Some Results on Generalized Exponents

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Abstract

A digraph $G = (V, E)$ is primitive if, for some positive integer k, there is a $u \to v$ walk of length k for every pair u, v of vertices of V. The minimum such k is called the exponent of G, denoted $\exp(G)$. The exponent of a vertex $u \in V$, denoted $\exp(u)$, is the least integer k such that there is a $u \to v$ walk of length k for each $v \in V$. For a set $X \subseteq V$, $\exp(X)$ is the least integer k such that for each $v \in V$ there is a $X \to v$ walk of length k, i.e., a $u \to v$ walk of length k for some $u \in X$. Let $F(G, k) := \max\{\exp(X) : |X| = k\}$ and $F(n, k) := \max\{F(G, k) : |V| = n\}$, where $|X|$ and $|V|$ denote the number of vertices in X and V, respectively.

Recently B. Liu and Q. Li proved $F(n, k) = (n-k)(n-1)+1$ for all $1 \leq k \leq n-1$. In this paper, for each k, $1 \leq k \leq n-1$, we characterize the digraphs G such that $F(G, k) = F(n, k)$, thereby answering a question of R. Brualdi and B. Liu. We also find some new upper bounds on the (ordinary) exponent of G in terms of the maximum outdegree of G, $\Delta^+(G) = \max\{\mathrm{d}^+(u) : u \in V\}$, and thus obtain a new refinement of the Wielandt bound $(n-1)^2 + 1$.

1 Introduction and Notation

Let $G = (V, E)$ denote a digraph on n vertices. Loops are permitted but no multiple arcs. A $u \to v$ walk in G is a sequence of vertices $u, u_1, \ldots, u_p = v$ and a sequence of arcs $(u, u_1), (u_1, u_2), \ldots, (u_{p-1}, v)$ where the vertices and the arcs are not necessarily distinct. A closed walk is a $u \to v$ walk where $u = v$. A path is a walk with distinct vertices. A cycle is a closed $u \to v$ walk with distinct vertices except for $u = v$. The length of a walk W is the number of arcs in W. The girth s of G is the length of a shortest cycle in G. An r-cycle is a cycle of length r.

The notation $u \xrightarrow{k} v$ (resp. $u \overset{k}{\not\to} v$) is used to indicate that there is a $u \to v$ walk (resp. no $u \to v$ walk) of length k and the notation $u \xrightarrow{k^+} v$ to indicate that $u \xrightarrow{m} v$ for all $m \geq k$. Similarly, for a set $X \subseteq V$, the notation $X \xrightarrow{k} v$ (resp. $X \overset{k}{\not\to} v$) means that $u \xrightarrow{k} v$ for some $u \in X$ (resp. $u \overset{k}{\not\to} v$ for all $u \in X$), and the notation $X \xrightarrow{k^+} v$ means that $X \xrightarrow{m} v$ for all $m \geq k$.

A digraph G is strongly connected if there is a $u \to v$ path for each pair u, v of vertices of G. A digraph G is primitive if there exists some positive integer k such that $u \xrightarrow{k} v$ for every pair u, v of vertices of G. The minimum such k is called the exponent of G, denoted $\exp(G)$. A theorem of Moon and Moser [6] shows that almost all digraphs are primitive. The exponent of a vertex $u \in V$, denoted $\exp(u)$, is the least integer k such that $u \xrightarrow{k} v$ for each $v \in V$. It is easy to see that if $u \xrightarrow{k} v$ for each $v \in V$, then also $u \xrightarrow{k^+} v$ for each $v \in V$.

Interest in primitive digraphs grew out of work done by Perron [7] and Frobenius [3] at the turn of the century on the combinatorial properties of matrices with non-negative real entries. We say G is the digraph associated with a non-negative matrix $A = (a_{ij})$ if the adjacency matrix of G is the zero-nonzero pattern of A; that is, (i, j) is an arc of G if and only if $a_{ij} > 0$ in A. If A^k has all positive entries for some $k \geq 1$, then A is called primitive and the minimum such k is called the exponent of A, denoted $\exp(A)$. Clearly, $\exp(A) = \exp(G)$.

It is well-known that a strongly connected digraph G is primitive if and only if the greatest common divisor of the lengths of the cycles in G is one. Another equivalent formulation is given by Lewin [4]: A strongly connected graph G is primitive if and only if there is an integer $k \geq 1$ such that $u \xrightarrow{k,k+1} v$ for some vertices $u, v \in V(G)$.

Let $G = (V, E)$ be a primitive digraph on n vertices. In 1950, H. Wielandt [9] found that $\exp(G) \leq (n-1)^2 + 1$ and showed there is a unique (up to isomorphism) digraph, W_n, that attains this bound. The digraph $W_n = (V, E)$ is defined as follows: $V = \{u_i : 1 \leq i \leq n\}$ and $E = \{(u_i, u_{i+1}) : 1 \leq i \leq n-1\} \cup \{(u_{n-1}, u_1), (u_n, u_1)\}$. Since then much work has been done on finding exponents for various classes of digraphs and on improving the Wielandt upper bound by introducing parameters such as the girth s and the diameter d of G.

In 1990, R. Brualdi and B. Liu [2] generalized the concept of exponent by considering $X \to v$ walks, where $X \subseteq V$ and X does not necessarily have cardinality one. We use their notations in the following definitions.

Suppose G is primitive. For a set $X \subseteq V$, the exponent of the set X, denoted $\exp(X)$, is the least integer k such that $X \xrightarrow{k} v$ for each $v \in V$. It is easy to see that if $X \xrightarrow{k} v$ for each $v \in V$, then $X \xrightarrow{k^+} v$ for each $v \in V$. Let $F(G,k) := \max\{\exp(X) : |X| = k\}$, where $|X|$ denotes the number of vertices in X. Define $F(n,k) := \max\{F(G,k) : |V| = n\}$, where the maximum is taken over all primitive digraphs on n vertices.

Brualdi and Liu [2] prove that $F(W_n,k) = (n-1)(n-k)+1$ for all $1 \leq k \leq n-1$. When $k=1$, $F(G,1)$ is the ordinary exponent of G and so W_n is the unique digraph for which $F(W_n,1) = F(n,1) = (n-1)^2+1$. Brualdi and Liu also show $F(W_n,n-1) = F(n,n-1) = n$. They conjecture [2, Conjecture 5.6] that $F(W_n,k) = F(n,k)$ for all $2 \leq k \leq n-2$ and remark elsewhere [2, p. 486] that they believe, for each k, $1 \leq k \leq n-1$, W_n is the unique digraph G for which $F(G,k) = F(n,k)$. Their conjecture [2, Conjecture 5.6] was proved recently by B. Liu and Q. Li [5]. B. Liu and Q. Li also showed that for each integer k, $1 \leq k \leq n-1$, $F(G,k) \leq s(n-k-1)+n$ for all primitive digraphs G with n vertices and girth s.

In Section 2 of this paper, we prove that, for each k, $1 \leq k \leq n-2$, $F(G,k) = F(n,k)$ only for $G = W_n$. However, when $k = n-1$, we show there are digraphs G other than W_n such that $F(G,k) = F(n,k) = n$, thereby answering the question of R. Brualdi and B. Liu [2, p. 486]. Also in Section 2 we find sufficient conditions for $F(G,k) = s(n-k-1)+n$, where G is a primitive digraph with n vertices and girth s.

In Section 3, we obtain a new upper bound $\exp(G) \leq (n-\Delta^+ + 1)^2 + 1$ for all primitive digraphs G with maximum outdegree $\Delta^+(G) \leq (n+1)/2$, and a characterization of digraphs attaining this bound. We also obtain a new proof of the Dulmage-Mendelsohn bound $n + s(n-2)$ which implies the Wielandt bound $(n-1)^2+1$.

2 The Numbers $F(G,k)$

Let $G = (V,E)$ be a digraph and choose a vertex $u \in V$. Let $R_i(u) = \{v \in V : u \xrightarrow{i} v\}$ and $R'_i(u) = \{v \in V : v \xrightarrow{i} u\}$ for all $i \geq 1$. For $i = 0$, we define $R_i(u) = R'_i(u) = \{u\}$. We begin with the following lemma.

Lemma 1 *Let G be a primitive digraph on n vertices and let $u \in V$. Choose a positive integer r_1 and a non-negative integer r_2 and let $A_i = R_{ir_1+r_2}(u)$ (resp. $A'_i = R'_{ir_1+r_2}(u)$)*

for all $i \geq 0$. Then,

$$\left| A_i \setminus \bigcup_{j=0}^{i-1} A_j \right| \geq 1 \text{ provided } \left| \bigcup_{j=0}^{i-1} A_j \right| < n.$$

Similarly,

$$\left| A'_i \setminus \bigcup_{j=0}^{i-1} A'_j \right| \geq 1 \text{ provided } \left| \bigcup_{j=0}^{i-1} A'_j \right| < n.$$

Proof. It may be supposed that $i \geq 1$. Let $G^{r_1} = (V', E')$ be the digraph with $V' = V$ such that, for any two vertices u_1 and u_2 in V', $(u_1, u_2) \in E'$ if and only if $u_1 \xrightarrow{r_1} u_2$ in G. Let $X = \cup_{j=0}^{i-1} A_j$. Then $X \neq \emptyset$. Suppose $|X| < n$. Since G^{r_1} is strongly connected, there exist some $v_1 \in X$ and some $v_2 \in V \setminus X$ such that $(v_1, v_2) \in E'$; i.e., $v_1 \xrightarrow{r_1} v_2$ in G. Then $v_1 \in A_{i-1}$; otherwise if $v_1 \in \cup_{j=0}^{i-2} A_j$, then $v_2 \in R_{r_1}(v_1) \subseteq \cup_{j=0}^{i-1} A_j = X$, a contradiction. Therefore $v_2 \in R_{r_1}(v_1) \subseteq A_i$ and so $|A_i \setminus X| \geq |\{v_2\}| = 1$. By a similar argument we also have $|A'_i \setminus \cup_{j=0}^{i-1} A'_j| \geq 1$ provided $|\cup_{j=0}^{i-1} A'_j| < n$. □

The following lemma is an extension of the Dulmage-Mendelsohn bound to the generalized exponent $F(G, k)$.

Lemma 2 ([5]) *Let G be a primitive digraph with n vertices and girth s. Then,*

$$F(G, k) \leq s(n - k - 1) + n \text{ for each integer } k, 1 \leq k \leq n - 1.$$

The following two theorems characterize the extreme digraphs G for which $F(G, k) = F(n, k) = (n-1)(n-k) + 1$, where $1 \leq k \leq n - 1$.

Theorem 3 *Let G be a primitive digraph on n vertices and let $1 \leq k \leq n - 2$. Then, $F(G, k) = (n-1)(n-k) + 1$ if and only if $G = W_n$, the Wielandt digraph on n vertices.*

Proof. Brualdi and Liu [2] have shown $F(W_n, k) = (n-1)(n-k)+1$ for all integers k, $1 \leq k \leq n-2$. Now suppose $F(G, k) = (n-1)(n-k)+1$ for some integer k, $1 \leq k \leq n-2$. If $s \leq n-2$, then by Lemma 2, $F(G, k) \leq (n-2)(n-k-1)+n = (n-1)(n-k)+k+2-n < (n-1)(n-k)+1$, a contradiction. Thus $s = n-1$. There are only two primitive digraphs with girth $s = n-1$. If $G \neq W_n$, then every vertex $u \in V$ is on a cycle of length $s = n-1$.

Let X be a set of vertices of G with $|X| = k$. Choose $u \in V$. Let $A_i = R'_{is}(u)$ for all $0 \leq i \leq n-k$. By Lemma 1, $| \cup_{j=0}^{n-k} A_j| + |X| > n - k + k = n$ and so there is an A_i,

$0 \leq i \leq n-k$, which contains a vertex $v \in X$. Thus, $X \stackrel{is}{\to} u$ and so $X \stackrel{s(n-k)}{\longrightarrow} u$ since u is on some s-cycle. Since this holds for all vertices of G and all sets X with $|X| = k$, $F(G,k) \leq s(n-k) = (n-1)(n-k) < F(W_n,k)$, also a contradiction. □

Theorem 4 *Let $\mathcal{F}$ be the family of digraphs $G = (V,E)$ on n vertices satisfying the following three conditions:*

(i) G contains a hamiltonian cycle $u_0, u_1, u_2, \ldots, u_{n-2}, u_{n-1}, u_0$.

(ii) The remaining arcs of E consist of a non-empty subset of $\{(u_i,u_0) : 0 \leq i \leq n-2\}$.

(iii) The arcs in Condition (ii) are chosen so that G is primitive.

Then, $F(G,n-1) = n$ if and only if $G \in \mathcal{F}$.

Proof. Suppose $G \in \mathcal{F}$. Let $X = \{u_1, u_2, \ldots, u_{n-1}\}$. Then $X \stackrel{n-1}{\not\longrightarrow} u_{n-1}$ and hence $F(G,n-1) \geq n$. Since $F(G,n-1) \leq n$ by Lemma 2, we have $F(G,n-1) = n$.

Suppose $F(G,n-1) = n$. Let X be a set of vertices of G such that $|X| = n-1$ and $F(G,n-1) = \exp(X) = n$. Choose $u_0 \in V$ such that $u_0 \notin X$. If there is a cycle C which does not contain u_0, then all the vertices of C are in X and so $X \stackrel{n-|C|}{\longrightarrow} v$ for all $v \in V$, which contradicts $\exp(X) = n$. Therefore, u_0 is contained in all cycles of G. By [2, Lemma 5.2], $F(G,n-1) \leq \max\{n-s,t\}$, where s and t are, respectively, the lengths of shortest and longest cycles in G. Thus, G must contain a hamiltonian cycle, i.e., condition (i) holds. Let $u_0, u_1, \ldots, u_{n-2}, u_{n-1}, u_0$ be this hamiltonian cycle. We note that G contains no (u_i,u_j) arc for all $1 \leq i+1 < j \leq n-1$, or else $X \stackrel{n-1}{\to} u_{n-1}$ and hence $X \stackrel{n-1}{\to} u$ for each $u \in V$ which implies $\exp(X) < n$, a contradiction. Now since u_0 must be contained in all the cycles of G and G contains at least two cycles and is primitive we have $G \in \mathcal{F}$. □

Let $F(n,s,k) := \max_G\{F(G,k)\}$, where $1 \leq k \leq n-1$ and the maximum is taken over all primitive digraphs G with n vertices and girth s. It is shown in Lemma 2 that $F(n,s,k) \leq s(n-k-1)+n$. We are interested in finding digraphs for which equality holds in the previous inequality.

In the next lemma, we find for each integer k, where $1 \leq k \leq n-1$, and for each integer s, where $1 \leq s \leq n-1$ and $\gcd(n,s) = 1$, a primitive digraph on n vertices and girth s such that $F(G,k) = F(n,s,k) = s(n-k-1)+n$.

Lemma 5 *$F(n,s,k) = s(n-k-1)+n$ whenever $1 \leq k \leq n-1$, $1 \leq s \leq n-1$ and $gcd(n,s) = 1$.*

Proof. Let G be the primitive digraph consisting of the hamiltonian cycle $0, 1, \ldots, n-1, 0$ and the arc $(0, n-s+1)$ for some integer s, where $1 \leq s \leq n-1$ and $\gcd(n, s) = 1$. Thus, G has girth s. It is convenient to regard the vertices of G as integers modulo n represented by $\{0, 1, \ldots, n-1\}$. Therefore, it is easy to see that $R_{rs}(0) = \{0, s, 2s, \ldots, rs\} = \{ts : 0 \leq t \leq r\}$ and $R_{n-i+rs}(i) = R_{rs}(0) = \{ts : 0 \leq t \leq r\}$, where $1 \leq i \leq n$.

Let $X = \{1+ts : 0 \leq t \leq k-1\}$. Then, $R_{s(n-k-1)+n-1}(1+ts) = R_{n-(1+ts)+s(n-k-1+t)}(1+ts) = R_{s(n-k-1+t)}(0) = \{js : 0 \leq j \leq n-k-1+t\}$ which implies $R_{s(n-k-1)+n-1}(X) = \{js : 0 \leq j \leq n-2\} \neq V$. Thus, $\exp_G(X) > s(n-k+1)+n-1$ and so $F(G, k) = s(n-k-1)+n$ by Lemma 2. Therefore, $F(n, s, k) = s(n-k-1)+n$. □

Note that Lemma 5 is not satisfied by a unique digraph. In the next theorem we find sufficient conditions on a primitive digraph G with n vertices and girth s for which $F(G, k) = F(n, s, k) = s(n-k-1)+n$. These conditions are very similar to those given by J. Shao [8, Theorem 3.3] in characterizing primitive digraphs with exponent $n + s(n-2)$ and in fact imply the conditions given in [8]. We first mention the relevant lemma of J. Shao.

Lemma 6 ([8]) *Let G be a primitive digraph on n vertices and girth $s \geq 2$. Then, $\exp(G) = n + s(n-2)$ if and only if the following four conditions hold.*

(i) *$G = K_{s,n} \cup E'$, where $K_{s,n}$ is the digraph consisting of the hamiltonian cycle $1, 2, \ldots, n$ and the arc $(s, 1)$, and E' is a subset (possibly empty) of E where $E = \{(j, i) : s+1 \leq i \leq j \leq n\}$.*

(ii) *Let $n = r_1 > r_2 > \cdots > r_\lambda = s$ be the set of distinct cycle lengths in G. Then, $\gcd(n, s) = 1$ and $r_2, r_3, \ldots, r_\lambda$ are all multiples of s.*

(iii) *G contains a unique cycle of length s.*

(iv) *If $E' \neq \emptyset$, then $n-1$ is not a non-negative integral combination of $r_2/s, r_3/s, \ldots, r_{\lambda-1}/s$, i.e.,*

$$n - 1 \neq a_2 \frac{r_2}{s} + a_3 \frac{r_3}{s} + \cdots + a_{\lambda-1} \frac{r_{\lambda-1}}{s}$$

for all nonnegative integers $a_2, a_3, \ldots, a_{\lambda-1}$.

Theorem 7 *Let G be a primitive digraph with n vertices and girth s. If G satisfies conditions (i),(ii), and (iii) in Lemma 6 and condition (v) below, then,*

$$\exp(G) = s(n-2) + n \quad \textit{and} \quad F(G, k) = s(n-k-1) + n.$$

Condition (v): If $E' \neq \emptyset$, then $n-k+i-1$ is not a non-negative integral combination of $r_2/s, r_3/s, \ldots, r_{\lambda-1}/s$ for all $1 \leq i \leq \min\{t,k\}$, where $t = \lfloor (n-1)/s \rfloor$, i.e., $n-k+i-1 \neq \sum_{j=2}^{\lambda-1} a_j r_j/s$ for all non-negative integers $a_2, a_3, \ldots, a_{\lambda-1}$.

Proof. We first show that condition (v) implies condition (iv) of Lemma 6. If $k \leq t$ then we get condition (iv) from condition (v) immediately. Suppose $k > t$. We note $r_j \leq n-1-s$ and hence $r_j/s \leq \lfloor (n-1)/s \rfloor - 1 = t-1$, $2 \leq j \leq \lambda - 1$. Suppose condition (iv) does not hold, i.e., $\sum_{j=2}^{\lambda-1} a_j r_j/s = n-1$ for some non-negative integers $a_2, a_3, \ldots, a_{\lambda-1}$. Then $\sum_{j=2}^{\lambda-1} b_j r_j/s \in \{n-k, n-k+1, \ldots, n-k+t-1\}$ for some non-negative integers $b_2, b_3, \ldots, b_{\lambda-1}$ since each $r_j/s \leq t-1$, $2 \leq j \leq \lambda-1$, and $\{n-k, n-k+1, \ldots, n-k+t-1\}$ contains t consecutive integers. Thus, condition (v) is violated.

Therefore, condition (v) implies condition (iv) and hence if G satisfies conditions (i),(ii),(iii), and (v), then $\exp(G) = n + s(n-2)$ by Lemma 6.

Now suppose conditions (i), (ii), (iii), and (v) are satisfied. Then from condition (i) $G = K_{s,n} \cup E'$. It is easy to verify that $s+1 \overset{n+s(n-2)}{\longrightarrow} n$ and $s+1 \overset{n-1+s(n-2)}{\not\longrightarrow} n$. Let $X = \{1+is : 1 \leq i \leq k\}$. Suppose $1+is \overset{n-1+s(n-k-1)}{\longrightarrow} n$ for some integer i, $1 \leq i \leq k$. Then clearly $1+s \overset{n-1+s(n-k+i-2)}{\longrightarrow} n$. Let $t = \lfloor (n-1)/s \rfloor$.

Case 1: $i > t$. Then $1+s \overset{ts}{\longrightarrow} 1+(t+1)s \overset{(i-t-1)s}{\longrightarrow} 1+is \overset{n-1+s(n-k-1)}{\longrightarrow} n$. Since $1+(t+1)s$ lies on the cycle of length s, we have $1+s \overset{n-1+s(n-2)}{\longrightarrow} n$, a contradiction.

Case 2: $i \leq t$. By assumption there is an $s+1 \to n$ walk W of length $n-1+s(n-k+i-2)$. If any vertex of W lies on the cycle of length s, then we also have $1+s \overset{n-1+s(n-2)}{\longrightarrow} n$, a contradiction. Therefore, $W \cap \{1,2,\ldots,s\} = \emptyset$ and so there exist non-negative integers $a_2, a_3, \ldots, a_{\lambda-1}$ such that $n-s-1+\sum_{j=2}^{\lambda-1} a_j r_j = n-1+s(n-k+i-2)$, i.e., $\sum_{j=2}^{\lambda-1} a_j r_j/s = n-k+i-1$, contradicting condition (v). □

We note that while conditions (i),(ii),(iii), and (v) are sufficient for $F(G,k) = s(n-k-1)+n$, they are not necessary. Consider the digraph G consisting of a path $u_1, u_2, \ldots, u_7$ and arcs $(u_2,u_1), (u_5,u_1), (u_7,u_3)$. Then G contains cycles of lengths 2 and 5 and so is primitive, and G does not have a hamiltonian cycle violating condition (i) of Lemma 6. Let $k=5$ and let $X = \{u_2, u_3, u_5, u_6, u_7\}$. Then, $X \overset{s(n-k-1)+n-1}{\not\longrightarrow} u_7$, i.e., $X \overset{8}{\not\to} u_7$ and so $F(G,k) = s(n-k-1)+n = 9$. It seems difficult to give a complete characterization of the primitive digraphs G with n vertices and girth s such that $F(G,k) = s(n-k-1)+n$.

3 Exponent results based on maximum outdegree

In this section, we consider the exponent of a primitive digraph with n vertices and maximum outdegree $\max\{\mathrm{d}^+(u) : u \in G\} = \Delta^+$. Since $\exp(G) = \exp(G^t)$, where $V(G^t) = V(G)$ and $E(G^t) = \{(u,v) : (v,u) \in E(G)\}$, similar results hold for primitive digraphs with n vertices and maximum indegree Δ^-. For simplicity of notation, in the following we let $\Delta := \Delta^+$.

Theorem 8 *Suppose G is a primitive digraph on n vertices and $u \in V$ is a vertex with outdegree $\mathrm{d}^+(u)$. Let r be the length of a shortest cycle containing u. Then,*

$$\exp(u) \le r(n - \mathrm{d}^+(u)) + 1.$$

Proof. Let $A_i = R_{ir+1}(u)$ for all $i \ge 0$. Since $|A_0| = \mathrm{d}^+(u)$, and by Lemma 1 $\left|A_i \setminus \bigcup_{j=0}^{i-1} A_j\right| \ge 1$ for $i \ge 1$ provided $\left|\bigcup_{j=0}^{i-1} A_j\right| < n$, we obtain $|\cup_{i=0}^{n-\mathrm{d}^+(u)} A_i| \ge \min\{n, |A_0| + n - \mathrm{d}^+(u)\} = n$. Thus, $|\cup_{i=0}^{n-\mathrm{d}^+(u)} A_i| = n$ and $u \stackrel{1+r(n-\mathrm{d}^+(u))}{\longrightarrow} v$ for each vertex $v \in V$ since u lies on a r-cycle in G. Therefore $\exp(u) \le r(n - \mathrm{d}^+(u)) + 1$. □

Corollary 9 *Suppose G is a primitive digraph on n vertices and $u \in V$ is a vertex with outdegree Δ. Let r be the length of a shortest cycle containing u. Then,*

$$\exp(G) \le r(n - \Delta) + n.$$

Proof. By Theorem 8, $\exp(u) \le r(n - \Delta) + 1$, i.e., $u \stackrel{r(n-\Delta)+1^+}{\longrightarrow} v$ for each $v \in V$. Let w be any vertex of G and let P be a shortest $w \to u$ path. Then $w \stackrel{|P|}{\to} u \stackrel{r(n-\Delta)+1}{\longrightarrow} v$, which implies $w \stackrel{r(n-\Delta)+n}{\longrightarrow} v$ for each $v \in V$ since $|P| \le n - 1$. Thus, $\exp(G) \le r(n - \Delta) + n$. □

The following corollary is a new proof of the Dulmage-Mendelsohn bound.

Corollary 10 *Let G be a primitive digraph on n vertices and girth s. Then,*
$\exp(G) \le s(n-2) + n \le (n-1)^2 + 1$.

Proof. Let C be a cycle of length s. Since G is primitive, there is a vertex $u \in V(C)$ with outdegree $\mathrm{d}^+(u) \geq 2$. Thus, by Theorem 8 and Corollary 9, $\exp(G) \leq \exp(u)+n-1 \leq s(n-\mathrm{d}^+(u))+n \leq s(n-2)+n \leq (n-1)(n-2)+n = (n-1)^2+1$. $\square$

Let $a_1 < a_2 < \cdots < a_p$ be positive integers. The Frobenius-Schur index, $\phi(a_1, a_2, \ldots, a_p)$, is the least integer such that the equation $x_1a_1 + x_2a_2 + \cdots + x_pa_p = n$ has a solution in non-negative integers $x_1, x_2, \ldots, x_p$ for all $n \geq \phi(a_1, \ldots, a_p)$. We first state a result due to A. Brauer in 1942.

Lemma 11 ([1]) *Let m be a positive integer. Then,*

$$\phi(m, m+1, \ldots, m+j-1) = m\left(1 + \left\lfloor \frac{m-2}{j-1} \right\rfloor\right).$$

Theorem 12 *Let G be a primitive digraph on n vertices with maximum outdegree $\Delta \leq (n+1)/2$. Then,*

$$\exp(G) \leq (n-\Delta+1)^2+1.$$

Proof. Let $u \in V$ be a vertex with outdegree $\mathrm{d}^+(u) = \Delta$ and r be the length of a shortest cycle C containing u. It is easy to verify that $r \leq n-\Delta+1$. Let $Y = R_1(u) = \{b_1, b_2, \ldots, b_\Delta\}$, $b_1 \in V(C)$, and $R_1(b_1) \cap V(C) = \{v\}$. By Corollary 9, $\exp(G) \leq r(n-\Delta)+n$. We consider two cases.

Case 1: $r \leq n-\Delta$. Then,

$$\exp(G) \leq (n-\Delta)^2 + n < (n-\Delta+1)^2+1.$$

Case 2: $r = n-\Delta+1$. Then $V = V(C) \cup Y$. Let P_i be a shortest $b_i \to v$ path, $1 \leq i \leq \Delta$. Note that there is no arc from any vertex in Y to any vertex of C other than b_1 and v since r is the length of a shortest cycle containing u. Note also that because $V(C) \cup Y = V$, $|P_i|$ is the number of vertices of P_i which are in the set Y. Let $t = \max_{1 \leq i \leq \Delta}\{|P_i|\}$. If $t = 1$, then by Theorem 8, $\exp(G) \leq n-\Delta+\exp(u) \leq n-\Delta+1+r(n-\Delta) = (n-\Delta+1)^2$. Suppose $t \geq 2$. Let $w \in C \setminus Y$. Then w is contained in cycles of lengths $n-\Delta+1, n-\Delta+2, \ldots, n-\Delta+t$ and $\exp(w) \leq n-\Delta+\phi(n-\Delta+1, n-\Delta+2, \ldots, n-\Delta+t) = n-\Delta+(n-\Delta+1)(1+\lfloor (n-\Delta-1)/(t-1) \rfloor)$, applying Lemma 11. Since $v \in C \setminus Y$ we also have $\exp(G) \leq n-\Delta+t+(n-\Delta+1)(1+\lfloor (n-\Delta-1)/(t-1) \rfloor)$.

Suppose $t \geq 3$. Then $n \geq 5$ since $t \leq \Delta$ and $\Delta \leq (n+1)/2$ by assumption. Thus, $\exp(G) \leq n-\Delta+3+(n-\Delta+1)(1+\lfloor (n-\Delta-1)/2 \rfloor) \leq 2(n-\Delta+2)+(n-\Delta)^2/2-1/2 = (n-\Delta+1)^2+5/2-(n-\Delta)^2/2 < (n-\Delta+1)^2+1$, since $n \geq 5$.

Suppose $t = 2$. Then $\exp(G) \leq n-\Delta+2+(n-\Delta+1)(n-\Delta) = (n-\Delta+1)^2+1$. □

We next characterize those primitive digraphs with n vertices and maximum outdegree $\Delta \leq (n+1)/2$ which have exponent $\exp(G) = (n-\Delta+1)^2+1$. For $\Delta = 2$, only one digraph, W_n, satisfies $\exp(G) = (n-1)^2+1$.

Theorem 13 *Let $\mathcal{F}$ be the family of primitive digraphs $G = (V, E)$ satisfying the following four conditions:*

(i) $V = \{u_i : 1 \leq i \leq n-\Delta\} \cup B_1 \cup B_2$, *where B_1, B_2 are non-empty and* $2 \leq |B_1|+|B_2| = \Delta \leq (n+1)/2$.

(ii) $E_0 \subseteq E \subseteq E_0 \cup E_1$, *where* $E_0 = \{(u_i, u_{i+1}) : 1 \leq i \leq n-\Delta-1\} \cup \{(u_{n-\Delta}, b) : b \in B_1 \cup B_2\} \cup \{(b, u_1) : b \in B_1\}$ *and* $E_1 = \{(b_i, b_j) : b_i \in B_1 \cup B_2, b_j \in B_1\} \cup \{(u_{n-\Delta-1}, b) : b \in B_2\}$.

(iii) $R_2(B_2) = \{u_1\}$ *and* $R_2(B_1) \subseteq \{u_1, u_2\}$, *where $R_2(B_i)$ denotes the set of vertices of G which can be reached by a walk of length 2 from some vertex in B_i.*

(iv) Either there exist some $b_1 \in B_1$ and $b_2 \in B_2$ such that the subdigraph of G induced by the set of vertices $\{b_1, b_2, u_1, u_2, \cdots, u_{n-\Delta}\}$ is isomorphic to $W_{n-\Delta+2}$, the Wielandt digraph of order $n-\Delta+2$, or there exists some $b_1 \in B_1$ such that there is no (b, b_1) arc for all $b \in B_2$.

Suppose G is a primitive digraph with n vertices and maximum outdegree Δ, where $2 \leq \Delta \leq (n+1)/2$. Then, $\exp(G) = (n-\Delta+1)^2+1$ *if and only if $G \in \mathcal{F}$.*

Proof. If $G \in \mathcal{F}$, then G is primitive and by conditions (i) and (ii), G contains cycles of lengths $n-\Delta+1$ and $n-\Delta+2$, has a vertex $u = u_{n-\Delta}$ with $\mathrm{d}^+(u) = \Delta \leq (n+1)/2$ and the shortest cycle containing u has length $n-\Delta+1$. Thus, by Theorem 12, $\exp(G) \leq (n-\Delta+1)^2+1$. By condition (iii), G contains no cycle with length other than $n-\Delta+1$ or $n-\Delta+2$. By conditions (iii) and (iv), $R_2(b_2) = \{u_1\}$ and so either there is no $b_2 \to b_2$ walk or no $b_2 \to b_1$ walk of length $\phi(n-\Delta+1, n-\Delta+2)+n-\Delta+1 = (n-\Delta)(n-\Delta+1)+n-\Delta+1 = (n-\Delta+1)^2$. Thus, $\exp(G) \geq (n-\Delta+1)^2+1$ whence we conclude $\exp(G) = (n-\Delta+1)^2+1$.

On the other hand, suppose $\exp(G) = (n-\Delta+1)^2+1$. Let $u \in V$ be a vertex with outdegree Δ and let $Y = R_1(u) = \{b_i : 1 \leq i \leq \Delta\}$. By the proof of Theorem 12, a shortest cycle containing u has length $n-\Delta+1$. Let $C : b_1, u_1, u_2, \cdots, u_{n-\Delta} = u$ be such a shortest cycle. Also, by the proof of Theorem 12, $\max_{1 \leq i \leq \Delta} |P_i| = 2$, where P_i is a shortest $b_i \to u_1$ path in G. Let $B_j := \{b_i : |P_i| = j\}$ for $1 \leq j \leq 2$. Then, B_1 and B_2 are non-empty, and so condition (i) holds.

Again, by the proof of Theorem 12, no vertex $w \in C \setminus Y$ can be contained in cycles of length greater than $n-\Delta+2$ or length less than $n-\Delta+1$, so condition (iii) holds and G contains no arc in the set $\{(u_i,u_j) : 1 \leq i \leq n-\Delta, j \neq i+1\} \cup \{(b,u_i) : b \in B_1, 2 \leq i \leq n-\Delta\} \cup \{(b,u_i) : b \in B_2, 1 \leq i \leq n-\Delta\} \cup \{(u_i,b) : b \in B_2, 1 \leq i \leq n-\Delta-2\} \cup \{(u_i,b) : b \in B_1, 1 \leq i \leq n-\Delta-1\}$, so condition (ii) holds.

If condition (iv) does not hold, then $\{(u_{n-\Delta-1},b) : b \in B_2\} \subseteq E$. Thus every vertex in B_2 is contained in cycles of lengths $n-\Delta+1$ and $n-\Delta+2$. Also for each vertex $b_1 \in B_1$ there is a vertex $b_2 \in B_2$ such that $(b_2,b_1) \in E$. Thus, $u \overset{\phi(n-\Delta+1,n-\Delta+2)}{\longrightarrow} v$ for each pair u, v of vertices of B_2 and so $b_2 \overset{1+\phi(n-\Delta+1,n-\Delta+2)}{\longrightarrow} b_1$ for each vertex $b_2 \in B_2$ and each vertex $b_1 \in B_1$. Then it is not difficult to verify that $\exp(G) \leq n-\Delta+1+\phi(n-\Delta+1,n-\Delta+2) = (n-\Delta+1)^2$, a contradiction. Thus, condition (iv) holds and therefore $G \in \mathcal{F}$. □

Remark. If $\Delta > (n+1)/2$, then we do not necessarily have $\exp(G) \leq (n-\Delta+1)^2+1$ (Theorem 12). Here is a counterexample. Let $G = (V,E)$, $V = \{1,2,\cdots,n\}$ and $E = \{(i,i+1) : 1 \leq i \leq n-1\} \cup \{(n-1,1)\} \cup \{(n,i) : 1 \leq i \leq \Delta\}$. Then vertex n is contained in cycles of lengths $n-\Delta+1, n-\Delta+2, \ldots, n-1, n$. We note by Lemma 11 $\phi(n-\Delta+1, n-\Delta+2, \ldots, n-1, n) = n-\Delta+1$ since $\Delta \geq (n+1)/2$. Then, $\exp(G) \geq \exp(1) = n-1+\phi(n-\Delta+1, n-\Delta+2, \cdots, n-1, n) = 2n-\Delta > (n-\Delta+1)^2+1$ provided that Δ is sufficiently large.

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B. Liu, Generalized exponents of Boolean matrices, Proceedings of international workshop on discrete mathematics and algorithms, 181-191, 1994.

B. Liu and Z. Bo, The k^{th} upper multiexponent of primitive matrices, Graphs and Combinatorics (Japan), to appear.

References

[1] A. Brauer, On a problem of partitions, *Amer. J. Math.* **64** (1942), 299-312.

[2] R. A. Brualdi and B. Liu, Generalized exponents of primitive directed graphs, *J. Graph Theory* **14** (1990), 483–499.

[3] G. Frobenius, Über Matrizen aus nicht negativen Elementen, *Sitzungsberichte der Königlich Preußischen Akademie der Wissenschaften zu Berlin* (1912), 456-477.

[4] M. Lewin, On exponents of primitive matrices, *Numerische Mathematik* **18** (1971), 154-161.

[5] B. Liu and Q. Li, On a conjecture about the generalized exponent of primitive matrices, *J. Graph Theory* **18** (1994), 177-179.

[6] J. W. Moon and L Moser, Almost all (0,1) matrices are primitive, *Studia Scientiarum Mathematicarum Hungarica* **1** (1966), 153-156.

[7] O. Perron, Zur Theorie der Matrices, *Mathematische Annalen* **64** (1907), 248-263.

[8] J. Shao, On the exponent of a primitive digraph, *Linear Algebra Appl.* **64** (1985), 21–31.

[9] H. Wielandt, Unzerlegbare, nicht negative Matrizen, *Math. Zeit.* **52** (1950), 642-645.