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A method for defuzzification based on centroid point

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Abstract

A method for ranking fuzzy numbers based on the centroid point is proposed and some of its desirable properties are studied. Many different methods have been proposed to deal with ranking fuzzy numbers. Constructing ranking indexes based on centroides is an important. But some weaknesses are found in these indexes. The purpose of this article is to give a new ranking index to rank various numbers effectively.

Keywords: Fuzzy number, Defuzzification, Ranking, Centroid point, Distance.

1. Introduction

In the decision analysis of the fuzzy environment, fuzzy numbers need to be compared and discriminated by decision makers. Ranking fuzzy numbers is not the total order relations under the ordinary meaning, but the partial order under the lattice structure. Thus the ranking theories and methods of fuzzy numbers become one of the important and difficult task of fuzzy decision related problems. As the fuzzy number is determined by the membership function, to achieve the purpose of ranking, the sort of fuzzy numbers is to construct various order relationships from the standpoint of membership function to some extent. Practically, the centroids of fuzzy numbers are some properties of fuzzy numbers which are extracted from geometric aspects. Based on this information, a comprehensive index $R=\sqrt{\left(\bar{x}\right)^2+\left(\bar{y}\right)^2}$ is proposed in (Cheng, 1998) to sort the fuzzy numbers; in 2002, (Chu, 2002) found that this index still had some defects and gave a ranking index function $S=\bar{x}.\bar{y}$ to improve it; in 2008, (Zhao, 2008) pointed out that the criterions provided by Cheng and Chu were incorrect for some fuzzy numbers' ranking and proposed an index function $D=\frac{1}{2}\mu(\bar{x})\left(\left|\bar{x}\right|+\left|\bar{y}\right|+\left|\bar{x}.\bar{y}\right|\right)$

(where $\bar{\mu}(x) = \begin{cases} 1, & x \geq 0, \\ -1, & x < 0. \end{cases}$) to improve the first two methods; in the same year, in (Wang, 2008) pointed out that $\bar{x}_A$ indicates the representative location of fuzzy number A on the real axis, and $\bar{y}_A$ represents the average height of the fuzzy number. Besides this, the degree of representative location is more important than average height in order to ranking fuzzy numbers more conveniently. Based on this concept, a revision of Cheng and Chu's method was presented as follows. For any two fuzzy numbers A and B, (a) if $\bar{x}_A > \bar{x}_B$, then $B \prec A$. (b) if $\bar{x}_A < \bar{x}_B$, then $B \succ A$. (c) if $\bar{x}_A = \bar{x}_B$, then if $\bar{y}_A > \bar{y}_B$, then $B \prec A$; else if $\bar{y}_A < \bar{y}_B$, then $B \succ A$; else $\bar{y}_A = \bar{y}_B$, then $A \sim B$. By studying, we found that these indexes have some obvious shortcomings for some fuzzy numbers' ranking. To compensate for these shortcomings, a new index of ranking fuzzy numbers have been constructed in this paper.

The rest of this paper is organized as follows. In the next section some basic definitions are reviewed. In section 3, a new approach is proposed for the ranking of generalized trapezoidal fuzzy numbers. In this section the ranking results of the proposed approach are compared with different existing approaches. The conclusions are given in section 4.

2. Preliminaries

The basic definition of a fuzzy number given in (Heilpern, 1992; Kauffman, et al. 1991; Saneifard, 2009) as follow:

Definition 2.1. Let U be a universe set. A fuzzy set of U is defined by a membership function $\mu_A(x) \to [0,1]$, where $\mu_A(x)$ indicates the degree of x in A.

Definition 2.2. An extended fuzzy number A is described as any fuzzy subset of the universe set U with membership function μ_A defined as follow:

- μ_A is a continuous mapping from U to the closed interval $[0, w], 0 < w \leq 1$.
- $\mu_A(x) = 0$, for all $x \in (-\infty, a_1]$.
- μ_A is strictly increasing between $[a_1, a_2]$.
- $\mu_A(x) = w$, for all $x \in [a_2, a_3]$, w is a constant and $0 < w \leq 1$.
- μ_A is strictly decreasing between $[a_3, a_4]$.
- $\mu_A(x) = 0$, for all $x \in [a_4, +\infty)$.

In above situations a_1, a_2, a_3 and a_4 are real numbers. If $a_1 = a_2 = a_3 = a_4$, A becomes a crisp real number.

Definition 2.3. The membership function μ_A of extended fuzzy number A is expressed by 1 as:

$$\mu_A = \begin{cases} f_A^L(x), & when \ a_1 \le x \le a_2, \\ w, & when \ a_2 \le x \le a_3, \\ f_A^R(x), & when \ a_3 \le x \le a_4, \\ 0, & otherwise. \end{cases} \quad (1)$$

Where $f_A^L(x):[a_1,a_2]\to[0,w]$ and $f_A^R(x):[a_3,a_4]\to[0,w]$. Based on the basic theories of fuzzy numbers, A is a normal fuzzy number if $w=1$, whereas A is a non-normal fuzzy number if $0<w\le 1$. Therefore, the extended fuzzy number A in Definition 2.3 can be denoted as $(a_1,a_2,a_3,a_4;w)$. The image $-A$ of A can be expressed by $(-a_1,-a_2,-a_3,-a_4;w)$ (Kauffman, et al. 1991). With Zadeh's extension principle, the arithmetic operation of fuzzy sets especially the fuzzy numbers can be defined. Here, this article recalls the two simplest cases of scalar addition and scalar multiplication. Let $g_A^L(y):[0,w]\to[a_1,a_2]$ and $g_A^R(y):[0,w]\to[a_3,a_4]$ be the inverse functions of f_A^L and f_A^R, respectively. Then $g_A^L(y)$ and $g_A^R(y)$ should be integrable on the closed interval $[0,w]$. In other words, both $\int_0^w g_A^L(y)dy$ and $\int_0^w g_A^R(y)dy$ should exist. In the case of trapezoidal fuzzy number, the inverse functions $g_A^L(y)$ and $g_A^R(y)$ can be analytically expressed as:

$$g_A^L(y) = a_1 + \frac{(a_2-a_1)y}{w}, \ 0 \le y \le w. \quad (2)$$

$$g_A^R(y) = a_4 - \frac{(a_4-a_3)y}{w}, \ 0 \le y \le w. \quad (3)$$

In order to determine the centroid point $(\bar{x}_0, \bar{y}_0)$ of a fuzzy number A, Wang et al. (2006) provided the following centroid formulae:

$$\bar{x}_0(A) = \frac{\int_{a_1}^{a_2} x\, f_A^L(x)dx + \int_{a_2}^{a_3} (x\,w)dx + \int_{a_3}^{a_4} x\, f_A^R(x)dx}{\int_{a_1}^{a_2} f_A^L(x)dx + \int_{a_2}^{a_3} (w)dx + \int_{a_3}^{a_4} f_A^R(x)dx} \quad (4)$$

$$\bar{y}_0(A) = \frac{\int_0^w y\left(g_A^R(y) - g_A^L(y)\right)dy}{\int_0^w \left(g_A^R(y) - g_A^L(y)\right)dy} \quad (5)$$

For this trapezoidal fuzzy number, the following results are derived from 4 and 5,

$$\bar{x}_0(A) = \frac{1}{3}\left[a_1 + a_2 + a_3 + a_4 - \frac{a_4 a_3 - a_1 a_2}{(a_4 + a_3) - (a_1 - a_2)}\right], \tag{6}$$

$$\bar{y}_0 = w\frac{1}{3}\left[1 + \frac{a_3 - a_2}{(a_4 + a_3) - (a_1 + a_2)}\right]. \tag{7}$$

Definition 2.4. Let $A = (a_1, a_2, a_3, a_4; w)$ is arbitrary fuzzy number; the centroid point is denoted as $(\bar{x}_0(A), \bar{y}_0(A))$ (Calculated by 4 and 5). The new measurement of fuzzy numbers is defined as follow:

$$M(A) = sign\left(\left|\bar{x}_0(A)\right| + w\left|\bar{y}_0(A)\right| + w\left|\bar{x}_0(A).\bar{y}_0(A)\right|\right)\int_0^w \left|g_A^L(y)\right| dy. \tag{8}$$

Where; $sign(.)$ is symbolic function.

Proposition 2.1. Let A is a fuzzy number, for $\forall y \in [0, w]$.

1. If $\inf Supp\, A \geq 0 (\inf_{y\in[0,w]} g_A^L(y) \geq 0)$, then $M(A) \geq 0$.
2. If $\inf Supp\, A \leq 0 (\sup_{y\in[0,w]} g_A^R(y) \leq 0)$, then $M(A) \leq 0$.

Proof (1) As $\inf Supp\, A \geq 0, a_1 \geq 0$ and $g_A^L(y) \geq 0$, for $\forall y \in [0, w]$. Since $g_A^R(y) \geq g_A^L(y)$ and $a_1 \leq a_2 \leq a_3 \leq a_4$, and also $\bar{x}_0(A) > 0$, then $M(A) \geq 0$. Similarly we can prove 2.

3. A new method for ranking fuzzy numbers

In this section, we present a new approach for ranking fuzzy numbers based on the distance method. The method not only considers the centroid point of a fuzzy number, but also the minimum crisp value of fuzzy numbers. For ranking fuzzy numbers, this study firstly defines a minimum crisp value $\tau_{\min}$ to be the benchmark and its characteristic function $\mu_{\tau_{\min}}(x)$ is as follow:

$$\mu_{\tau_{\min}}(x) = \begin{cases} 1 & when \;\; x = \tau_{\min}, \\ 0 & when \;\; x \neq \tau_{\min}. \end{cases} \tag{9}$$

When ranking n fuzzy numbers $A_1, A_2, \ldots, A_n$ the minimum crisp value $\tau_{\min}$ is defined as:

$$\tau_{\min} = \min\{x | x \in Domain(A_1, A_2, \cdots, A_n)\}. \tag{10}$$

The advantages of the definition of minimum crisp value are two-fold: first, the minimum crisp values will be obtained by themselves, and another is it is easy to compute.

Example 3.1. Three fuzzy numbers A, B and C have been illustrated by (Chen, 1985). The fuzzy numbers and the minimum crisp value are illustrated in Figure 1 By 10, this study obtains $\mu_{\tau_{\min}}$ and inverse functions as follow:

$\tau_{\min} = \min\{x | x \in Domain(A, B, C)\} = \min\{0.01, 0.1, 0.2, 0.4, 0.5, 0.6, 0.7, 0.8, 0.9\} = 0.01$,
and

$$g_{\min}(x) = \begin{cases} g_{\min}^{L}(x) = 0.01, \\ g_{\max}^{L}(x) = 0.01. \end{cases}$$

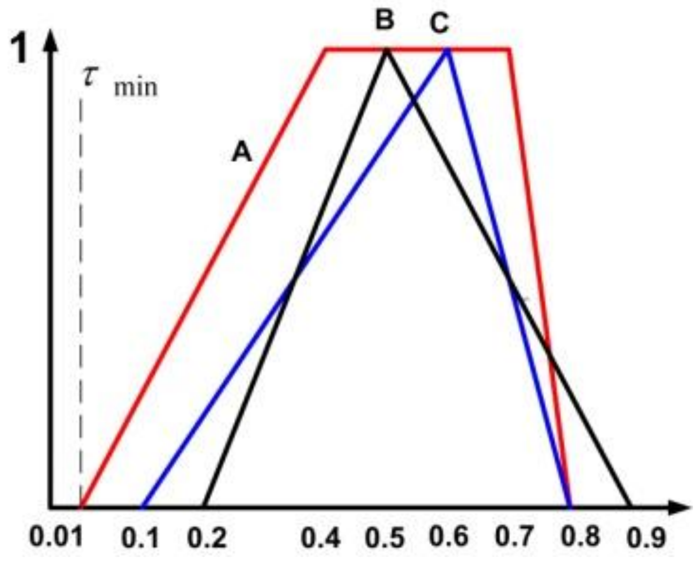

Figure 1. Fuzzy numbers A, B, C and $\tau_{\min}$

Assume that there are n fuzzy numbers $A_1, A_2, \ldots, A_n$. The proposed method for ranking fuzzy numbers $A_1, A_2, \ldots, A_n$ is now presented as follow:

Step 1. Use formulas 4 and 5 to calculate the centroid point $(\bar{x}_0(A_j), \bar{y}_0(A_j))$ of each fuzzy numbers A_j, where $1 \le j \le n$.

Step 2. Calculate the minimum crisp value $\tau_{\min}$ of all fuzzy numbers A_j, where $1 \le j \le n$.

Step 3. Use the point $(\bar{x}_0(A_j), \bar{y}_0(A_j))$ to calculate the ranking value $dist(A_j, \tau_{\min})$ of the fuzzy numbers A_j, where $1 \le j \le n$, as follow:

$$dist(A_j, \tau_{\min}) = \left| M(A_j) - \tau_{\min} \right|. \qquad (11)$$

From 11, one may that $dist(A_j, \tau_{\min})$ could be considered as the Euclidean distance between the point $(M(A_j), 0)$ and the point $(\tau_{\min}, 0)$. We can see that the smaller the value of $dist(A_j, \tau_{\min})$, the better the ranking of A_j, where $1 \le j \le n$.

Let A_j be a fuzzy number characterized by 1 and $dist(A_j, \tau_{\min})$ is the Euclidean distance between the point $(M(A_j),0)$ and the point $(\tau_{\min},0)$.

Since this article wants to approximate a fuzzy number by a scalar value, the researchers have to use an operator $dist: F \rightarrow \Re$ (A space of all fuzzy numbers will be denoted by F) which transforms fuzzy numbers into a family of real line. $dist$ is a crisp approximation operator. Since ever above defuzzification can be used as a crisp approximation of a fuzzy number, therefore the resultant value is used to rank the fuzzy numbers. Thus, $dist$ is used to rank fuzzy numbers. The smaller $dist$, the larger fuzzy number.

Let $A_1, A_2 \in F$ be two arbitrary fuzzy numbers. Define the ranking of A_1 and A_2 by $dist$ on F as follow:

(1) $dist(A_1, \tau_{\min}) > dist(A_2, \tau_{\min})$ if only if $A_1 \prec A_2$,
(2) $dist(A_1, \tau_{\min}) < dist(A_2, \tau_{\min})$ if only if $A_1 \succ A_2$,
(3) $dist(A_1, \tau_{\min}) = dist(A_2, \tau_{\min})$ if only if $A_1 \sim A_2$.

Then, this article formulates the order $\succeq$ and $\preceq$ as $A_1 \succeq A_2$ if and only if $A_1 \succ A_2$ or $A_1 \sim A_2$, $A_1 \preceq A_2$, if and only if $A_1 \prec A_2$ or $A_1 \sim A_2$. The new ranking index can sort many different fuzzy numbers simultaneously. In addition, the calculation is simple, and the index also satisfies the common properties of ranking fuzzy numbers:
(a) Transitivity of the order relation, i.e. if $A_1 \preceq A_2$ and $A_2 \preceq A_3$, then we should have $A_1 \preceq A_3$.

(b) Compatibility of addition, that is if there is $A_1 \preceq A_2$ on $\{A_1, A_2\}$, then there is $A_1 + A_3 \preceq A_2 + A_3$ on $\{A_1 + A_3, A_2 + A_3\}$.

Remark 3.1. If $A_1 \preceq A_2$, then $-A_1 \succeq -A_2$.
Hence, this article can infer ranking order of the images of the fuzzy numbers. To present rationality of this method, some examples are proposed to illustrate these methods and compared with others method.(Saneifard, 2010; Saneifard et. al. 2007; Ezatti, et al. 2010).

Example 3.2. Consider the data used in (Saneifard, 2009), i.e. the three fuzzy numbers, $A = (5,6,6,7)$, $B = (5.9,6,6,7)$ and $C = (6,6,6,7)$, as shown in Figure 2. According to Eq. (11), the ranking index values are obtained i.e. $dist(A, \tau_{\min}) = 2.69$, $dist(B, \tau_{\min}) = 1.52$ and $dist(C, \tau_{\min}) = 1.59$. Accordingly, the ranking order of fuzzy numbers is $A \prec C \prec B$. However, by Chu and Tsao's approach (Chu et al. 2002), the ranking order is $A \prec C \prec B$. which is the same as the one obtained by the writers approach. However, their approach is simpler in the computation procedure.Meanwhile, using CV proposed index (Cheng, 1999), the ranking order is $C \prec B \prec A$. From Figure 2, it is easy to see that the ranking results obtained by CV index approach are unreasonable and are not consistent with human intuition.

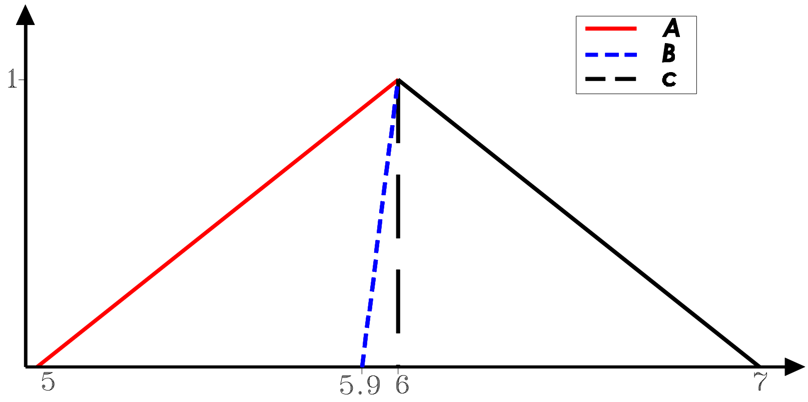

Figure 2. Fuzzy numbers A, B, C of Example 3.2.

Example 3.3. Consider the following sets: $A = (1,2,2,5)$, $B = (0,3,3,4)$ and $C = (2,2.5,2.5,3)$, (see Figure 3). By using this new approach, $dist(A, \tau_{\min}) = 2.5$, $dist(B, \tau_{\min}) = 0.79$ and $dist(C, \tau_{\min}) = 1.03$. Hence, the ranking order is $A \prec C \prec B$ too. It seems that, the result obtained by "Distance Minimization" method is unreasonable. To compare with some of the other methods in (Chu et al. 2002), the readers can refer to Table 1. Furthermore, in the mentioned example, $dist(-A, \tau_{\min}) = 2.88$, $dist(-B, \tau_{\min}) = 5.26$ and $dist(-C, \tau_{\min}) = 3.22$, consequently the ranking order of the images the three fuzzy numbers is $-B \prec -C \prec -A$. Clearly, this proposed method has consistency in ranking fuzzy numbers and their images, which could not be guaranteed by CV-index method. Through Figure 3, it is easy to see that neither of them is consistent with human intuition.

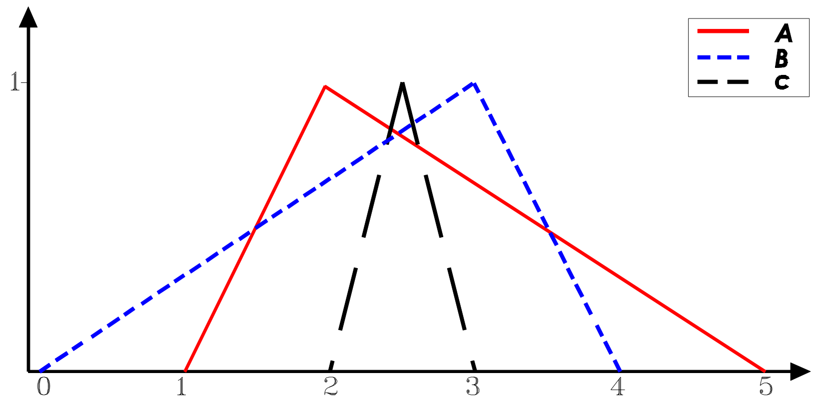

Figure 3. Fuzzy numbers A, B, C of Example 3.3.

Table 1. Comparative results of Example (3.3)

Fuzzy number	New approach	Sign Distance with $p = 2$	Distance Minimization	Chu and Tsao (Revisited)	CV index	Magnitude method
$\boldsymbol{A}$	2.50	3.91	2.5	0.74	0.32	2.16
$\boldsymbol{B}$	0.79	3.91	2.5	0.74	0.36	2.83
$\boldsymbol{C}$	1.03	3.55	2.5	0.75	0.08	2.50
Results	$\boldsymbol{A \prec C \prec B}$	$\boldsymbol{C \prec A \sim B}$	$\boldsymbol{C \sim A \sim B}$	$\boldsymbol{A \sim B \prec C}$	$\boldsymbol{B \prec A \prec C}$	$\boldsymbol{A \prec C \prec B}$

All the above examples show that this method is more consistent with institution than the previous ranking methods.

4. Conclusions

Fuzzy systems have gained more and more attention from researchers and practitioners in various fields. In such systems, the output represented by a fuzzy set sometimes needs to be transformed into a scalar value, and this task is known as the defuzzification process. Several analytic methods have been proposed for this problem, but in this paper, the researchers suggest a new approach to the problem of defuzzification using the new measure of fuzzy numbers. In this study some preliminary results on properties of such defuzzification reported.

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References

Cheng, C. H., A new approach for ranking fuzzy numbers by distance method. Fuzzy Sets and Systems. 95, 307–317, 1998.

Chu, T. C., Tsao, C. T., Ranking fuzzy numbers with an area between the centroid point and original point, Computers and Mathematics with Applications. 43, 111–117, 2002.

Saneifard, R., Allahviranloo, T., Hosseinzadeh, F., Mikaeilvand, N., Euclidean ranking DMUs with fuzzy data in DEA, Applied Mathematical Sciences. 60, 2989–2998, 2007.

Saneifard, R., Ranking L-R fuzzy numbers with weighted averaging based on levels, International Journal of Industrial Mathematics. 2, 163–173, 2009.

Saneifard, R., A method for defuzzification by weighted distance, International Journal of Industrial Mathematics. 3, 209–217, 2009.

Saneifard, R., Defuzzification method for solving fuzzy linear systems, International Journal of Industrial Mathematics. 4, 321–331, 2009.

Saneifard, R., Ezatti, R., Defuzzification through a bi-Symmetrical weighted function. Aust. J. Basic Appl. Sci, 10, 4976–4984, 2010.

Ezzati, R., Saneifard, R., A new approach for ranking of fuzzy numbers with continuous weighted quasi-arithmetic means, Mathematical Sciences. 4, 143–158, 2010.

Ezzati, R., Saneifard, R., Defuzzification trough a novel approach, Proc.10th Iranian Conference on Fuzzy Systems. 343–348, 2010.

Zhao, J., Liu, Q. S., Ranking fuzzy numbers based on the centroid of fuzzy numbers, Fuzzy Systems and Mathematics. 22, 142–146, 2008.

Whang, Y. J., Lee, H. S., The revised method of ranking fuzzy numbers with an area between the centroid and original points, Computers and Mathematics with Applications. 55, 2033–2042, 2008.

Heilpern, S., The expected value of a fuzzy number, Fuzzy Sets and Systems. 47, 81–86, 1992.

Kauffman, A., Gupta, M.M., Introduction to Fuzzy Arithmetic: Theory and Application, Van Nostrand Reinhold, New York, 1991.

Wang, Y. M., Yang, J. B., Xu, D. L., and Chin, K. S., On the centroids of fuzzy numbers, Fuzzy Sets and Systems. 157, 919–926, 2006.

Chen, S. H., Ranking fuzzy numbers with maximizing set and minimizing set, Fuzzy Sets and Systems. 17, 113–129, 1985.

Cheng, C. H., Ranking alternatives with fuzzy weights using maximizing set and minimizing set, Fuzzy Sets and System. 105, 365–375, 1999.

Alexander, C., A fuzzy optmization approach to production planning. International Journal of Computers in Fuzzy Systems Enginering, 34, 258-272, 2008.

Ercan, T., Cetinkaya, S., Yilmaz, G., Future recognition with fuzzy sets. Proceedings of 1st Fuzzy Systems Symposium, FUZZYSS'09, 1-2 October, Ankara, Turkey, pp.138–143, 2009.

Turkeroglu, M. Z., Fuzzy sets in service industry, Kelebek Press, Istanbul, 1993.