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Shakespearian Modal Logic: A Labelled Treatment of Modal Identity

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To be or not to be: that is the question
(Hamlet,III,i,56)

ABSTRACT. In this paper we describe a modal proof system arising from the combination of a tableau-like classical system, which incorporates a restricted ("analytical") version of the cut rule, with a label formalism which allows for a speicialised, logic dependant unification algorithm. The system provides a uniform proof-theoretical treatment of first-order (normal) modal logics with identity, with and without Barcan formula and/or its converse

1 Introduction

Artosi Benassi Governatori and Rotolo (1996) present a proof method for normal systems of first-order modal logic based on KEM. In this paper we extend such a method to first-order modal logic with identity; in particular, the introduction of the identity symbol is relevant to the treatment of existential presuppositions (see Hintikka 1969): according to whether $\exists x(x = a_n)$ is true or not, we shall be able to establish whether the individual denoted by a_n exists in the domain of quantification. KEM (see Artosi and Governatori 1994, Governatori 1995) is a labelled analytic proof system based on a combination of tableau and natural deduction inference rules which allows for a suitably restricted ("analytic") application of the cut rule; the label scheme arises from an alphabet of constant and variable "world" symbols. A "world" label is a world-symbol or a "structured"

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sequence of world-symbols we call a "world-path". Constant and variable world-symbols denote worlds and sets of worlds respectively (in a Kripke model), while a world-path conveys information about access between the worlds in it. We attach labels to signed formulas (i.e., formulas prefixed with a "T" or "F") to yield *labelled signed formulas* (*LS-formulas*). A LS-formula TA, i (FA, i) means that A is true (false) at the (last) world (on the path) i. Individuals are attached to world symbols, according to whether they belong to the associated domain, in such a way as to keep trace of the existential (non-existential) status of terms. In the course of proof search, labels and domains are manipulated in a way closely related to the quantified modal semantics and "matched" using (specialised, logic-dependent) unification algorithms. That two structured labels i and k are unifiable means that they virtually represent the same path, i.e., any world which you could get to by the path i could be reached by the path k and vice versa.

As we shall see, our label formalism allows us to treat each of the variants of the first-order modal logics $K, D, T, S4, B, S5$ with varying domains in a broader sense, by using the quantifier rules of the tableau method in combination with corresponding appropriate versions of the modal rules. In what follows we assume a Modal First-Order Language with identity L defined in the usual way. Let $\mathfrak{C} = \{a_1, a_2, \ldots\}$ and $\mathfrak{V} = \{x_1, x_2, \ldots\}$ be the sets of individual symbols (resp. constants and variables of L), and $\mathfrak{P} = \{P_1, P_2, \ldots\}$ is the set of predicates. We shall use $t, t', \ldots$ to denote arbitrary terms, and $A, B, C, \ldots$ to denote arbitrary formulas of L.

The systems we shall consider are extensions of that given by Kripke (1963) with constants and the usual axioms of modal identity, namely:

- $t = t$
- $(t = s) \rightarrow (A(t) \rightarrow A(s))$
- $(t \neq s) \rightarrow \Box(t \neq s)$

and possibly by either of (or both) the following formulas (Barcan Formula and its Converse):

BF $\forall x \Box A(x) \rightarrow \Box \forall x A(x)$

CBF $\Box \forall x A(x) \rightarrow \forall x \Box A(x)$

All the systems of Quantified Modal Logic (QML) we shall be concerned with include *modus ponens*, *necessitation*, and *universal generalization.*

By a *First-Order Kripke Model* $\mathcal{M}$ we shall mean a 5-tuple

$$\langle \mathcal{W}, R, \mathcal{D}, e, v \rangle$$

where $\mathcal{W}$, R, $\mathcal{D}$ are as usual, e is a mapping $e : \mathcal{W} \rightarrow \wp(\mathcal{D})$ which assigns to each possible world a domain of individuals, and v is the usual evaluation function such that 1) for any $a_n \in \mathfrak{C}$, any $x \in \mathfrak{V}$ and any $w_i, w_j \in \mathcal{W}$,

$v(a_n, w_i) = v(a_n, w_j)$, $v(x, w_i) = v(x, w_j)$ (rigidity condition[1]); 2) for any n-ary predicate $P_m \in \mathfrak{P}$ and any $w_i \in \mathcal{W}$ $v(P_m, w_i) \subseteq (e(w_i))^n$; 3) "=" and open formulas are evaluated globally; 4) closed formulas are interpreted with respect to the domain of the world where they are evaluated (see Kripke 1963); finally connectives and operators are as usual.

2 KEM Language and Label Formalism

As usual with refutation methods, a KEM-proof of a formula A consists of successful attempt to construct a countermodel for A by assuming that A is false in some arbitrary model $\mathcal{M}$.

The set $\Im$ of labels arises from two (non empty) sets $\Phi_C = \{w_1, w_2, \ldots\}$ (the set of *constant world symbols*), and $\Phi_V = \{W_1, W_2, \ldots\}$ (the set of *variable world symbols*) through the following definition:

$$
\begin{aligned}
\Im &= \bigcup_{1 \leq i} \Im_i \text{ where } \Im_i \text{ is :} \\
&\Im_1 = \Phi_C \cup \Phi_V; \\
&\Im_2 = \Im_1 \times \Phi_C; \\
&\Im_{n+1} = \Im_1 \times \Im_n \ (n > 1).
\end{aligned}
$$

That is, a world-label is either (i) an element of the set Φ_C, or (ii) an element of the set Φ_V, or (iii) a path term (k', k) where (iiia) $k' \in \Phi_C \cup \Phi_V$ and (iiib) $k \in \Phi_C$ or $k = (i', i)$ where (i', i) is a label. From now on we shall use $i, j, k, \ldots$ to denote arbitrary labels.

For any label $i = (k', k)$ we shall call k' the *head* of i, k the *body* of i, and denote them by $h(i)$ and $b(i)$ respectively. Notice that these notions are recursive (they correspond to projection functions): if $b(i)$ denotes the body of i, then $b(b(i))$ will denote the body of $b(i)$, $b(b(b(i)))$ will denote the body of $b(b(i))$; and so on. We call each of $b(i)$, $b(b(i))$, etc., a *segment* of i. Let $s(i)$ denote any segment of i (obviously, by definition every segment $s(i)$ of a label i is a label); then $h(s(i))$ will denote the head of $s(i)$. We shall call a label i *restricted* if $h(i) \in \Phi_C$, otherwise *unrestricted*.

For any label i, we define the length of i, $l(i)$, as the number of world-symbols in i, i.e., $l(i) = n \Leftrightarrow i \in \Im_n$. $s^n(i)$ will denote the segment of i of length n, i.e., $s^n(i) = s(i)$ such that $l(s(i)) = n$. We shall use $h^n(i)$ and i^n indifferently as abbreviations for $h(s^n(i))$.

For any label $i, l(i) > n$, we define the *countersegment-n* of i, as follows:

$$c^n(i) = h(i) \times (\cdots \times (h^k(i) \times (\cdots \times (h^{n+1}(i), w_0))))(n < k < l(i))$$

where w_0 is a dummy label, i.e., a label not appearing in i (the context in which such a notion is applied will tell us what w_0 stands for).

Example 2.1 If $i = (w_4, (W_3, (w_3, (W_2, w_1))))$, then $l(i) = 5$, $s^3(i) =$

[1] For a discussion about rigidity and identity see Kripke 1971.

$(w_3, (W_2, w_1)$, and its countersegment-3 is $c^3(i) = (w_4, (W_3, w_0))$; intuitively $c^n(i)$, is what remains of i after deleting $s^n(i)$.

We stipulate that if $i \in \Im_1$ and $a_1, \ldots, a_n \in \mathfrak{C}$, then $i[\![a_1, \ldots, a_n]\!]$, $i[a_1, \ldots, a_n] \in \Im_1$ and, we shall use $i[\![a_1, \ldots, a_n]\!][a'_1, \ldots, a'_m]$ to denote

$$(h(i)[\![a_1, \ldots, a_n]\!][a_m, \ldots, a_p], b(i)[\![a'_1, \ldots, a'_r]\!][a''_1, \ldots, a''_s]) \ .$$

As an intuitive explanation, we may think of a label $i \in \Phi_C$ as denoting a world (a *given* one), and a label $i \in \Phi_V$ as denoting a set of worlds (*any* world) in some Kripke model. A label $i = (k', k)$ may be viewed as representing a path from k to a (set of) world(s) k' accessible from k (or, equivalently, the world(s) denoted by k). A constant a_n occurring in a label may be thought of as an element of the universe for which we have enough information to decide whether it belongs to the domain of the world(s) in the path represented by the label(s) it is attached to; given a label $i[\![a_1, \ldots, a_m]\!][a'_1, \ldots, a'_q]$, if a_n is in $[\![a_1, \ldots, a_m]\!]$ then a_n denotes an element of the domain of i; otherwise (a_n is in $[a'_1, \ldots, a'_q]$) a_n denotes an element of the complement.

Example 2.2 The label $(W_1, w_1[\![a_1]\!])$ represents a path which takes us to the set W_1 of worlds accessible from w_1, and a_1 denotes an element of the domain of w_1; $(w_2[a_1], (W_1[\![a_2, a_3]\!], w_1))$ represents a path which takes us to a world w_2 accessible via any world accessible from w_1 (i.e., accessible from the sub-path $(W_1[\![a_2, a_3]\!], w_1)$), where the domain of w_2 does not contain the element denoted by a_1, and a_2, a_3 stand for individuals "shared" by the worlds denoted by W_1 (notice that labels are read from right to left).

3 Unifications

Labels and terms are manipulated according to rules (label-, and term-*unifications*) which simulate respectively the accessibility and domain conditions for the given L. The KEM label unification scheme involves two kinds of unifications, "high" and "low" unifications respectively. A high unification, σ^L, is meant to mirror a single constraint on R, and a low unification, σ_L, is used to simulate the full range of conditions governing the accessibility relation which characterises L.

3.1 Label Unifications

High unifications: We define a substitution in the usual way as a mapping

$$\sigma \quad : \quad \Phi_V \longrightarrow \Im$$

For two labels i and k, and a substitution σ, if σ is a unifier of i and k then we shall say that i, k are σ-*unifiable.* We shall (somewhat unconventionally) use $(i, k)\sigma$ to denote both that i and k are σ-unifiable and the result of their unification. On this basis we define several specialised, logic-dependent notions of σ-unification. In particular, the notion of two labels i, k being σ^K-, σ^D-, σ^T-, σ^4-, σ^B-unifiable is defined as follows:

$$(i,k)\sigma^K = (i,k)\sigma \quad \begin{array}{l}\text{at least one of } i \text{ and } k \text{ is restricted, and}\\ \forall n \leq l(i), (s^n(i), s^n(k))\sigma^K\end{array}$$

$$(i,k)\sigma^D = (i,k)\sigma$$

$$(i,k)\sigma^T = \begin{cases} (s^{l(k)}(i),k)\sigma & l(i) > l(k), \text{ and} \\ & \forall n \geq l(k), (h^n(i), h(k))\sigma = (h(i), h(k))\sigma \\ (i, s^{l(i)}(k))\sigma & l(k) > l(i), \text{ and} \\ & \forall n \geq l(i), (h(i), h^n(k))\sigma = (h(i), h(k))\sigma \end{cases}$$

$$(i,k)\sigma^4 = \begin{cases} c^{l(i)}(k) & l(k) > l(i), h(i) \in \Phi_V \text{ and} \\ & w_0 = (i, s^{l(i)}(k))\sigma \\ c^{l(k)}(i) & l(i) > l(k), h(k) \in \Phi_V \text{ and} \\ & w_0 = (s^{l(k)}(i), k)\sigma \end{cases}$$

$$(i,k)\sigma^B = \begin{cases} (b(b(i)),k)\sigma & \text{if } h(i) \in \Phi_V \text{ and} \\ & (h(i), h(k))\sigma = (h(b(b(i))), h(k))\sigma \\ (i, b(b(k)))\sigma & \text{if } h(k) \in \Phi_V \text{ and} \\ & (h(i), h(k))\sigma = (h(i), h(b(b(k))))\sigma \end{cases}$$

For the notion of σ^T-unification, take for example $i = (w_3, (W_1, w_1))$ and $k = (w_3, (W_2, (w_2, w_1)))$. Here $(W_2, w_3)\sigma = (w_3, w_3)\sigma$. Then i and k σ^T-unify to $(w_3, (w_2, w_1))$. This intuitively means that the world w_3, accessible from a sub-path $s(k) = (W_2, (w_2, w_1))$, after the deletion of W_2 from k, is accessible from any path i which turns out to denote the same world(s) as $s(k)$; in fact the step from w_2 to W_2 is irrelevant because of the reflexivity relation of the model. For the notion of σ^4-unification, take for example $i = (W_3, (w_2, w_1))$ and $k = (w_5, (w_4, (w_3, (W_2, w_1))))$. Here $s^{l(i)}(k) = (w_3, (W_2, w_1))$. Then i and k σ^4-unify to $(w_5, (w_4, (w_3, (w_2, w_1))))$ since $(i, s^{l(i)}(k))\sigma = ((W_3, (w_2, w_1)), (w_3, (W_2, w_1)))\sigma$. This intuitively means that all the worlds accessible from a sub-path $s^{l(i)}(k)$ of k are accessible from any path i which leads to the same world(s) denoted by $s^{l(i)}(k)$.

Low Unification: We are now able to combine the above unifications in a single low unification for $L = K, D, T, S4, B, S5$.

$$(i,k)\sigma_L = \begin{cases} (c^n(i), c^m(k))\sigma^{L_1 \cdots L_n} \\ (i,k)\sigma^{L_1 \cdots L_n} \end{cases}$$

where $w_0 = (s^n(i), s^m(k))\sigma_L$ and

$$(i,k)\sigma^{L_1 \cdots L_n} = \begin{cases} (i,k)\sigma^{L_1} \\ \vdots \\ (i,k)\sigma^{L_n} \end{cases}$$

where $L_1 \cdots L_n$ stand for the axioms characterising L.

For $S5$ we provide the following specialised σ_{S5}-unification:

$$(i,k)\sigma_{S5} = \begin{cases} (h(i),h(k))\sigma & \min\{l(i),l(k)\} = 1 \\ ((h(i),h(k))\sigma,(s^1(i),s^1(k))\sigma) & \text{otherwise} \end{cases}$$

We shall say that i *extends* k iff there exists an $s(i)$ such that either (i) $s(i) = k$ or (ii) $(s(i),k)\sigma_L$; and that i *extends immediately* k iff i extends k and $s(i) = b(i)$. We now provide a useful property of labels and unifications.

Lemma 3.1 *If* $(i,k)\sigma_L = l$ *then* $(i,l)\sigma_L$ *and* $(l,k)\sigma_L$.

Proof. The proof will be by induction on the number of applications of $\sigma^{L_1\cdots L_n}$ in a σ_L-unification. Let n be the number of such applications. If $n = 1$ then we have to prove the property for $\sigma^{L_1\cdots L_n}$,[2] which means

(1) $$(i,k)\sigma^{L_1\cdots L_n} = l \Rightarrow (i,l)\sigma^{L_1\cdots L_n},(k,l)\sigma^{L_1\cdots L_n}$$

We then provide the definition of σ^{DT4}

$$(i,k)\sigma^{DT4} = \begin{cases} (i,k)\sigma^D & l(i) = l(k) \\ (i,k)\sigma^T & l(i) < l(k), h(i) \in \Phi_C \\ (i,k)\sigma^4 & l(i) < l(k), h(i) \in \Phi_V \end{cases}$$

At this point we prove the property stated in (1) by induction on the length of labels.

If $\min\{l(i),l(k)\} = 1$ then we assume that $l(i) = 1$ (the proof for $l(k) = 1$ is similar). 1) $i \in \Phi_C$. If also $l(k) = 1$, we apply σ^D; in every case, by obvious considerations about σ, $l = (i,k)\sigma^D = i$, but $(i,i)\sigma^D$ and $(i,k)\sigma^D$. If $l(k) > 1$ and $(i,k)\sigma^T$, then $l = (i,k)\sigma^T = (i,s^1(k))\sigma^T = i$, hence $(i,i)\sigma^D$ and $(i,k)\sigma^T$. If $l(k) > 1$ and $(i,k)\sigma^B$, then $l = (i,k)\sigma^B = (i,s^1(k))\sigma = i$, therefore $(i,i)\sigma^D$ and $(i,k)\sigma^B$. 2) $i \in \Phi_V$ then by the definition of σ it unifies with any label, in particular $(i,k)\sigma^D = k = l$, whence $(i,k)\sigma^D$ and $(k,k)\sigma^D$.

Let us suppose now that $\min\{l(i),l(k)\} = n > 1$, and that the property holds up to n for $\sigma^{L_1\cdots L_n}$. Thus we have the following cases.

$L_1\cdots L_n = D$ and $L_1\cdots L_n = K$. If $l(i) = l(k)$ then $(i,k)\sigma^D = l$; by the inductive hypothesis $(b(i),b(l))\sigma^D$, $(b(k),b(l))\sigma^D$, $(h(i),h(l))\sigma^D$ and $(h(k),h(l))\sigma^D$; therefore $(i,l)\sigma^D$ and $(k,l)\sigma^D$. The proof for K follows from the fact that l contains only constants, which implies that each single element of i and k is either a variable or the constant occurring in the corresponding place in l.

$L_1\cdots L_n = DT$. If $l(i) < l(k)$ and $(i,k)\sigma^T = l$, by the inductive hypothesis $(b(i),b(l))\sigma^D$, $(s^{l(b(i))}(k),b(l))\sigma^D$. By the definition of σ^T, we know that $l^n = (h(i),h(k))\sigma = (h(i),h^{l(i)}(k))\sigma$; therefore $(i,l)\sigma^D$ and $(k,l)\sigma^T$. The case $l(i) = l(k)$ is the same as the case for D above.

[2]Hereafter, in order to shorten proofs, when we have to consider labels of different lengths, we shall assume, unless specified, the first to be the shorter. Obviously proofs for the other cases can be carried out in the same way.

$L_1 \cdots L_n = DT4$. If $l(i) < l(k)$ and $h(i) \in \Phi_V$, then $(i,k)\sigma^4 = c^{l(i)}(k)$ where $w_0 = (i, s^{l(i)}(k))\sigma$. By the inductive hypothesis and the definition of σ we have $(i, s^{l(i)}(l))\sigma$ and $(s^{l(i)}(k), s^{l(i)}(l))\sigma$ and therefore $(i,l)\sigma^4$ and $(k,l)\sigma^D$. The other clauses of σ^{DT4} are respectively the cases for T and D above.

$L_1 \cdots L_n = DTB$. If $l(i) < l(k)$ and $(i,k)\sigma^B = l$, by inductive hypothesis $(b(i), b(l))\sigma^D$, $(s^{l(b(i))}(k), b(l))\sigma^D$; by the definition of σ^B, we know that $l^n = (h(i), h(k))\sigma = (h(i), h(b(b(i))))\sigma)$; therefore $(i,l)\sigma^D$ and $(k,l)\sigma^B$. The other cases of the σ^{DTB} are respectively the cases for T and D above.

We have thus proved the inductive base for the lemma. We can now assume that the lemma holds up to the n-th application of $\sigma^{L_1 \cdots L_n}$. By the definition of σ_L, $(s^n(i), s^m(k))\sigma_L = w_0 = s^l(l)$ and $(c^n(i), c^m(k)j)\sigma^{L_1 \cdots L_n} = c^l(l)$, but, by the inductive hypothesis, $(s^n(i), s^l(l))\sigma_L$ and $(s^m(k), s^l(l))\sigma_L$. By the property we have just proved for $\sigma^{L_1 \cdots L_n}$ $(c^n(i), c^l(l))\sigma^{L_1 \cdots L_n}$ and $(c^m(k), c^l(l))\sigma^{L_1 \cdots L_n}$, which implies $(i,l)\sigma_L$ and $(k,l)\sigma_L$.

For $S5$ if $\min\{l(i), l(k)\} = 1$, we have $(i,k)\sigma_{S5}$ iff $(h(i), h(k))\sigma$, whence, if i is restricted, then $(i,k)\sigma_{S5} = h(i) = l$ and thus $(i,l)\sigma_{S5}$, i.e., $(h(i), h(i))\sigma$, and similarly for k; otherwise $(i,k)\sigma_{S5} = h(k) = l$, therefore for the same reason as in the previous case $(k,l)\sigma_{S5}$ and $(i,l)\sigma_{S5}$. If $\min\{l(i), l(k)\} > 1$ we can repeat the same argument of the other case with the difference that $l = (h(i), h(k)\sigma, (s^1(i), s^1(k))\sigma)$. □

3.2 Term Unifications

In proving formulas of L we use terms associated to world domains and complements. Therefore we need a tool which tells us when two such symbols denote the same individual(s) relative to a given world.

Given a set of labels $\mathcal{L}$ the ρ-unification is just the usual unification with the constraint that an indexed variable $(x_n)_i$, $i \in \Im$, ρ-unifies with a term t either if such a term is a constant a_m attached to a label $k \in \mathcal{L}$ (i.e., $k[\![a_m]\!]$), such that $(i,k)\sigma_L$, or t is a variable; two indexed variables, for example $(x_m)_i$ and $(x_m)_k$, ρ-unify iff $(i,k)\sigma_L$-unify. Formally:

$$((t)_i, (t')_k)\rho = \begin{cases} t & \text{if } t = t' \\ (t)_i & \text{if } i = k \\ (t)_{(i,k)\sigma_L} & \text{if } (i,k)\sigma_L \\ t & \text{if } t \in D(k\sigma_L^{\mathcal{L}}) \\ t' & \text{if } t' \in D(i\sigma_L^{\mathcal{L}}) \end{cases}$$

where $D(i\sigma_L^{\mathcal{L}})$ is the set of constants extracted from the $[\![\,]\!]$ part of the head of i and from the $[\![\,]\!]$ part of the head of the labels in $\mathcal{L}$ unifying with i.

4 Inference Rules

In displaying the rules of KEM we shall use Smullyan-Fitting (Fitting 1983) $\alpha, \beta, \gamma, \delta, \nu, \pi$ unifying notation that classifies signed formulas as shown in the following tables.

α	α_1	α_2	β	β_1	β_2
$TA \wedge B$	TA	TB	$FA \wedge B$	FA	FB
$FA \vee B$	FA	FB	$TA \vee B$	TA	TB
$FA \rightarrow B$	TA	FB	$TA \rightarrow B$	FA	TB
$T\neg A$	FA	FA	$F\neg A$	TA	TA

TABLE 1 Classification of α- and β-formulas

ν	ν_0	π	π_0
$T\Box A$	TA	$T\Diamond A$	TA
$F\Diamond A$	FA	$F\Box A$	FA

TABLE 2 Classification of ν- and π-formulas

γ	γ_0	δ	δ_0
$T\forall xA$	$TA(t)$	$T\exists xA$	$TA(t)$
$F\exists xA$	$FA(t)$	$F\forall xA$	$FA(t)$

TABLE 3 Classification of γ- and δ-formulas

X^C denotes the *conjugate* of X, i.e., the result of changing the sign of X to its opposite; two LS-formulas $X(t), i$ and $X^C(t'), k$ such that $(i,k)\sigma_L$ and $(t,t')\rho$ will be called $\sigma_L\rho$-*complementary*. We shall write a β-formula also as $[\beta_1, \beta_2]$.

Propositional Rules

α-rules

$$\frac{\alpha, k}{\alpha_n, k}[n = 1, 2]$$

β-rules

$$\frac{\begin{array}{c}[\beta_1(t^1), \beta_2(t^2)], k\\ \beta_n^C(t), l\end{array}}{\beta_{3-n}(t^{3-n})(k,l)\sigma_L}[(k,l)\sigma_L \text{ and } (t^n, t)\rho]$$

The α rules are just the familiar linear branch-expansion rules of the tableau method. In place of the usual tableau branching β rules we have a set of linear 2-premise β rules which represent well-known natural inference principles (such as disjunctive syllogism and its dual, modus ponens, and modus tollens).

Quantifier Rules

γ-rules

$$\frac{\gamma, i[\![a_1, \ldots, a_n]\!]}{\gamma_0(x_m)_i, i[\![a_1, \ldots, a_n]\!]}[x_m \text{ new}]$$

δ-rules

$$\frac{\delta, i[\![a_1, \ldots, a_n]\!]}{\delta_0(a_m), i[\![a_1, \ldots, a_n, a_m]\!]}[h(i) \in \Phi_C, a_m \text{ new}]$$

$$\frac{\delta, i[\![a_1, \ldots, a_n]\!]}{\delta_0(a_m), i[\![a_1, \ldots, a_n]\!]}[h(i) \in \Phi_V, a_m \text{ new}]$$

The γ rules are the usual "universal" rules of tableaux method but the variable is indexed by the current label; this means that the range of quantification is the domain of the world(s) denoted by i.

The δ rules are the usual "existential" rules of the tableau method modified in such a way as to add the instantiation of the quantified variable to the "domain" of the current label, i.e., the constant names an individual in the domain of world denoted by i; notice that constants cannot be attached to unrestricted labels according to their intuitive interpretation: the individual rigidly designated by the constant might not be in all the worlds named by the label.

Modal Rules
ν-rules

$$\frac{\nu, i}{\nu_0, (i', i)}[i' \text{ unrestricted and new}]$$

π-rules

$$\frac{\pi, i}{\pi_0, (i', i)}[i' \text{ restricted and new}]$$

The ν and π rules are as usual where "new" in the proviso means that the label does not occur previously in the tree.

Domains Rules
Atomic domain rule

$$\frac{TPa'_1, \ldots, a'_m, i[\![a_1, \ldots, a_n]\!]}{TPa'_1, \ldots, a'_m, i[\![a_1, \ldots, a_n, a'_1, \ldots, a'_m]\!]}$$

This rule corresponds to the basic semantic condition concerning atomic formulas: a predicate is satisfied in a world only by n-tuples of individuals existing in the domain; therefore, the constants are denoting in the domain associated to the world whence we take care of it discharging the constants in the label. On the other hand, a false atomic formula does not entail the existence of individuals in the domain. According to the semantic interpretation of "=" we do not discharge the constants occurring in such a predicate in the label.

Binding rules

$$\frac{TX(x_n), i}{TX(x_n)_{W_m}, i}[W_m \text{ new}] \qquad \frac{FX(x_n), i}{FX(a_m), i}[a_m \text{ new}]$$

These rules correspond to the basic assumptions concerning the semantics of open formulas; notice that $(x_n)_{W_m}$ ranges over the whole D. In this way we *bind* open formulas.

Domains rule

$$\frac{\begin{array}{c} i[\![a_1, \ldots, a_n]\!][a_q, \ldots, a_r] \\ k[\![a'_1, \ldots, a'_m]\!][a'_s, \ldots, a'_t] \end{array}}{(i,k)\sigma_L[\![a_1, \ldots, a_n, a'_1, \ldots, a'_m]\!][a_q, \ldots, a_r, a'_s, \ldots, a'_t]}$$

Increasing domains rule

$$\frac{\begin{array}{c} i[\![a_1, \ldots, a_n]\!][a_q, \ldots, a_r] \\ k[\![a'_1, \ldots, a'_m]\!][a'_s, \ldots, a'_t] \end{array}}{\begin{array}{c} i[\![a_1, \ldots, a_n]\!][a_q, \ldots, a_r, a'_s, \ldots, a'_t] \\ k[\![a_1, \ldots, a_n, a'_1, \ldots, a'_m]\!][a'_s, \ldots, a'_t] \end{array}}[k \text{ extends } i]$$

Decreasing domains rule

$$\frac{\begin{array}{c} i[\![a_1, \ldots, a_n]\!][a_q, \ldots, a_r] \\ k[\![a'_1, \ldots, a'_m]\!][a'_s, \ldots, a'_t] \end{array}}{\begin{array}{c} i[\![a_1, \ldots, a_n, a'_1, \ldots, a'_m]\!][a_q, \ldots, a_r] \\ k[\![a'_1, \ldots, a'_m]\!][a_q, \ldots, a_r, a'_s, \ldots, a'_t] \end{array}}[k \text{ extends } i]$$

Symmetric domains rules

$$\frac{\begin{array}{c} i[\![a_1, \ldots, a_n]\!][a_q, \ldots, a_r] \\ k[\![a'_1, \ldots, a'_m]\!][a'_s, \ldots, a'_t] \end{array}}{\begin{array}{c} i[\![a_1, \ldots, a_n, a'_1, \ldots, a'_m]\!][a_q, \ldots, a_r, a'_s, \ldots, a'_t] \\ k[\![a_1, \ldots, a_n, a'_1, \ldots, a'_m]\!][a_q, \ldots, a_r, a'_s, \ldots, a'_t] \end{array}}[i \text{ immediately extends } k]$$

The domains rules remind of Gabbay's (1994) visa rules and allow us to "move" individuals through worlds according to the domains conditions. Symmetric domains rule holds only for increasing and decreasing domains.

Identity rules

Complement rule

$$\frac{T(x)_i \neq a_m, k}{i[a_1, \ldots, a_n, a_m]}$$

Indexed variables range over the domain of the attached world(s), but the formula states that no value of $(x)_i$ is a_m, so a_m is not in the domain whence it is discharged in the label as an element of the complement.

Identity PNC

$$\frac{Fa_n = a_n, i}{\times}(i, i)\sigma_L$$

This rule states, intuitively, that no element is different from itself.

Substitution rules
Identity substitution

$$\frac{\begin{array}{c}Tt = t', i\\ Xt, k\end{array}}{Xt', k}[i\sigma_L^{\mathcal{L}} \neq \emptyset]$$

Where $i\sigma_L^{\mathcal{L}} = \{j \in \mathcal{L} : (i, j)\sigma_L\}$. Identity substitution corresponds to Leibniz's principle.

Domain substitution

$$\frac{X(x)_i, k}{X(x)_i/a_n, k}[a_n \in D(i\sigma_L^{\mathcal{L}})]$$

Domain substitution states that a variable indexed with a label i may be substituted by any constant whose denotation is in the domain of the world denoted by the label.

Structural Rules
PNC

$$\frac{\begin{array}{c}X(t), i\\ X^C(t'), k\end{array}}{\times, (i, k)\sigma_L}[(i, k)\sigma_L \text{ and } (t, t')\rho]$$

PNC (for Principle of Non-Contradiction) corresponds to the familiar branch-closure rule of the tableau method, saying that from the occurrence of a pair of $\sigma_L\rho$-complementary LS-formulas $X(t), i$ and $X^C(t'), k$ we may infer the closure ("×") of the branch. The $(i, k)\sigma_L$ in the "conclusion" of PNC means that the contradiction holds "in the same world" for the same individual.

Terms PNC

$$\frac{i[\![a_1,\ldots,a_n,a]\!][a'_1,\ldots,a'_m,a]}{\times}(i,i)\sigma_L$$

Besides the usual closure rules, PNC and Identity PNC we use the above rule stating that no individual can be at the same time in the domain and in the complement of a given (existing) world.

PB

$$\frac{}{X,i \qquad X^C,i}[i \text{ restricted}]$$

When we split with respect to $X(x_n)_k$, after the application of PB, x_n should be instantiated, in both branches, to the same constant. PB cannot be applied to formulas containing variables not bounded by a label.

PB (for Principle of Bivalence) is a 0-premise branching rule which plays the role of the cut rule of the sequent calculus (intuitive meaning: a formula A is either true or false in any *given* world, whence the requirement that i should be restricted). We restrict ourselves to an analytic version of PB as exposed by D'Agostino and Mondadori (1994), Artosi and Governatori (1994) and Governatori (1996). In particular, PB will be applied to subformulas of unanalysed β formulas already occurring in the branch; informally an *LS*-formula is unanalysed in a branch if its appropriate components do not occur in it. As it is well known (Smullyan 1968), what destroys analyticity is losing the (weak) subformula property (Fitting1990), and not having a cut rule restricted to subformulas. Otherwise each tableau system would not be analytic, since from the formula $\neg(A \to B)$ we obtain two nodes containing respectively $\neg A$ and B, but, obviously, $\neg A$ is not an immediate (strong) subformula of $\neg(A \to B)$. Nevertheless, a clever and ruled use of the cut could reduce sharply the complexity of the proof (Boolos 1984, D'Agostino and Mondadori 1994). Moreover it can be used to check closure in a modal setting. In fact we might define a procedure called "modal PB" which is applied when two complementary, but not σ_L-complementary, formulas occur in a branch to verify whether a restricted label unifying with both the labels of the complementary formulas occurs previously in the branch; if such a label exists, or can be built using already existing labels and the unification, then the branch closes (Governatori 1996).

5 Examples

1.	$F\forall x\Diamond\exists y\Box((\neg A \wedge (x \neq y)) \to \Box\Diamond\neg(A \to Px))$	w_1
2.	$F\Diamond\exists y\Box((A \wedge (a_1 \neq y)) \to \Box\Diamond\neg(A \to Pa_1))$	w_1
3.	$F\exists y\Box((A \wedge (a_1 \neq y)) \to \Box\Diamond\neg(A \to Pa_1))$	(W_1, w_1)
4.	$F\Box((A \wedge (a_1 \neq y_{(W_1,w_1)})) \to \Box\Diamond\neg(A \to Pa_1))$	(W_1, w_1)
5.	$F((A \wedge (a_1 \neq y_{(W_1,w_1)})) \to \Box\Diamond\neg(A \to Pa_1))$	$(w_2, (W_1, w_1))$
6.	$T(A \wedge (a_1 \neq y_{(W_1,w_1)}))$	$(w_2, (W_1, w_1))$
7.	$F\Box\Diamond\neg(A \to Pa_1))$	$(w_2, (W_1, w_1))$
8.	TA	$(w_2, (W_1, w_1))$
9.	$Ta_1 \neq y_{(W_1,w_1)}$	$(w_2[a_1], (W_1, w_1))$
10.	$F\Diamond\neg(Pa_1 \to A))$	$(w_3, (w_2[a_1], (W_1, w_1)))$
11.	$TA \to Pa_1$	$(W_2, (w_3, (w_2[a_1], (W_1, w_1))))$
12.	TPa_1	$(w_2[a_1], (w_3, (w_2[a_1], (W_1, w_1))))$
13.	TPa_1	$(w_2[\![a_1]\!][a_1], (w_3, (w_2[a_1], (W_1, w_1))))$
14.	$\times$	

The steps from 1 to 8 are straightforward; step 9 is obtained from the formula in 6 by an application of an α-rule and the complement rule to the resulting formula discharging then the constant in the label; steps 10 and 11 are obtained by the usual π- and ν-rules. At this point we may apply a β-rule and the Domain rule on the LS-formulas in 8 and 11 (notice that their labels σ_{S4}-unify) from which 12 follows; finally, by an application of the Atomic domain rule on TPa_1 we infer that a_1 belongs to the domain of the label, obtaining a contradiction (terms PNC).

1.	$F\Box\forall x(x \neq a_1) \to \forall x\Box(x \neq a_1)$	w_1
2.	$T\Box\forall x(x \neq a_1)$	w_1
3.	$F\forall x\Box(x \neq a_1)$	w_1
4.	$T\forall x(x \neq a_1)$	(W_1, w_1)
5.	$F\Box(a_2 \neq a_1)$	$w_1[\![a_2]\!]$
6.	$T(x_{(W_1,w_1)} \neq a_1)$	(W_1, w_1)
7.	$F(a_2 \neq a_1)$	$(w_2, w_1[\![a_2]\!])$
8.	$T(x_{(W_1,w_1)} \neq a_1)$	$(W_1[a_1], w_1[\![a_2]\!])$
9.	$T(x_{(W_1,w_1)} \neq a_1)$	$(W_1[\![a_2]\!][a_1], w_1[\![a_2]\!])$
10.	$\times$	

Steps 1–4 are straightforward; 5 is derived from 3 by an application of a δ-rule, discharging the new constant in the domain; step 8 is obtained by the complement rule on 6; at step 9, the increasing domain rule allows us to move a_2 from w_1 to (W_1, w_1); at this point the LS-formulas in 7 and 9 are $\sigma_L\rho$-complementary since $((W_1, w_1), (w_2, w_1))\sigma_L$ and $(x_{(W_1,w_1)}, a_2)\rho$ since $a_2 \in D((W_1, w_1)\sigma_L^{\mathcal{L}})$, where $\mathcal{L}$ is the set of labels occurring in the tree.

1.	$F\forall x(Qa_1 \rightarrow \Box(Px \rightarrow \exists y(a_1 = y)))$	w_1
2.	$F(Qa_1 \rightarrow \Box(Px_{w_1} \rightarrow \exists y(a_1 = y_))$	w_1
3.	TQa_1	$w_1[\![a_1]\!]$
4.	$F\Box(Px_{w_1} \rightarrow \exists y(a_1 = y))$	$w_1[\![a_1]\!]$
5.	$FPx_{w_1} \rightarrow \exists y(a_1 = y)$	$(w_2, w_1[\![a_1]\!])$
6.	TPx_{w_1}	$(w_2, w_1[\![a_1]\!])$
7.	$F\exists y(a_1 = y)$	$(w_2, w_1[\![a_1]\!])$
8.	$Fa_1 = y_{(w_2,w_1)}$	$(w_2, w_1[\![a_1]\!])$
9.	$Fa_1 = y_{(w_2,w_1)}$	$(w_2[a_1], w_1[\![a_1]\!])$
10.	TPa_1	$(w_2[a_1], w_1[\![a_1]\!])$
11.	TPa_1	$(w_2[\![a_1]\!][a_1], w_1[\![a_1]\!])$
12.	$\times$	

In this proof at step 3 the atomic domain rule discharges a_1 in the label, whereas at step 9 the complement rule ensures that a_1 is not in the domain of (w_2, w_1), but x_{w_1} can be substituted by any constant in the domain of w_1 $(D(w_1)\sigma_L^{\mathcal{L}})$, therefore we replace it with a_1 thus obtaining 10, from which 11 follows by an application of the atomic domain rule. The LS-formula in 11 states that a_1 belongs at the same time both to the domain of (w_2, w_1) and to its complement, whence terms PNC closes the tree.

6 Soundness and Completeness

In this section we sketch a proof of the soundness and completeness results in the same way as that given by Artosi Benassi Governatori and Rotolo (1996). Let $\mathcal{M} = \langle \mathcal{W}, R, \mathcal{D}, e, v\rangle$ be an L-model where: $\mathcal{W} = \Phi_C$; R is a binary relation on $\mathcal{W}$; $\mathcal{D}$ is a non empty set of individuals; e and v are as before. In what follows we define some functions translating labels and terms into elements of $\mathcal{M}$ according to the structure of the labels.

Let g be a function from $\Im$ to $\wp(\mathcal{W})$ thus defined:

$$g(i) = \begin{cases} h(i) = \{h(i)\} & \text{if } h(i) \in \Phi_C \\ h(i) = \{w_i \in \mathcal{W} : g(b(i))Rw_i\} & \text{if } h(i) \in \Phi_V \\ i = \mathcal{W} & \text{if } i \in \Phi_V \end{cases}$$

Let ι be a function of value assignment from $\mathfrak{C} \cup \mathfrak{V}$ to $\wp(D)$ thus defined:

$$\iota(t) = \begin{cases} e(g(i)) \in \wp(D) & \text{if } t = (x_n)_i \in \mathfrak{V} \\ \{d_n\} \in \wp(D) & \text{otherwise} \end{cases}$$

intuitively, ι assigns locally to a variable $(x_n)_i$ the set of individuals associated to the world denoted by the label i by means of wich x is indexed. If i is restricted $\iota((x_n)_i) = e(g(i))$, otherwise $\iota((x_n)_i) = \cap e(g(i))$.

Let r be a function from $\Im$ to R thus defined:

$$r(i) = \begin{cases} \emptyset & \text{if } l(i) = 1 \\ g(i^1)Rg(i^2), \ldots, g(i^{n-1})Rg(h(i)) & \text{if } l(i) = n > 1 \end{cases}$$

Let f be a function from LS-formulas to v thus defined:

$$f(SA, i) =_{def} v(A, w_j) = S$$

for all $w_j \in g(i)$.

Lemma 6.1 *For any $i, k \in \Im$ if $(i, k)\sigma_L$ then $g(i) \cap g(k) \neq \emptyset$.*

Proof. The proof is by induction on the number of applications of $\sigma^{L_1 \cdots L_n}$ in σ_L. We need first to prove the following:

Lemma 6.2 *For any $i, k \in \Im$ if $(i, k)\sigma^L$ then $g(i) \cap g(k) \neq \emptyset$.*

Proof. The proof is by induction on the length of labels. If $\min\{l(i), l(k)\} = 1$, then at least one of i and k is either a constant or a variable, so that five cases will be present. By the definition of unifications i, k are either: i) two constants, or ii) a variable and a constant, or iii) two variables, or iv) a variable and a label, or v) a constant and a label.[3]

Case i) Two constants unify if and only if they are the same constant, and so $i = k$; therefore from the definition of g, $g(i) = g(k)$ and so $g(i) \cap g(k) \neq \emptyset$.

Case ii) If i (resp. k) is a variable and k (resp. i) is a constant, then $g(i) = \mathcal{W}$ and $g(k) \in \wp(\mathcal{W})$ therefore also in this case $g(i) \cap g(k) \neq \emptyset$.

Case iii) and iv) These cases are identical to the previous ones because: 1) $\mathcal{W}$ is not empty, and 2) the variable is mapped to $\mathcal{W}$ and the label to some world(s) in it.

Case v) This case implies that $(i, k)\sigma^T$ or $(i, k)\sigma^B$. Let us assume, for the sake of economy, that $l(i) = 1$ and $l(k) = n > 1$. If $(i, k)\sigma^T$, then for each $h(s(k))$ such that $l(s(k)) > 1$ either $h(s(k)) \in \Phi_V$, or $h(s(k)) = i$; therefore $r(k) = iRk^2, \ldots, k^{n-1}Rk^n$. If $k^2 \in \Phi_V$, then k^2 denotes the set of worlds accessible from i; if $k^2 \in \Phi_C$, then $i = k$, but, through reflexivity $i \subseteq k^2$, so we take i as a representative of the set denoted by k^2, which implies iRk^3. We repeat the same argument until we arrive at iRk^n: if $k^n \in \Phi_C$, then $i = k^n$ and so they denote the same world; if $k^n \in \Phi_V$, then it denotes the set of worlds accessible from i; but i belongs to such a set, therefore, in all cases $g(i) \cap g(k) \neq \emptyset$. If $(i, k)\sigma^B$, then $h(k) \in \Phi_V$, $(i, h(k))\sigma$ and $(i, b(b(k)))\sigma$; moreover $r(k) = k^1Rk^2, k^2Rk^3$, but $k^1 = i$, and, by symmetry k^2Rk^1, which implies $k^1 \cap k^3 \neq \emptyset$, therefore $g(i) \cap g(k) \neq \emptyset$.

For the inductive step we have $\min\{l(i), l(k)\} = n > 1$. Let us assume inductively that the lemma is valid up to n; if $l(i) = l(k)$ we shall write i and k as $(h(i), b(i))$ and $(h(k), b(k))$, respectively. If $(i, k)\sigma^D$, by the definition of σ^D we get $(b(i), b(k))\sigma^D$, for which the lemma holds; let w_j be one of the worlds shared by $b(i)$ and $b(k)$, whence $w_jRh(i)$ and $w_jRh(k)$. We have now only to analyse what kind of labels are $h(i)$ and $h(k)$, which

[3]Cases ii), iii), and iv) are not found in KEM proofs, but they are useful both for dealing with cases in the inductive step and for case v).

falls under the cases i), ii), and iii). Cases i) and ii) are the same as the inductive base. We have thus to examine case iii). Both $h(i)$ and $h(k)$ denotes the set of worlds accessible from w_j, but such a set is not empty because of the seriality of R. If $(i,k)\sigma^K$ we repeat the argument for D apart from cases iii), iv), and v) of the base which are not allowed in σ^K.

If $l(i) \neq l(k)$, we shall assume that $l(i) < l(k)$ (the case $l(k) < l(i)$ is dealt with in the same way). If $(i,k)\sigma^T$ and $h(i) \in \Phi_C$ then $(i, s^{l(i)}(k))\sigma^D$, therefore, combining the proofs of the previous case and case v) of the inductive base we obtain the desired result. If $h(i) \in \Phi_V$, then for all $k^n, n \leq l(i)$, $(h(i), h(k))\sigma = (h(i), k^n)\sigma$ which means $g(i) \cap g(s^n(k)) \neq \emptyset$, and in particular $g(i) \cap g(s^{l(i)}(k)) \neq \emptyset$.

If $(i,k)\sigma^4$ then $h(i) \in \Phi_V$ and $(b(i), s^{l(i)-1}(k))\sigma^D$, for which the inductive hypothesis holds; let w_j be such a shared world. $h(i)$ denotes all the worlds accessible from w_j, but, due to transitivity, the world(s) denoted by $h(k)$ belong(s) to $h(i)$ and so $g(i) \cap g(k) \neq \emptyset$.

If $(i,k)\sigma^B$ and $l(i) \leq l(k)$ then $h(k) \in \Phi_V$ and $(i, b(b(k)))\sigma$, for which the inductive hypothesis holds; let w_j be such a shared world. By repeating the same argument as for case v) of the base for B we get $g(i) \cap g(k) \neq \emptyset$. □

We return to the proof of the main lemma. If σ_L consists of a single step of $\sigma^{L_1 \cdots L_n}$, then $(i,k)\sigma_L = (i,k)\sigma^{L_1 \cdots L_n}$; by Lemma 6.2 we obtain $g(i) \cap g(k) \neq \emptyset$.

Let us assume, inductively, that the lemma holds up to n. If σ_L consists of $n+1$ $\sigma^{L_1 \cdots L_n}$-unifications, $(i,k)\sigma_L = (c^i(i), c^k(k))\sigma^{L_1 \cdots L_n}$ where $(s^i(i), s^k(k))\sigma_L$, which contains n applications of $\sigma^{L_1 \cdots L_n}$, and the lemma holds for it. We can now repeat the argument of Lemma 6.2 with respect to $(c^i(i), c^k(k))\sigma^{L_1 \cdots L_n}$, proving thus that $g(i) \cap g(k) \neq \emptyset$. □

Lemma 6.3 *For any t, t', if $(t,t')\rho$ then $\iota(t) \cap \iota(t') \neq \emptyset$.*

Proof. The proof is obtained by assuming the non-emptyness of the domain of the worlds, by using as a preliminary condition the Lemma 6.1 and finally by checking the cases: i) both t, and t' are constants, ii) t is a constant (variable) and t' is a variable (constant) iii) both t, and t' are variables. □

Lemma 6.4 *For any $i,k \in \Im$ and t,t', if $f(SA(t),i)$, $(i,k)\sigma_L$ and $(t,t')\rho$ then $f(SA(t'),(i,k)\sigma_L)$.*

Proof. Let us suppose, by contradiction, that the lemma does not hold, so the proof trivially follows from Lemma 6.1, Lemma 6.3 and the definition of f. □

Theorem 6.5 $\models_L A \iff \vdash_L A$.

Proof. The proof is similar to that given by Hughes and Cresswell (1968) and Gabbay (1976). □

Theorem 6.6 $\vdash_L A \Rightarrow \vdash_{KEM(L)} A$.

Proof. For a detailed proof for the propositional, first-order and modal fragments see D'Agostino and Mondadori (1994), Artosi Benassi Governatori and Rotolo (1996), and Governatori (1996). We prove the axiom of modal identity:

1. $F(t \neq s) \to \Box(t \neq s)$	w_1
2. $Tt \neq s$	w_1
3. $F\Box(t \neq s)$	w_1
4. $Ft \neq s$	(w_2, w_1)
5. $Tt = s$	(w_2, w_1)
6. $Tt \neq t$	w_1
7. $\times$	

The others identity axioms are easily derivable in KEM in a similar way. □

Theorem 6.7 $\vdash_{KEM(L)} A \Rightarrow \models_L A$.

Proof. The α-rules and PB are obviously sound rules in $\mathcal{M}$. For the β-rules and PNC: by the hypothesis $(l,k)\sigma_L$ and $(d,d')\rho$, then, by Lemma 3.1, $(i,(i,k)\sigma_L)\sigma_L$ and $(k,(i,k)\sigma_L)\sigma_L$ hence, by Lemma 6.4, the formulas involved have the same value in $g(i)$, $g(k)$ and $g((i,k)\sigma_L)$; after that these rules become rules of KE, and thus they are sound rules in $\mathcal{M}$.

For the Domains Rule. If $(i,k)\sigma_L$ then by Lemma 6.1 $g(i) \cap g(k) \neq \emptyset$. We have thus to consider three cases: Case i) $h(i), h(k) \in \Phi_C$; then $g(i) = g(k)$ and so $(i,k)\sigma_L[\![a_1,\ldots,a_n,a'_1,\ldots,a'_m]\!][a_q,\ldots,a_r,a'_s,\ldots,a'_t]$. Case ii) $h(i) \in \Phi_V$ and $h(k) \in \Phi_C$ (or vice versa); then $g(i) \cap g(k) = g(k)$. Each $w_i \in g(i)$ is such that $w_i[\![a_1,\ldots,a_n]\!][a_q,\ldots,a_r]$, and $g(k)$ is $g(k)[\![a'_1,\ldots,a'_m]\!][a'_s,\ldots,a'_t]$, and so $(i,k)\sigma_L[\![a_1,\ldots,a_n,a'_1,\ldots,a'_m]\!][a_q,\ldots,a_r,a'_s,\ldots,a'_t]$. Case iii) $h(i)$, $h(k) \in \Phi_V$; $g(i) \cap g(k) = g((i,k)\sigma_L)$. Any worlds $w_i \in g(i)$ and $w_k \in g(k)$ are such that $w_i[\![a_1,\ldots,a_n]\!][a_q,\ldots,a_r]$ and $w_k[\![a'_1,\ldots,a'_m]\!][a'_s,\ldots,a'_t]$, so $(i,k)\sigma_L[\![a_1,\ldots,a_n,a'_1,\ldots,a'_m]\!][a_q,\ldots,a_r,a'_s,\ldots,a'_t]$.

For the Increasing Domains Rule. We know that k extends i, so $g(i)R^n g(k)$ or $g((i,s(k))\sigma_L)R^n g(k)$ by Lemma 6.1; however both cases implies $e(g(i)) \subseteq e(g(k))$, and so $k[\![a_1,\ldots,a_n,a'_1,\ldots,a'_m]\!][a'_s,\ldots,a'_t]$.

The proofs for the Decreasing- Increasing Symmetric- Domains Rule are similar to that for the Increasing Domain Rule.

For the Atomic domain rule. By the semantic conditions, for any $w \in g(i)$, $v(P^m,w) \subseteq e(w)^m$. Therefore, if $Pa'_1,\ldots,a'_m$ is true in w, then $\langle a'_1,\ldots,a'_m\rangle \in v(P^m,w)$ and so each a_k, $1 \leq k \leq m$, is in w.

For the Binding rules. The proof for these rules is straightforward according to their semantic interpretation.

For the γ-rules. We show the proof only for $\gamma = T\forall x A(x)$ (the other case follows by the usual interdefinability of quantifiers). Let us suppose that it does not hold, then $v(\forall x A(x), g(i[\![a_1,\ldots,a_n]\!])) = T$ and $v(A(x_n)_i, g(i[\![a_1,\ldots,a_n]\!])) = F$. By the semantic conditions we have $v(A(a_h), g(i[\![a]\!])) = F$, $1 \leq h \leq n$, which falsifies the hypothesis.

For the δ-rules. We show the proof only for $\delta = T\exists x A(x)$ (the other case follows by the usual interdefinability of quantifiers). Let us suppose that $i \in \Phi_C$, $v(\exists x A(x), g(i[\![a_1,\ldots,a_n]\!])) = T$ and $v(A(a_m), g(i[\![a_1,\ldots,a_n,a_m]\!])) = F$, thus we have $T\neg A(a_m)$. Since a_m is new to the branch, then $T\forall x \neg A(x)$ and so $T\neg\exists x A(x)$, contrary to the hypothesis. The proof for i unrestricted is similar.

For the Complement rule. Let us suppose that this rule is not sound. Then a_m does not belong to the complement and so a_m exists in i. Therefore, $\langle \iota(a_m), \iota(a_m)\rangle \notin v(=, g(i))$ contrary to the semantic definition of $=$.

For Identity PNC. The proof for this rule is trivial. It is worth noting that such a rule is sound because of the global interpretation of identity. A formula as $a_n = a_n$ is true in any world w even if a_n does not exist in w.

For the Identity substitution. Since $Tt = t', i$, by the rigidity condition t and t' denote the same individual in every world, so $v(Xt, g(k)) = v(Xt', g(k))$.

For the Domain substitution. Since $(x)_i$ ranges over the domain of individuals existing in i and a_n exists in i, so $v(X(x)_i, g(k)) = v(X(x)_i/a_n, g(k))$.

For Terms PNC. The proof for this rule is straightforward. In fact, it is contradictory to state that an individual exists and, at the same time, does not exist in a given world.

For the ν-rules. Let us suppose $\nu = T\Box A$; for all $w_j \in g(i)$ and for all $w_m \in g((i', i))$, $v(\Box A, w_j) = T$; but $v(\Box A, w_j) = T$ iff $\forall w_m : w_j R w_m, v(A, w_m) = T$, and $(\forall w_m : w_j R w_m, v(A, w_m) = T) = f(\nu_0, i')$ with i' unrestricted. The proof for the π-rules is similar. $\Box$

From theorems 6.5, 6.6, and 6.7 we obtain:

Theorem 6.8 $\vdash_{KEM(L)} A \iff \models_L A$.

7 Final Remarks

The interest in the system just presented is that it provides a uniform "natural" treatment of QML augmented with the identity predicate, by solving the difficulties coming from this assumption with a simple "semantic" engine. In the last ten years several theorem proving systems for first-order modal logic have been proposed, but all suffer of severe limitations. The use of the resolution-like methods (such as in Abadi and Manna 1986, Auffray and Enjalbert 1992) constraints one to treat the modalities in a non-uniform manner by defining *ad hoc* rules (modal dependent) to pre-process the input formulas. For this reason, sequent/tableau inference techniques (Fitting 1993, Jackson and Reichgelt 1989) appear more adequate. However, both resolution and sequent/tableau inference rules of the systems quoted above fail to solve the problem associated with the non-permutability of the quantifier and modal rules. Wallen's (1990) matrix proof method overcomes all these shortcomings. Nevertheless, it works only for a few standard modal logics since it does not cover the "symmetric" B logics. Jackson and Reichgelt's (1989) sequent based proof method is the most similar to ours. This system indexes the terms chosen to instantiate the quantified variables with the path that names the world in which the inference is done; so to find the (possible) countermodel to halt the proof, it is necessary to match the paths of the contradictory formulas, occurring in the proof-tree, and the ones of the terms. This operation is by means of a unification algorithm that needs some pieces of "external", "intelligent" reasoning about the appropriate accessibility restrictions. On the contrary, the method we propose is actually a natural answer to these problems. The inferencial scheme runs without normal-forms or translation procedures. Furthermore, it offers a simple solution to the permutation problem by making the search-space wholly insensitive to the application order of modal and quantifier rules. The label unification scheme avoids skolemization and recursively embodies the conditions on the accessibility relation for the various modal logics, thus dispensing proof search from any piece of "external" reasoning. KEM language and rules allow us to take care not only of the first-order and modal part but also of the structure of labels (worlds and domains in Kripke models) and the relationship between labels and formulas: the Complement rule points out the conditions under which individuals are (non-)existing in a world; in this perspective it is possible to infer closure from labels corresponding to worlds that cannot exist, i.e., worlds that both contain and do not contain one and the same individual (terms PNC).

References

Alberto Artosi, Paola Benassi, Guido Governatori, Antonino Rotolo. 1996. Labelled Proofs for Quantified Modal Logic. In *Logics in Artificial Intelligence*, ed. J.J. Alferes, L. M. Pereira, and E. Orlowska. 70–86. LNAI 1126, Berlin: Springer-Verlag.

Auffray, Yves, and Patrice Enjalbert. 1992. Modal Theorem Proving: An Equational Viewpoint. *Journal of Logic and Computation* 2:247–259

Artosi, Alberto, and Guido Governatori. 1994. Labelled Model Modal Logic. In *Workshop on Automated Model Building*. 11–17. Nancy: CADE 12.

Abadi, M., and Z. Manna. 1986. Modal Theorem Proving. In *Proceedings of 8th International Conference on Automated Deduction*, ed. J.H. Siekmann. 172–189. LNCS 230. Berlin: Springer-Verlag.

Boolos, George. 1984. Don't Eliminate Cut. *Journal of Philosophical Logic* 13:373–378.

D'Agostino, Marcello, and Marco Mondadori. 1994. The Taming of the Cut. *Journal of Logic and Computation* 4:285–319.

Fitting, Melvin. 1983. *Proof Methods for Modal and Intuitionistic Logics*. Dordrecht: Reidel.

Fitting, Melvin. 1990. *First-Order Logic and Automated Theorem Prover*. Berlin: Springer-Verlag.

Fitting, Melvin. 1993. Basic Modal Logic. In *Handbook of Logic in Artificial Intelligence and Logic Programming, Volume 1*, ed. C.J. Hogger, Dov M. Gabbay, and J.A. Robinson. 368–448. Oxford: Oxford University Press.

Gabbay, Dov M. 1976. *Investigations in Modal and Tense Logics*. Dordrecht: Reidel.

Gabbay, Dov M. 1994. Classical vs non-Classical Logics. In *Handbook of Logic in Artificial Intelligence and Logic Programming, Volume 1*, ed. C.J. Hogger, Dov M. Gabbay, and J.A. Robinson. 359–500. Oxford: Oxford University Press.

Governatori, Guido. 1995. Labelled Tableaux for Multi-Modal Logics. In *Theorem Proving with Analytic Tableaux and Related Methods*, ed. P. Baumgartner, R. Hähnle and J. Posegga. 79–94. LNAI 918. Berlin: Springer-Verlag.

Governatori, Guido. 1996. A Duplication and Loop Checking Free System for S4. In *5^{th} Workshop on Theorem Proving with Analytic Tableaux and Related Methods (Short Papers)*, ed. P. Miglioli, U. Moscato, D. Mundici, and M. Ornaghi. 19–32. Technical report 154-96, Università di Milano.

Hintikka, Jakko. 1969. Existential Presuppositions and Uniqueness Presuppositions. In *Models for Modalities*. 112–147. Dordrecht: Reidel.

Hughes, G.E., and M.J. Cresswell. 1968. *An Introduction to Modal Logic*. Metethuen: London.

Jackson, Peter, and Han Reichgelt. 1989. A General Proof Method for Modal Predicate Logic *Logic-Based Knowledge Representation*, ed. P. Jackson, H. Reichgel, and F. von Hamelin. 177–228. Cambridge, MA: MIT Presss.

Kripke, Saul. 1963. Semantical Considerations on Modal Logics. *Acta Philosophica Fennica* 16:83–94.

Kripke, Saul. 1971. Identity and Necessity. In *Identity and Individuation*, ed. M. K. Munitz. 135–164. New York: New York University Press.

Smullyan, Raymond. 1968. Analytic Cut. *Journal of Symbolic Logic* 33:560–564.

Wallen, Lincoln. 1990. *Automated Deduction in Nonclassical Logics*. Cambridge, MA: MIT Press.