that it must follow. The path \( \gamma_i, i \in \{1, \ldots, N\} \), has a smooth, unit-speed, parameterization \( \sigma_i : \mathbb{R} \to \mathbb{R}^2 \) with \( \gamma_i = \sigma_i(\mathbb{R}) \).

Assumption 2. For \( i \in \{1, \ldots, N\} \), the path \( \gamma_i \subset \mathbb{R}^2 \) is a one-dimensional embedded submanifold. Furthermore, there exists a smooth map \( s_i : \mathbb{R}^2 \to \mathbb{R} \) such that \( \gamma_i = s_i^{-1}(0) \) and \( ds_i \neq 0 \) for all \( z_i \in \gamma_i \).

The objective of path following is to design \( N \) smooth feedback controllers, one for each agent, to drive the output of each closed-loop agent towards \( \gamma_i \). Moreover, we ask that \( \gamma_i \) be output invariant in the sense defined in [10]. Let

\[
P_i := \{ x_i \in \mathbb{R}^4 : \alpha_i(x_i) := s_i \circ h_i(x_i) = 0 \}
\]

and \( \mathcal{P} := \mathcal{P}_1 \times \cdots \times \mathcal{P}_N \). Stabilizing the set \( P_i \) corresponds to sending the output \( z_i \) of agent \( i \) to its desired path. We seek to stabilize the largest controlled-invariant subset of \( P_i \), denoted \( P_i^* \).

Assumption 3. There exists, for all \( i \in \{1, \ldots, N\} \), a point \( x_i^* \in P_i \) such that the maximal controlled-invariant subset of \( P_i \) containing \( x_i^* \) is a non-empty, closed embedded submanifold with dimension \( n_i^* > 0 \).

Assumption 3 implies that path \( \gamma_i \) is feasible for unicycle \( i \). The set \( P_i^* \) is called the path following manifold of the unicycle \( i \) with respect to \( \gamma_i \) [10].

Definition II.1. The multi-agent path following manifold for the paths \( \gamma_1, \ldots, \gamma_N \), in a neighbourhood of \( x^* = \text{col}(x_1^*, \ldots, x_N^*) \in \mathcal{P} \), is

\[
P^* := \mathcal{P}_1^* \times \cdots \times \mathcal{P}_N^*
\]

and, under Assumption 3, its dimension is \( n^* = \sum n_i^* \geq N \).

C. Coordination

The multi-agent path following manifold (2) plays a key role in formulating and solving the coordination portion of our problem. The coordination specification along the paths is modeled as a constraint on the allowable motions on the multi-agent path following manifold \( \mathcal{P}^* \). To this end, we model coordination as a smooth constraint map \( \beta : \mathcal{P}^* \to \mathbb{R}^c \) with \( c \leq n^* \).

Definition II.2. A coordination function on a multi-agent path following manifold \( \mathcal{P}^* \) is a smooth map \( \beta : \mathcal{P}^* \to \mathbb{R}^c \), \( c \leq \dim(\mathcal{P}^*) \), where \( \beta \) has rank \( c \) everywhere.

Let \( \beta : \mathcal{P}^* \to \mathbb{R}^c \) be a coordination function and consider the set \( C = \beta^{-1}(0) \). In this paper we take the view that stabilizing the set \( C \) corresponds to achieving coordination. We also require that the coordination specification be invariant in the sense that, if the agents are initially coordinated, they remain coordinated indefinitely. This motivates us to characterize the largest controlled-invariant subset of \( C \).

Definition II.3. The coordination set \( C^* \) associated to a coordination function \( \beta \) is the largest controlled-invariant subset of \( C = \beta^{-1}(0) \).

D. Problem statement

The coordinated path following control design problem considered in this paper entails finding \( N \) feedback control laws ensuring that the closed-loop multi-agent system meets the following two objectives.

PF For each initial condition \( x(0) \) in a neighbourhood of \( P^* \), the corresponding solution \( x(t) \) is defined for all \( t \geq 0 \) and \( x(t) \to P^* \) as \( t \to +\infty \).

CO For each initial condition in a neighbourhood of \( C^* \), contained in \( P^* \), the corresponding solution \( x(t) \in P^* \) for all \( t \geq 0 \) and \( x(t) \to C^* \).

Here we have cast the coordinated path following problem as two set stabilization problems; namely the stabilization of \( P^* \) and \( C^* \). For general multi-agent systems, even if PF and CO hold, there is no guarantee that when \( x(0) \not\in P^* \), \( x(t) \not\to C^* \). In this paper we show that in the case of unicycles this problem does not occur.

III. PATH FOLLOWING

We begin by characterizing the multi-agent path following manifold. Since \( P^* \) is a product manifold, it is sufficient to characterize \( P_i^* \) for each unicycle. The zero dynamics algorithm applied to (1) with output \( \alpha_i(x_i) \) yields

\[
P_i^* = \{ x_i \in \mathbb{R}^4 : \alpha_i(x_i) = 0, w_i + v \neq 0 \}
\]

(3)
The set \( P^*_i \) is not connected, it consists of four disjoint components. These four components are distinguished by the fact that \( w_i + v \neq 0 \) on \( P^*_i \). Each component of \( P^*_i \) corresponds to a distinct type of motion for unicycle \( i \) along its path. The four different types of motion are depicted in Figure 1 for unicycle \( i \). In Figure 1 \( \sigma'_i(\lambda) \) is the tangent vector to \( \gamma_i \) at \( \sigma_i(\lambda) \) and indicates the parameterization direction. Since \( w_i + v \neq 0 \) on \( P^*_i \), unicycle \( i \) cannot switch between these four situations. Specifications PF is satisfied by making (3) attractive for each unicycle.

A. Feedback linearization

If path \( i \) is closed then \( D_i = \mathbb{R} \mod L_i \) where \( L_i > 0 \) is the length of the curve. If path \( i \) is non-closed then \( D_i = \mathbb{R} \). We now introduce a projection, similar to that used in [12], in the output space of the unicycle that associates the coordinate transformation \( P^*_i \) with \( \pi \) and the output \( \mathbb{R} \) is employed to prove this result. According to the generalized inverse function theorem [14, p.56] and, using arguments similar to those used in [12, Lemma 3.2], that \( \partial\pi_i = [\partial x_i, \partial y_i, 0 \ 0] \) and \( d\pi_i = [\partial x_i, \partial y_i, 0 \ 0] \) are orthogonal.

To show that 1) holds, we show that \( \det(dT_i) \) is non-zero on \( P^*_i \). This follows from the fact that \( \det(dT_i)|_{P^*_i} = -(w_i + v)(\partial x_i, \pi_i, \partial y_i, \alpha_i - \partial x_i, \alpha_i, \partial y_i, \pi_i)^2 \) and, using arguments similar to those used in [12, Lemma 3.2], that \( d\pi_i = [\partial x_i, \partial y_i, 0 \ 0] \) and \( d\pi_i = [\partial x_i, \partial y_i, 0 \ 0] \) are orthogonal.

To show that 2) holds, note that the restriction of \( T_i \) to \( P^*_i \) is given by

\[
T_i|_{P^*_i} = \text{col}(\eta_1^i, \eta_2^i, \xi_1^i, \xi_2^i) = \text{col}(\pi_i, L_f\pi_i, 0, 0).
\]

By definition, on \( P^*_i \), the unicycle is on the path so, by the definition of \( \pi_i \), we have \( \text{col}(x_i, y_i) = \sigma_i(\eta_1^i) \). On the component \( P^*_i+ \) of \( P^*_i \), the unicycle’s heading angle is collinear with \( \sigma'(\pi_i(x_i)) \) and so \( \theta_i = \text{arg}(\sigma'(\eta_1^i) + j\sigma'(\eta_2^i)) \) where \( j = \sqrt{-1} \). Using the equation \( \eta_2^i = L_f\pi_i \) one finds

\[
T_i|_{P^*_i+} = \text{col}(\sigma_1^i(\eta_1^i), \sigma_2^i(\eta_1^i), \text{arg}(\sigma_1^i(\eta_1^i) + j\sigma_2^i(\eta_1^i)), \eta_2^i - \nu).
\]

This shows that the inverse map, \( T_i^{-1}|_{P^*_i+} \), is well-defined and smooth, hence \( T_i|_{P^*_i} \) is a diffeomorphism onto its image. 

By Lemmas III.1 and III.2, in a neighborhood of each component of \( P^*_i \), the dynamic unicycle (1) is feedback equivalent to the system

\[
\text{col}(\eta_1^i, \eta_2^i, \xi_1^i, \xi_2^i) = \text{col}(\eta_2^i, v_i, \xi_2^i, v_i^i).
\]

Let \( \xi_i := (\xi_1^i, \xi_2^i) \) and \( \eta_i := (\eta_1^i, \eta_2^i) \). We call the \( \xi_i \)-subsystem in (4) the transversal dynamics of unicycle \( i \) to each component of \( P^*_i \). This is motivated by the fact that for system (4), stabilizing each component of \( P^*_i \) is equivalent, under mild assumptions, to stabilizing the \( \xi_i \)-subsystems, \( i \in \{1, \ldots, N\} \). The \( \eta_i \)-subsystem is called tangential dynamics of unicycle \( i \) with respect to each component of \( P^*_i \). The \( \eta_i \) states are a natural parameterization of each component of \( P^*_i \) with strong physical meaning for the coordinated path following problem. The state \( \eta_1^i \) represents the position of unicycle \( i \) along the path and the state \( \eta_2^i \in \mathbb{R} \) represents the velocity of unicycle \( i \) along the path.

IV. COORDINATION

In this section the dynamics of the multi-agent system of \( N \) unicycles restricted to \( P^* \) is found and the topology of \( P^* \) is characterized. We also describe how the tangential states
can be utilized to define a coordination set. Sufficient conditions for a coordination task to be feasible are presented.

A. Characterization of $C^*$

The overall dynamics of the $N$ unicycle system restricted to the path following manifold are described by the stacked tangential dynamics each unicycle. Let $\eta^1 := \col (\eta_1^1, \ldots, \eta_N^1)$, $\eta^2 := \col (\eta_1^2, \ldots, \eta_N^2)$, $\eta := \col (\eta^1, \eta^2)$, and $v^o := \col (v_1^1, \ldots, v_N^o)$ then the dynamics restricted to $\mathcal{P}^*$ are

$$\dot{\eta} = \begin{bmatrix} 0_{N \times N} & I_{N \times N} \\ 0_{N \times N} & 0_{N \times N} \end{bmatrix} \eta + \begin{bmatrix} 0_{N \times N} \\ I_{N \times N} \end{bmatrix} v^o. \tag{5}$$

Having found the dynamics on $\mathcal{P}^*$ we determine the topology of $\mathcal{P}^*$. Assume, without loss of generality, that $\gamma_i$ for $i \in \{1, \ldots, r\}$ where $r \leq N$ are closed paths and $\gamma_i$ for $i \in \{r + 1, \ldots, N\}$ are non-closed paths. If the path $\gamma_i$ is closed, then $\eta^1_i \in \mathcal{D}_i = \mathbb{R} \mod L_i$ and if $\gamma_i$ is non-closed, then $\eta^2_i \in \mathcal{D}_i = \mathbb{R}$. The state $\eta^{0,2} \in \mathbb{R}$ as discussed earlier. This shows that the path following manifold of unicycle $i$ is $\mathcal{P}^*_i = \mathbb{R} \mod L_i \times \mathbb{R}$, a cylinder, if $i \in \{1, \ldots, r\}$ and $\mathcal{P}^*_i = \mathbb{R}^2$ if $i \in \{r + 1, \ldots, N\}$.

Since $\eta$ is a parameterization of $\mathcal{P}^*$ it can be used to define a coordination function, $\beta : \mathcal{P}^* \to \mathbb{R}^c$, introduced in Definition II.2. Accordingly, the coordination set introduced in Definition II.3 is the largest controlled-invariant subset of

$$\mathcal{C} = \{ \eta \in \mathcal{P}^* : \beta(\eta) = 0 \}. \tag{6}$$

Since, when there are closed-paths, the multi-agent path following manifold is not diffeomorphic to $\mathbb{R}^{2N}$, the constraint function may not be continuous when viewed as a function from $\mathbb{R}^{2N} \to \mathbb{R}^c$. In these cases one should cover the multi-agent path following manifold with coordinate charts so that, in each chart, the local representation of $\beta$ is a map from an open neighbourhood of $\mathbb{R}^{2N}$ to $\mathbb{R}^c$. In the subsequent discussion we implicitly work with these local representations of $\beta$.

B. Feasible coordination set

A coordination specification is feasible if its corresponding coordination set, see Definition II.3, is non-empty. We present sufficient conditions for both linear and nonlinear coordination specifications to be feasible.

1) Linear Coordination: Assume that the coordination function $\beta$ is a linear affine function of the form $\beta(\eta) = A\eta + b$, i.e.,

$$\mathcal{C} = \{ \eta \in \mathcal{P}^* : \beta(\eta) = A\eta + b = 0 \} \tag{7}$$

with $A \in \mathbb{R}^{c \times 2N}$, rank $(A) = c$, and $b \in \text{Im} A$. For clarity, we write the matrix $A = [A_1 \ A_2]$ and the vector $b = \col (b_1, b_2)$ as where the partitions correspond to $\eta = \col (\eta^1, \eta^2)$. The following proposition gives a condition under which the linear coordination function is feasible.

**Proposition IV.1.** Consider the linear coordination function $(7)$. Let $R$ be a full row rank matrix satisfying $RA_2 = 0$. If

1. $(\forall \eta_2 \in \mathcal{C}) \ RA_1 \eta_2 = 0$, or
2. $\text{Im} \begin{bmatrix} b \\ 0 \end{bmatrix} \subset \text{Im} \begin{bmatrix} A_1 & A_2 \\ 0 & RA_1 \end{bmatrix} \tag{8}$

then the coordination set $\mathcal{C}$ is nonempty.

**Proof.** By the definition of $(7)$, $b \in \text{Im} A$, therefore the set $\mathcal{C}$ is non-empty. For the set $\mathcal{C}$ itself to be controlled invariant there must exist a control law $v^o$ such that the derivative of $A\eta + b$ is identically zero. Taking the derivative of $\beta$ along solutions of the system $(5)$ we obtain

$$A_1 \eta_2 + A_2 v^o|_{C} = 0. \tag{9}$$

Note that the equation $(9)$ must be solved on the set $\mathcal{C}$. Left multiply equation $(9)$ by $R$ to obtain that the equation $(9)$ is solvable in $v^o$ if and only if $RA_1 \eta_2|_{\mathcal{C}} = 0$. If this condition holds then the largest controlled invariant subset, $C^*$ equals $\mathcal{C}$. Since the set $\mathcal{C}$ is non-empty the coordination set is also non-empty. On the other hand, if there exists $\eta_2 \in \mathcal{C}$ for which $RA_1 \eta_2^o \neq 0$, then we cannot find $v^o$ solving equation $(9)$. In this case we add the condition $RA_1 \eta_2^o = 0$ to the set $\mathcal{C}$ and we obtain a new set $\mathcal{C}_1 \subseteq \mathcal{C} = \{ \eta \in \mathcal{C} : RA_1 \eta_2 = 0 \}$.

In order for the set $\mathcal{C}_1$ to be controlled invariant there must exist a control law $v^o$ such that the derivative of $RA_1 \eta_2^o$ along solutions of the system $(5)$ is identically zero. Setting the derivative equal to zero we obtain $RA_1 \eta_2|_{\mathcal{C}_1} = 0$. Equation $(9)$ must be solved on the set $\mathcal{C}_1$. The control law $v^o = 0$ trivially solves this equation, so the coordination set, $C^*$, equals $\mathcal{C}_1$. The condition $(8)$ guarantees that there exist solution to the constraints imposed on the set $\mathcal{C}_1$, i.e., $A_1 \eta_2 = 0$ and $RA_1 \eta_2 = 0$, so it is non-empty. Thus, the coordination set is non-empty accordingly.

$\square$

2) Nonlinear Coordination: Consider a coordination specifications described by a nonlinear coordination function $\beta : \mathcal{P}^* \to \mathbb{R}^{c}$ with $c \leq 2N$. By Definition II.2 the set $(6)$ is a smooth $(2N - c)$-dimensional embedded submanifold of $\mathcal{P}^*$. Consider the partition $\partial \beta = [\partial_{\eta^1} \beta \ \partial_{\eta^2} \beta]$ for $d\beta$ corresponding to $\eta = \col (\eta^1, \eta^2)$. To check whether or not $\mathcal{C}$ is controlled-invariant we take the derivative of $\beta$ along solutions of the system $(5)$ to obtain

$$\partial_{\eta^1} \beta \eta^2 + \partial_{\eta^2} \beta \eta^o = 0. \tag{10}$$

Fix $\eta \in \mathcal{C}$ and suppose that $R(\eta)$ is a full row rank matrix satisfying $R(\eta)\partial_{\eta^2} \beta = 0$ in neighbourhood $U \subseteq \mathcal{C}$ of $\eta$. Then, equation $(10)$ is solvable in $v^o$ in the neighbourhood $U$ if and only if $(\forall \eta \in U) \ R(\eta)\partial_{\eta^2} \beta \eta^2 = 0$. If this condition holds then the largest controlled-invariant set is the open set $U$ and has the same dimension as $\mathcal{C}$. If this condition does not hold we impose the constraint $R(\eta)\partial_{\eta^2} \beta \eta^2 = 0$ on the set $U \subseteq \mathcal{C}$. Doing so, we give rise to a new set $\mathcal{C}_1 \subseteq U \subseteq \mathcal{C}$ given by $\mathcal{C}_1 = \{ \eta \in U : R(\eta)\partial_{\eta^2} \beta \eta^2 = 0 \}$. In order to check controlled-invariance of $\mathcal{C}_1$ we take the
derivative of the constraint \( R(\eta)\partial_\eta \frac{\partial }{\partial } \beta \eta^2 = 0 \) along solutions of the system (5) to get
\[
\partial_\eta \frac{\partial }{\partial } \beta \left( \partial_\eta R(\eta) \partial_\eta \beta \eta^2 \right) \eta^2 + \partial_\eta \beta \left( R(\eta) \partial_\eta \beta \eta^2 \right) \nu^\parallel = 0 \tag{11}
\]
The above expression is very general and it is not clear whether or not it is solvable in \( \nu^\parallel \) on \( C_1 \). In the following propositions we consider special cases of the coordination function for which, in a neighbourhood of \( \eta \in C_1 \), this equation is solvable in \( \nu^\parallel \).

**Proposition IV.2.** Let the coordination function be such that \( \partial_\eta \beta = 0 \). Then the coordination set, \( C^* \), is non-empty and equals an open subset \( U \subseteq C \).

**Proposition IV.3.** Let the coordination function be such that \( \partial_\eta \beta \equiv 0 \). Fix \( \eta \in C \) and suppose that in a neighbourhood \( U \subseteq C \), \( \partial_\eta \beta \left( \partial_\eta \beta \eta^2 \right) \eta^2 = \Im \partial_\eta \beta \) and
\[
\rank \begin{bmatrix} \partial_\eta \beta & \partial_\eta \beta \\ \partial_\eta \beta & \partial_\eta \beta \eta^2 \end{bmatrix} \leq 2N.
\]
Then, the coordination set, \( C^* \) is equal to \( C_1 \) and is non-empty.

The proof of above propositions are omitted due to space restrictions. Their proofs are conceptually the same as the proof of Proposition IV.1. The set \( C^* \subset \mathcal{P}^* = \mathcal{P}_1^* \times \cdots \times \mathcal{P}_N^* \) is a submanifold of a product manifold.

V. CONTROL DESIGN

We propose a method to accomplish coordinated path following. Since we view coordinated path following as a nested set stabilization problem, path following control design is decoupled from coordination control design.

A. Stabilizing \( \mathcal{P}^* \)

The multi-agent path following manifold, \( \mathcal{P}^* \), can be stabilized by stabilizing, for \( i \in \{1, \ldots, N\} \), the path following manifold \( \mathcal{P}_i^* \) of unicycle \( i \). It was seen in (4) that \( v_i^\perp \) can be utilized to stabilize \( \mathcal{P}_i^* \). Therefore, we design the transversal control input \( v_i^\perp \) for unicycle \( i \) as \( v_i^\perp = -k_i \xi_i^2 - k_i^2 \xi_i^2 \) with \( k_i, k_i^2 > 0, i \in \{1, \ldots, N\} \). With this choice the origin of each transversal subsystem is rendered exponentially stable. If the trajectory of the unicycle is bounded, then this is equivalent to stabilizing \( \mathcal{P}_i^* \). The component of \( \mathcal{P}_i^* \) the unicycle goes to, however, depends entirely on initial conditions. In other words, the desired component of \( \mathcal{P}_i^* \) cannot be selected by this control law. When \( \mathcal{D}_i = \mathbb{R} \) the path \( \gamma_i \) itself is unbounded and so traversing the path results in unbounded trajectories for the unicycle. In that case additional conditions are needed to ensure that \( \xi \rightarrow 0 \iff \chi_i \rightarrow \mathcal{P}_i^* \) [15].

B. Stabilizing \( C^* \)

We propose to employ TFL to render the coordination set \( C^* \) stable for (5). Note that this method is not possible for every coordination set. Checkable conditions to determine if TFL is feasible for a general coordination set are given in [11]. We introduce some special coordination sets which can be stabilized using TFL in the following corollaries.

These corollaries, which follow from the results in Section IV-B, provide conditions under which the coordination function that defines \( C \) yields a well defined relative degree and can be used as the transversal output of (5).

**Corollary V.1.** If \( \text{rank } A_2 = c \) then the output \( \beta(\eta) = A\eta + b \) yields a well-defined vector relative degree of \( \{1, \ldots, 1\} \) for the system (5). If \( A_2 \) is identically zero then the output \( \beta(\eta) = A\eta + b \) yields a well-defined vector relative degree of \( \{2, \ldots, 2\} \) for the system (5).

**Corollary V.2.** If the coordination function \( \beta : \mathcal{P}^* \rightarrow \mathbb{R}^c \) is such that \( \partial_\eta \beta = 0 \) and \( \text{rank } d\beta = c \) then the output \( \beta(\eta) \) yields a well-defined vector relative degree of \( \{1, \ldots, 1\} \) for the system (5) at each \( \eta \in C \).

**Corollary V.3.** If the coordination function \( \beta : \mathcal{P}^* \rightarrow \mathbb{R}^c \) is such that \( \partial_\eta \beta = 0 \) and \( \text{rank } d\beta = c \) then the output \( \beta(\eta) \) yields a well-defined vector relative degree of \( \{2, \ldots, 2\} \) for the system (5) at each \( \eta \in C \).

When a coordination set can be stabilized using TFL, the TFL normal form of the tangential dynamics (5) with respect to the set \( C^* \)
\[
\dot{\zeta} = f(\zeta, \mu) + g^\perp(\zeta, \mu)\tau^\perp + g^\parallel(\zeta, \mu)\tau^\parallel \tag{12}
\]
where \( (\zeta, \mu) \in \mathbb{R}^{c*} \times \mathbb{R}^{n*} - \mathbb{R}^c \) with \( c* := \dim C^* \) and \( n* := \dim \mathcal{P}^* \). It should be noted that even though (5) is linear the TFL normal form (12) may be nonlinear because the coordination function may be nonlinear. In (12) the \( \eta \) states are decomposed into \( \zeta \) and \( \mu \) states, and the control input, \( \nu^\parallel \) into two groups, \( \tau^\parallel \) and \( \tau^\perp \). The \( \mu \)-subsystem describes the motion transversal to the set \( C^* \) but contained in \( \mathcal{P}^* \). The linear control law \( \tau^\perp = -K\mu \) with \( K \) being a matrix of appropriate size can be employed to render the origin of the \( \mu \)-subsystem stable. In this way the CO specification is achieved. The \( \zeta \)-subsystem describes the dynamics tangent to \( C^* \). The multi-agent system restricted to \( C^* \) has dynamics
\[
\dot{\zeta} = f(\zeta, 0) + g^\parallel(\zeta, 0)\tau^\parallel. \tag{13}
\]
Therefore, in some cases, it may be possible to use the extra control input \( \tau^\parallel \) to control the dynamics restricted to \( C^* \). Consider, for a moment, a general multi-agent system with given sets \( C^* \subset \mathcal{P}^* \) in its state space. In the general case the decoupled design procedure employed above may not work. This is because, even if \( C^* \) is (globally) asymptotically stable for initial conditions on \( \mathcal{P}^* \), it does not follow that the set \( C^* \) is (globally) asymptotically stable for all initial conditions. For the multi-agent system of \( N \) unicycles this stability issue does not arise because in the TFL normal form of each unicycle (4) the \( \xi_i \)-subsystem and \( \eta_i \)-subsystem are decoupled. The stability issue, however, arises when one designs \( \tau^\parallel \) and \( \tau^\perp \) separately, and applies them to (12) as the \( \mu \)-subsystem and \( \zeta \)-subsystem are not, in general, decoupled [16].
VI. SIMULATION

Consider a multi-agent system of three unicycles which are required to follow the paths $\gamma_1 = z_2^1 = \sin(\frac{1}{2\pi} z_1^1) = 0$, $\gamma_2 = z_2^2 - 2 = 0$, $\gamma_3 = z_2^3 + \sin(\frac{1}{2\pi} z_1^3) - 4$. The path following controller is designed by first feedback linearizing each unicycle to bring them into the normal form given (4). We use PD controllers to stabilize each unicycle’s transversal dynamics.

While the unicycles are on their paths we ask that all three unicycles traverse the same amount of arc-length at any moment in time. Moreover, the unicycles are required to move with a prescribed velocity $v_d$. Since $\eta_i^1$ specifies the arc-length of unicycle $i$ along its path, one choice of coordination function that encodes this specification is $\beta(\eta) = \text{col}(\eta_1^1 - \eta_1^1, \eta_2^1 - \eta_1^2, \eta_3^1 - \eta_1^3)$.

One can verify that the coordination function satisfies the conditions of Proposition IV.3 thus $C^* = C = \beta^{-1}(0)$ and is non-empty. The dynamics on the multi-agent path following manifold $P^*$ are given by (5) with $N = 3$. By Corollary V.3, the function $\beta(\eta)$, when taken as the output of the tangential dynamics, yields a well-defined vector relative degree of $\{2,2,2\}$. As such TFL amounts to input-output feedback linearization for non-square MIMO systems and the multi-agent tangential dynamics, restricted to $P^*$, are feedback equivalent to a system of the form (12). However, in this particular example, if we take $\text{col}(\beta(\eta), \eta_1^1)$ as the output of the concatenated tangential dynamics, it can be shown to yield a well-defined vector relative degree of $\{2,2,2\}$. As such the tangential dynamics are feedback equivalent to

$$\text{col}(\zeta^1, \zeta^2, \mu_1^2, \mu_2^3, \mu_4^3) = \text{col}(\zeta^1, \tau^1, \mu_2^1, \mu_4^1, \tau^2)$$

Here, unlike in the general normal form (12), the $\mu$- and $\zeta$-subsystems are decoupled. Furthermore, the $\zeta$ dynamics, which govern the dynamics on $C^*$ and, in a sense, govern the dynamics of the entire platoon while synchronized, are linear and controllable. This is in contrast to the general case in (13). We design $\tau^1 = -k_1 \mu_1^1 - k_2 \mu_2^3$ and $\tau^2 = -k_2 \mu_2^3 - k_4 \mu_4^3$ with $k_1, k_2, k_3, k_4 > 0$ to stabilize the coordination set $C^*$. On $C^*$ we design $\tau^1$ to make unicycle 1, and hence the synchronized multi-agent system, move with constant velocity $v_d$. Let $\tau^1 = -k (\zeta^2 - v_d)$ with $k > 0$. Figure 2 shows the position of unicycles along their paths. Figure 3 shows that all $\eta_i^1$ for $i \in \{1,2,3\}$ have reached $v_d = 1$.

REFERENCES