The Turing Computability of the Solution of the \textit{DGH} Equation

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Abstract: In this paper, we study the Turing computability of the solution operator of the initial value problem for the \textit{DGH} equation. We define a nonlinear map $K_R : H^s(R) \rightarrow C(R, H^s(R))$ from the initial value to the solution of the equation. Based on the Type 2 theory of effectivity, we suggest that $K_R$ is $(\delta_{H^s}, [p \rightarrow \delta_{H^s}])$-computable.

Keywords: \textit{DGH} equation; Turing machines; computability

1 Introduction

Differential equations are very useful models of real problems. Not all differential equations have a well solution. What we mostly concern is how the solution is computed out. Thus, the computability of the solution operator for differential equations becomes the hot issue in effective analysis. We should find out these equations whose solution operators are computable. Based on the theory [1] of Turing computability, we have obtained some results. For example, Klaus Weihrauch and Ning Zhong [2] have shown that the initial value problem of the \textit{KdV} equation posed on real line has a computable solution, which indicates the great meaning of studying of the computability of differential equations. Since for, some men of learning have studied some other differential equations, such as \textit{mKdV} equation [3], \textit{Schrodinger} equation [4], generalized functions [5], and so on. The \textit{DGH} equation is also an important equation which is used as a physical, hydrodynamics and other fields.

In this paper, we investigate the Cathy problem of the \textit{DGH} equation [6–8], we will show that the solution operator of the \textit{DGH} equation is also computable. The proof of main result is given in Section 2.

2 Main result

In this section, we use the similar approach as that in [2] and construct two estimates to prove the main result. In various computable models, we regard Type 2 theory of effectivity as computability model to analysis, relevant details is given in [2].

The following is the initial value problem of \textit{DGH} equation:

$$
\begin{aligned}
&u_t - \alpha^2 u_{xxx} + 2 \omega u_x + 3 uu_x + \gamma u_{xxx} = \alpha^2 (2u_x u_{xx} + uu_{xxx}), \quad t > 0, x \in R,
&u(0, x) = \varphi(x) \in H^s(R).
\end{aligned}
$$

(1)

It has stated that the whole solution of (1) is unique and existing. For simplifying the process of computing, letting $\alpha = 1$, (1) turns out

$$
\begin{aligned}
u_t + uu_x + \partial_x (1 - \partial_x^2)^{-1}((u + \omega)^2 + \frac{1}{2}(u_x)^2 + \gamma u_{xx}) &= 0.
\end{aligned}
$$

(2)

Then

$$
\begin{aligned}
u_t - \gamma u_{xx} = -\frac{1}{2}(u_x)^2 + f(u),
\end{aligned}
$$

(3)

where $f(u) = -\partial_x (1 - \partial_x^2)^{-1}(u^2 + \frac{1}{2}(u_x)^2 + 2(\omega + \frac{\gamma}{2})u)$, $u \in H^s$, $s \geq 3$, $t \in [0, T)$, $u$ is the solution of (1).

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We have the equivalent integral equation of (1)

\[ u(t) = N(t)\varphi + \int_0^t N(t - \tau) (f(u) - \frac{1}{2}u^2) d\tau, \tag{4} \]

where \( u(t)(x) := u(t, x) \), \( N(t)u = \frac{1}{2\pi} \int_R e^{i(\xi x - \gamma t)}u(\xi) d\xi \). We use the following iterative sequence with the initial data \( \varphi \) as the seed:

\[
\begin{align*}
  v_0(t) &= N(t)\varphi,
  v_{j+1}(t) &= v_0(t) - \int_0^t N(t - \tau)(f(v_j)) - \frac{1}{2}(V^2)_{x\tau} d\tau.
\end{align*}
\tag{5}
\]

The iterative sequence (5) is contracting near \( t = 0 \), thus the sequence converges to a unique limit. Since the limit satisfies the integral equation (4), it is the solution of the initial value problem (1). To prove the solution operator is computable, we need to construct a Type 2 Turing machine to compute it.

Firstly, we define the operator:

\[
S(u, \varphi, t) = N(t)\varphi + \int_0^t N(t - \tau)(f(u) - \frac{1}{2}u^2) d\tau,
\]

which is \( [(\rho \to \delta_s), \delta_s, \rho, \delta_s] \)-computable, this follows in (1). Therefore, the function \( S(u, \varphi)(t) := S(u, \varphi, t) \) is \( [(\rho \to \delta_s), \delta_s, [(\rho \to \delta_s)] \)-computable. Then we define the function \( \nu : S(R) \times N \to C(R; S(R)) \), where \( S(R) \) is the Schwartz space:

\[
\begin{align*}
  \nu(\varphi, 0) &= S(0, \varphi),
  \nu(\varphi, j + 1) &= S(\nu(\varphi, j), \varphi).
\end{align*}
\]

It is easy to verify that \( \nu \) is \( \{\delta_s, \gamma N, [(\rho \to \delta_s)]\} \)-computable.

Now we can show several lemmas which lead to our main theorem.

Let

\[
W(t)\varphi = 1/(2\pi) \int_R (e^{i\xi x} e^{-i\gamma t} \varphi(\xi)) d\xi,
\]

then

\[
S(u, \varphi)(t) = W(t)\varphi - \int_0^t W(t - \tau) - \frac{1}{2}(u^2) x d\tau.
\]

Lemma 1 If \( u(x, t) \) is the solution of (1), then there is a computable function \( e : N \times R \times R \to R \) which is non-decreasing in the second and third argument such that

\[
\sup_{0 \leq t \leq T} \|u(x, t)\|_s \leq e^x_T(\|\varphi\|_s),
\]

where \( e^x_T(r) := e(s, T, r), s \) is an integer and \( s \geq 3 \).

**Proof.** Let \( e^x_T(r) := e(s, T, r) = e^{\alpha^x_T} \|r\|_s \), see [7]. □

Lemma 2 For all \( T > 0, \varphi \in H^s \), we have

\[
\begin{align*}
  \|W(t)\varphi\|_s &= \frac{1}{\sqrt{2\pi}} \|\varphi\|_s, \\
  \|W(t)\varphi\|_{X^s_T} &\leq (3 + T)^\frac{1}{2} \|\varphi\|_s,
\end{align*}
\]

where \( \|u\|_{X^s_T} = ((\Lambda^s_{1, T})^2 + (\Lambda^s_{2, T})^2 + (\Lambda^s_{3, T})^2)^\frac{1}{2}, X^s_T = \{u \in C[0, T]; H^s(R); \Lambda^s_T < \infty\} \) is a Banach space with the norm \( \|u\|_{X^s_T} \) (detail see Definition in [2]).

**Proof.** Let \( u(x, t) = W(t)\varphi \). We can obtain that

\[
D^m_T u(x, t) = \frac{1}{2\pi} \int_R e^{i\xi x} (i\xi)^m e^{-i\gamma t} \varphi(\xi) d\xi.
\]

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For $m \leq s$, $\|D_x^m u(x, t)\|^2 = \frac{1}{2\pi} \int_R |\xi|^m \varphi(\xi)^2 d\xi = \frac{1}{2\pi} \|F(\varphi^{(m)})\|^2 = \frac{1}{2\pi} \|\varphi^{(m)}\|^2$, then for $t \in R$, we have $\|W(t)\varphi\|_s = \frac{1}{2\pi} \|\varphi\|_s$, and

$$\Lambda_{1,T}(u) = \sup_{0 \leq t \leq T} \|u(x,t)\|_s = \sup_{0 \leq t \leq T} \|W(t)\varphi\|_s \leq \frac{1}{2\pi} \|\varphi\|_s.$$  

Moreover,

$$D_{x}^{s+1}u = \frac{1}{2\pi} \int_R e^{i\xi \cdot (i\xi)^{s+1}} \varphi(\xi) d\xi = \frac{1}{2\pi} \int_R e^{i\xi \cdot (i\xi)^{s+1}} e^{-\gamma t} \varphi(\xi) \gamma d\xi,$$

then

$$\|D_{x}^{s+1}\|^2 \leq \frac{1}{2\pi} \int_R \frac{\eta^{(s+1)}}{\gamma} |\varphi(\xi)|^2 d\xi = \gamma \int_R \frac{\eta^{2}}{\gamma} |\varphi^2 \varphi(\xi)|^2 d\xi \leq \frac{\gamma}{2\pi} \|\varphi\|_s^2,$$

and

$$\Lambda_{2,T}(u) = (\sup_{x \in R} \int_0^T |D_{x}^{s+1}W(t)\varphi|^2 dt)^{1/2} \leq (\sup_{x \in R} \int_{-\infty}^{+\infty} |D_{x}^{s+1}W(t)\varphi|^2 dt)^{1/2} \leq \sqrt{\frac{\gamma}{2\pi}} \|\varphi\|_s.$$  

Similar to, we have

$$\Lambda_{3,T} \leq (1 + T)^{1/2} \|\varphi\|_s.$$  

So, where $\gamma < 2\pi$, we obtain the result

$$\|W(t)\varphi\|_{X_T^s} \leq \left( \frac{1}{2\pi} \|\varphi\|_s^2 + \frac{\gamma}{2\pi} \|\varphi\|_s^2 + (1 + T)\|\varphi\|_s^2 \right)^{1/2} \leq (3 + T)^{1/2} \|\varphi\|_s.$$  

**Lemma 3** Let $f(u) - \frac{1}{2}(u^2)_x \in C(R, H^s(R))$, then

$$\|\int_0^T W(t - \tau)(f(u) - \frac{1}{2}(u^2)_x) d\tau\|_{X_T^s} \leq (3 + T)^{1/2} \int_0^T \|f(u) - \frac{1}{2}(u^2)_x\|_{X_T^s} d\tau.$$  

**Proof.** See lemma 4.10 in [2].

**Lemma 4** If $\varphi \in H^s(R), \|\varphi\|_{X_T^s} \leq \frac{1}{16}$. Let $y^0 := \bar{S}(0, \varphi), y^{j+1} := \bar{S}(y^j, \varphi)$, then

$$\|y^{j+1} - y^j\|_s \leq \left( \frac{3}{4} \right)^{j+1} \times \frac{5}{16} (3 + T)^{1/2} \|\varphi\|_s.$$  

**Proof.**

$$\|y^j\|_{X_T^s} = \|W(t)\varphi - \int_0^T W(t - \tau)(f(y^{j-1}) - \frac{1}{2}(y^{j-1})_x^2) d\tau\|_{X_T^s} \leq (3 + T)^{1/2} \|\varphi\|_s + (3 + T)^{1/2} \int_0^T \|f(y^{j-1}) - \frac{1}{2}(y^{j-1})^2_x\|_{X_T^s} d\tau \leq (3 + T)^{1/2} \|\varphi\|_s + (3 + T)^{1/2} \|\varphi\|_s + (3 + T)^{1/2} \alpha_T \|y^{j-1}\|_{X_T^s}.$$  

where $c_0$ is related to $\gamma$. In following discussion, let $T > 0$, satisfying $c_0 T^{1/2}$, and

$$T^{1/2} \alpha_T (3 + T)^{1/2} \|\varphi\|_s \leq \frac{1}{16}.$$  

Since $\|y^0\|_{X_T^s} = \|W(t)\varphi\|_{X_T^s} \leq (3 + T)^{1/2} \|\varphi\|_s$, we obtain that

$$\|y^j\|_{X_T^s} \leq 3(3 + T)^{1/2} \|\varphi\|_s, \forall j \in N,$$

then

$$\|y^j - y^0\|_{X_T^s} \leq \|\int_0^T W(t - \tau)(f(y^0) - \frac{1}{2}(y^0)_x^2) d\tau\|_{X_T^s} \leq (3 + T)^{1/2} (T^{1/2} c_0 \|y^0\|_s + T^{1/2} \alpha_T \|y^0\|_{X_T^s}) \leq 3 + T^{1/2} \frac{1}{16} \|\varphi\|_s + \frac{1}{16} \|\varphi\|_s \leq \frac{5}{16} (3 + T)^{1/2} \|\varphi\|_s.$$  

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For $j \geq 2$,
\[
\|y^j - y^{j-1}\|_{X_\star} = \left\| \int_0^T W(t)\varphi[f(y^j) - f(y^{j-1}) - \frac{1}{2}(y^{j-1})^2 + \frac{1}{2}(y^{j-2})^2] \|_{X_\star} \right. \\
\left. \leq (3 + T)^{\frac{1}{2}} \int_0^T \|f(y^j) - f(y^{j-2})\|_s + \frac{1}{2}(y^{j-1})^2 + \frac{1}{2}(y^{j-2})^2 \|_s \right\| dr.
\]

Based on [2], we have
\[
\|f(y^{j-1}) - f(y^{j-2})\|_s \leq \|\omega\|_s \frac{3}{2} \|y^{j-1}\|_s + \|y^{j-2}\|_s + 2\omega + |\gamma| \]
\[
= \|y^{j-1} - y^{j-2}\|_s \frac{3}{2} (\|y^{j-1}\|_s + \|y^{j-2}\|_s) + 2\omega + |\gamma| \]
\[
\leq \|y^{j-1} - y^{j-2}\|_s \frac{3}{2} \times 6(3 + T)^{\frac{1}{2}} \alpha_T \|\varphi\|_s + 2\omega + |\gamma|.
\]

While $\frac{1}{2}(y^{j-1})^2 + \frac{1}{2}(y^{j-2})^2 \|_s \leq \frac{1}{2} \times 6(3 + T)^{\frac{1}{2}} \alpha_T \|\varphi\|_s \|y^{j-1} - y^{j-2}\|_{X_\star}$, so
\[
\|y^j - y^{j-1}\|_{X_\star} \leq (3 + T)^{\frac{1}{2}} T^{\frac{1}{2}} [9(3 + T)^{\frac{1}{2}} \alpha_T \|\varphi\|_s + 2\omega + |\gamma| + 3(3 + T)^{\frac{1}{2}} \alpha_T \|\varphi\|_s] \|y^{j-1} - y^{j-2}\|_{X_\star}
\]
\[
\leq \frac{1}{2} \|y^{j-1} - y^{j-2}\|_{X_\star}.
\]

If $T^{\frac{1}{2}} \alpha_T (3 + T)^{\frac{1}{2}} \|\varphi\|_s \leq \frac{1}{16}$, then
\[
\|y^{j+1} - y^j\|_s \leq \|y^{j+1} - y^j\|_{X_\star} \leq \frac{3}{4} J^{j+1} \times \frac{5}{16} (3 + T)^{\frac{1}{2}} \|\varphi\|_s
\]

So, the sequence satisfying the contraction mapping principle.

Since for $t \leq T$, $(y^j(t))_{j \geq 1}$ converges to some $y(t) \in H^s(\mathbb{R})$ with respect to the norm of $\| \cdot \|_s$ and $\| \cdot \|_{X_\star}$, and satisfies $y(t) = \bar{S}(y, \varphi)(t)$, we have that from (11) and (13)
\[
\|y\|_{X_\star} \leq 3(3 + T)^{\frac{1}{2}} \|\varphi\|_s, \ i f T^{\frac{1}{2}} \alpha_T (3 + T)^{\frac{1}{2}} \|\varphi\|_s \leq \frac{1}{16}.
\]

**Lemma 5** If $v(t) = \bar{S} (v, \varphi)(t)$, $v_n(t) = \bar{S} (v_n, \varphi_n)(t)$, where $\varphi_n$ is the Cauchy sequence which converges to $\varphi$, then $v(t)$ is computable.

**Proof.**
\[
\|v - v_n\|_{X_\star} = \|W(t)(\varphi - \varphi_n) - \int_0^T W(t - \tau) [f(v) - f(v_n) - \frac{1}{2}(v_2 - v_n^2)_{x}]\|
\]
\[
\leq (3 + T)^{\frac{1}{2}} \|\varphi - \varphi_n\|_s + (3 + T)^{\frac{1}{2}} T^{\frac{1}{2}} \|v - v_n\|_s \frac{3}{2} (\|v\|_s + \|v_n\|_s) + 2\omega + |\gamma| + (3 + T)^{\frac{1}{2}} T^{\frac{1}{2}} \alpha_T \|v + v_n\|_{X_\star} \|v - v_n\|_{X_\star}.
\]

Observing that $\|\varphi_n\|_s \leq \|\varphi\|_s + 1$, apply (14) to $v$ and $v_n$. If $T^{\frac{1}{2}} \alpha_T (3 + T)^{\frac{1}{2}} (\|\varphi\|_s + 1) \leq \frac{1}{16}$, then
\[
\|v - v_n\|_{X_\star} \leq (3 + T)^{\frac{1}{2}} \|\varphi - \varphi_n\|_s + (3 + T)^{\frac{1}{2}} T^{\frac{1}{2}} \|v - v_n\|_s \frac{3}{2} (\|v\|_s + \|v_n\|_s) + 2\omega + |\gamma| + (3 + T)^{\frac{1}{2}} T^{\frac{1}{2}} \alpha_T \|v + v_n\|_{X_\star} \|v - v_n\|_{X_\star}
\]
\[
\leq (3 + T)^{\frac{1}{2}} \|\varphi - \varphi_n\|_s + (3 + T)^{\frac{1}{2}} T^{\frac{1}{2}} \|v - v_n\|_s \frac{1}{2} (4(3 + T)^{\frac{1}{2}} \alpha_T (2\|\varphi\|_s + 1))
\]
\[
+ 2(3 + T) \alpha_T T^{\frac{1}{2}} \frac{1}{2} \|\varphi\|_s + 1) \|v - v_n\|_s.
\]

Since
\[
9(3 + T)^{\frac{1}{2}} \alpha_T (\|\varphi\|_s + 1) + 3(3 + T) \alpha_T T^{\frac{1}{2}} (\|\varphi\|_s + 1) \leq 12(3 + T)^{\frac{1}{2}} \alpha_T (\|\varphi\|_s + 1)
\]
\[
\leq \frac{3}{4} \|v - v_n\|_{X_\star} \leq 4(3 + T)^{\frac{1}{2}} \|\varphi - \varphi_n\|_s.
\]

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So, if \( T^2 \alpha_1^c (3 + T)^{3/2} \left( \| \varphi \|_s + 1 \right) \leq \frac{1}{10}, 0 \leq t \leq T, \)

\[
\| v - v_n \|_s \leq 4(3 + T)^{3/2} \| \varphi - \varphi_n \|_s.
\]

Since \( v_n(t) \) is computable, then \( v \) is computable, i.e. the solution operator of (4) is computable. So we have the following conclusion.

**Theorem 6** The solution operator of DGH equation \( K_R : H^s(R) \to C(R, H^s(R)) \) is \( (\delta_{H^s}, [\rho \to \delta_{H^s}]) \)-computable regarding to \( s \geq 3 \).

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**References**


