

Size-Dependent Transition to High-Dimensional Chaotic Dynamics in a Two-Dimensional Excitable Medium

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Abstract

The spatiotemporal dynamics of an excitable medium with multiple spiral defects is shown to vary smoothly with system size from short-lived transients for small systems to extensive chaos for large systems. A comparison of the Lyapunov dimension density with the average spiral defect density suggests an average dimension per spiral defect varying between three and seven. We discuss some implications of these results for experimental studies of ventricular fibrillation.

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Much research in the nonequilibrium physics of excitable media has been motivated by the hypothesis that ventricular fibrillation (VF) in a human heart corresponds to irregular electrical activity involving topological defects, *e.g.*, spirals in two-space dimensions or filaments in three dimensions [1]. Experimental studies of the spatial electrical disorder in a fibrillating heart suggest that many such defects may coexist [2], but the connection of this spatial disorder to temporal dynamics is unclear. A second observation is that this disordered behavior occurs only for sufficiently large hearts [3]; the transition to disordered behavior with increasing system size is not well understood. Understanding the relationship between the dynamics, the spatial disorder, and the size of an excitable medium may thus be important for interpreting the results of experiments on fibrillating hearts [2, 4] and for analyzing other spatially-extended excitable media [5].

In this Letter, we numerically study a two-dimensional model of a homogeneous excitable medium [5, 6] with an emphasis on determining when spatiotemporal chaos occurs in the system, and on quantitatively analyzing basic time and length scales of observed chaotic states. Some initial conditions decay rapidly to a constant or to a periodic state. As the system size increases, however, the set of initial conditions leading to sustained, non-periodic dynamics increases smoothly, and we discuss the transition from periodic to non-periodic dynamics with increasing system size. The non-periodic dynamics sustained in sufficiently large systems are statistically stationary, and we compute Lyapunov exponents and dimensions, defect statistics, and two-point correlation lengths to characterize these states. These different statistics are compared, both to test previous conjectures [4] and to evaluate the complexity of the defects. Our results indicate that an excitable medium of moderate size with few defects on average can already sustain extensive, high-dimensional chaotic dynamics, a fact with important implications for control of excitable media by small parameter perturbations [7]. In the following, we explain the model, summarize our calculations, and discuss our results.

For our analysis, we used a model proposed by Bär and collaborators [5] for which there was prior evidence of nonperiodic behavior and an analysis of defect statistics [5, 6]. This

model expresses the interaction of an activator field $u(t, x, y)$ with an inhibitor field $v(t, x, y)$ via the partial differential equations

$$\frac{\partial u}{\partial t} = \nabla^2 u + \frac{1}{\epsilon} u(1 - u) \left(u - \frac{v + b}{a} \right), \quad (1a)$$

$$\frac{\partial v}{\partial t} = f(u) - v, \quad (1b)$$

$$(1c)$$

which we solve numerically in a square domain of side L with either biperiodic (BP) or no-flux (NF) boundary conditions on the field $u(t, x, y)$. The function $f(u)$ has the form

$$f(u) = \begin{cases} 0, & \text{if } u \leq 1/3, \\ 1 - 6.75u(u - 1)^2, & \text{if } 1/3 \leq u \leq 1, \\ 1, & \text{if } u > 1, \end{cases} \quad (2)$$

which was motivated by consideration of reacting and diffusing adsorbates on a metal surface [5]. Its nonlinear form leads to three fixed points (one stable, two unstable); the larger unstable fixed point (u^*, v^*) (which does not appear in the widely-used Fitzhugh-Nagumo model) seems necessary for the occurrence of spatiotemporal chaos. The parameter ϵ in Eq. 1a determines the ratio of time scales of the fast field u and slow field v and is the key bifurcation parameter in this paper. The positive parameters a and b were fixed at the values $a = 0.84$ and $b = 0.07$ to take advantage of substantial earlier work using these values [5, 6]. Spiral solutions are then known empirically to be unstable when ϵ exceeds a critical value $\epsilon_c \approx 0.069$ [5], and a large system indeed exhibits sustained complicated dynamics for certain initial conditions when $\epsilon > \epsilon_c$. A snapshot of such a disordered nonperiodic state with 31 spiral defects is shown in Figure 1.

Our calculations involved integrating Eq. 1, calculating the Lyapunov spectrum of the numerical trajectory, and counting the number of spiral defects at successive times. For both kinds of boundary conditions, Eq. 1 was solved numerically by first introducing second-order accurate finite difference approximations for the spatial derivatives on a uniform square mesh of spacing Δx and then using an implicit time-stepping algorithm proposed by Barkley [8].

For the calculations reported below, we used a spatial grid size $\Delta x = 0.50$ and time step $\Delta t = 0.05\epsilon$ [9]. The spectrum of Lyapunov exponents λ_i and the Lyapunov fractal dimension D were calculated by well-known algorithms based on linear variational equations [12] that were integrated by a forward-Euler algorithm with the same grid and time step. The time step chosen, $\Delta t = 0.05\epsilon$, was much smaller than that required by integration of only Eq. 1, but was necessary to compute Lyapunov exponents accurately to within a few percent. For given boundary conditions and initial data, Eq. 1 was integrated for 2000 time units to allow a statistically stationary state to be obtained, and then the full system with variational equations was integrated for an additional 1000 time units (≈ 200 spiral periods), during which statistics were calculated.

To study the dependence of the dynamics on initial conditions, we integrated Eq. 1 from an ensemble of 100 initial conditions generated by distributing the field values uniformly (at each grid point) in the ranges $u \in [0.8u^*, 1.2u^*]$, $v \in [0.8v^*, 1.2v^*]$. This procedure was repeated for both boundary conditions and for square systems of side length L varying from 5 to 40. For all initial conditions, the dynamics was short-lived in small systems ($L < 15$ (for NF boundary condition) or $L < 8$ (for BP)), decaying in less than 100 time units to either the stable uniform state or to a plane-wave state (only in the case of biperiodic boundary conditions). Sufficiently large systems ($L > 35$) sustained dynamics for at least 3000 time units. The fraction f of initial conditions which led to non-periodic dynamics sustained for a time T_{np} is shown in Figure 2 as a function of system size for both boundary conditions. For biperiodic boundary conditions (Figure 2(a)), the curve is independent of the cutoff time T_{np} , indicating that transients either die quickly or are sustained indefinitely (more than 50,000 time units). However, for no-flux boundary conditions (Figure 2(b)), all initial conditions eventually decay to the uniform resting state. The mean and median transient times for no-flux boundary conditions both scale exponentially with system size [13]. These results suggest a possible explanation for the observation that fibrillation occurs in large hearts and *sometimes* occurs in hearts of intermediate size; whether or not fibrillation is sustained may depend on the detailed structure of the initial state chosen.

The fact that a given state was transient was revealed only by an eventual abrupt change to the uniform state; the dynamics of the transient itself was found to be statistically stationary. For parameter values $\epsilon > \epsilon_c$ and for system sizes $L > 25$, these statistically stationary states were found to be high-dimensional ($D \geq 20$) and extensively chaotic [16, 17] as shown in Figure 3 by a linear dependence of D on L^2 . From the asymptotic slopes of the curve in Figure 3, an intensive dimension density $\delta = \lim_{L^2 \rightarrow \infty} \partial D / \partial(L^2)$ was obtained and then reexpressed as a dimension correlation length $\xi_\delta = \delta^{-1/d}$ for a $d = 2$ dimensional domain [16]. To test a speculation of Bayly *et. al.* [4] that knowledge of the experimentally-accessible two-point correlation length ξ_2 might provide knowledge of the dynamical length ξ_δ for a chaotic state of spiral defects (see also Ref. [17]), we computed ξ_2 and ξ_δ for several values of the parameter ϵ . For each ϵ value studied, the two-point correlation function $C(r)$ had a similar monotonically-decreasing but non-exponential form so we estimated ξ_2 by the position of the first zero crossing of $C(r)$ [9]. As shown in Figure 4(a), the two lengths agree within a factor of 1.5 or better but have opposing trends as ϵ increases, with ξ_δ decreasing over the range $0.07 \leq \epsilon \leq 0.12$ (corresponding to an increase in the fractal dimension D for a system of fixed size) and ξ_2 increasing slightly. Figure 4(a) suggests that neither a quantitative estimate nor a qualitative trend of the quantity ξ_δ can be obtained by measuring the length ξ_2 . Further analysis of more physiologically accurate models will be needed to relate ξ_δ and ξ_2 for the heart data of Bayly *et. al.* [4].

We also explored whether the fractal dimension D of the chaotic states was related to the statistics of the number $N(t)$ of spiral defects, *e.g.*, to its time average $\langle N \rangle$. The spirals were counted at successive times by locating their cores, which occur at those points (x, y) in the medium where the fields (u, v) take on the values (u^*, v^*) [5]. It has been shown previously that $N(t)$ is constant for $\epsilon < \epsilon_c$, in which case the time average $\langle N \rangle$ is fixed by the choice of initial condition [5]. For larger values $\epsilon > \epsilon_c$, $N(t)$ is no longer constant and its variance increases with ϵ until the fluctuations are comparable to the mean for $\epsilon \geq 0.095$ [6]. We found that the mean $\langle N \rangle$ scaled extensively with system area L^2 for both boundary conditions considered, so that the average defect density $n = \langle N \rangle / L^2$ was independent of

system size [6]. The ratio $d = D/\langle N \rangle$ of the Lyapunov dimension to the mean number of defects therefore defines an intensive quantity that measures the number of dynamical degrees of freedom associated with each defect on average. Because the extensive quantities D and $\langle N \rangle$ are of the form $\alpha L^2 + \beta$ rather than simply αL^2 (where α and β are constants), the ratio d asymptotes slowly to a constant value close to δ/n . We studied the dependence of $D/\langle N \rangle$ on area L^2 for two different values of ϵ , and found that we could estimate the ratio d accurately using a single system of side length $L = 40$ with periodic boundary conditions. For this system size, d increases smoothly with ϵ from less than 4 for $\epsilon \approx \epsilon_c$ to nearly 7 for $\epsilon \geq 0.095$ at which point it becomes approximately constant (see Figure 4(b)). Thus a fixed number of degrees of freedom can not, in general, be associated with each spiral defect in an excitable medium.

In summary, for a particular model of a two-dimensional excitable medium [5, 6], we have demonstrated by numerical calculations a *smooth* transition from short-lived transient dynamics to extensive, high-dimensional chaotic dynamics with increasing system size L . It is unclear whether the dynamics observed were rigorously chaotic or transient, but the non-periodic dynamics observed were statistically stationary on such long time scales (for large systems) that Lyapunov spectra of the dynamics were readily computed. Nevertheless, small systems do not exhibit sustained chaotic dynamics, which may help to explain the observation that fibrillation is sustainable only in sufficiently large animal hearts [3]. The transition to sustained chaos with system size is smooth rather than sudden for both boundary conditions studied. Our results differ from some previous time-series based work [10, 11] by using variational equations that allow all positive Lyapunov exponents and the Lyapunov dimension D to be calculated [12], without restriction to small values of D . In systems large enough to sustain non-periodic dynamics, there is not a quantitative or even a qualitative relation between the dimension correlation length ξ_δ and the widely used two-point length ξ_2 ; this complicates the interpretation of recent spatiotemporal data taken on fibrillating hearts [2, 4]. The mean Lyapunov dimension per defect of 3 to 7 suggests that the dynamics of excitable media with even a few defects, such as large animal hearts in

ventricular fibrillation [2], may be quite high-dimensional. High-dimensionality of the observed chaotic states further suggests that it will be difficult to stabilize such states by small variations of parameters [7], and may explain why some previous attempts to analyze the dynamics of fibrillation with low-dimensional time series embedding techniques have been inconclusive [18].

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FIGURES

FIG. 1. Density plot at time $t = 500$ of the slow field $v(t, x, y)$ for a spatiotemporal chaotic state with 31 spiral defects present. Dark and light regions correspond to values less or greater than the value $v^* = 0.484$ corresponding to the stable uniform state; the field values span the range $v \in [0, a - b]$. Parameter values were $\epsilon = 0.074$, $a = 0.84$, $b = 0.07$, $L = 50$, $\Delta x = 0.5$ and $\Delta t = 0.0037$.

FIG. 2. Fraction f of 100 random initial conditions still exhibiting non-periodic dynamics after a given time. **(a)** For bi-periodic boundary conditions with cutoff time $T_{np} = 100$ (\circ), 1000 (\square), or any larger value, the transition has the same form, with systems of side length $L > 25$ nearly always exhibiting sustained dynamics. **(b)** For no-flux boundary conditions with cutoff time $T_{np} = 100$ (\bullet) and $T_{np} = 1000$ (\blacksquare), we see that the median transient time depends on the cutoff. Comparison of these graphs shows that dynamics are substantially less likely to be sustained for a given time with no-flux boundary conditions. The parameters used were the same as in Figure 1.

FIG. 3. Lyapunov dimension D versus system area $A = L^2$ of Eq. 1 for the parameter values of Figure 1. Extensive (linear) scaling is found for two different boundary conditions, no-flux (\blacksquare) and periodic (\bullet). Data for $L \leq 25$ did not exist since all initial conditions decayed quickly to the uniform state. The dimension extrapolates to zero for a positive system size, so the ratio D/L^2 of the dimension to system area asymptotes slowly to the dimension density δ .

FIG. 4. **(a)** Dimension correlation length ξ_δ (\bullet) and two-point correlation length ξ_2 (\blacksquare) for different ϵ values. **(b)** Degrees of freedom per mean defect, $D/\langle N \rangle$ as a function of ϵ . This ratio increases steadily with ϵ above the transition to chaos at $\epsilon = \epsilon_c$, and varies little in the region $\epsilon > 0.095$ when the variance of the defect number $N(t)$ equals its mean.



