MOSCOW MATHEMATICAL JOURNAL
Volume 5, Number 2, April–June 2005, Pages 415–441

ISOMONODROMIC DEFORMATIONS OF $\mathfrak{sl}(2)$ FUCHSIAN SYSTEMS ON THE RIEMANN SPHERE

SERGEY OBLEZIN

To the memory of A. A. Bolibruch

Abstract. This paper is devoted to the two geometric constructions provided by the isomonodromic method for Fuchsian systems. We develop the subject in the sense of geometric representation theory following Drinfeld's ideas (see [8]). Thus we identify the initial data space of the $\mathfrak{sl}(2)$ Schlesinger system with the moduli space of the Frobenius–Hecke (FH-)sheaves originally introduced by Drinfeld (see [6]). First, we perform the procedure of separation of variables in terms of the Hecke correspondences between moduli spaces. In this way we present a geometric interpretation of the Flashka–McLaughlin, Gaudin and Sklyanin formulas. In the second part of the paper, we construct the Drinfeld compactification of the initial data space and describe the compactifying divisor in terms of certain FH-sheaves. Finally, we give a geometric presentation of the dynamics of the isomonodromic system in terms of deformations of the compactifying divisor and explain the role of apparent singularities for Fuchsian equations. To illustrate the results and methods, we give an example of the simplest isomonodromic system with four marked points known as the Painlevé-VI system.

2000 Math. Subj. Class. 15A54, 32G02, 34B02.

Key words and phrases. Isomonodromic deformation, separation of variables, the Drinfeld compactification, the Frobenius–Hecke sheaves, the Painlevé-VI equation.

1. Introduction

In this work we present a geometric interpretation of the isomonodromic deformation of a non-resonant $\mathfrak{sl}(2)$ Fuchsian system on the Riemann sphere; such deformation is called the Schlesinger system. Given a generic Fuchsian differential equation of order N with singularities at $S := \{a_1, \ldots, a_n\}$ on $\mathbb{P}^1$, let us put it into an isomonodromic analytical family in the following way.

Received February 13, 2004.

Supported in part by the CRDF Grant RM1-2545, by the program for support of scientific schools NSh-1999.2003.2 and by the RFBR Grant 04-01-00642.

Consider the Fuchsian system of differential equations

$$\frac{d}{dz}Y(z) = \left(\sum_{i=1}^{n} \frac{B_i}{z-a_i}\right) Y(z)$$

with matrix coefficients $B_i \in \mathrm{Mat}(N, \mathbb{C})$. Note from the very beginning that we consider only non-resonant systems, that is we assume $\lambda_i^{(a)} - \lambda_i^{(b)} \notin \mathbb{N}$ for the eigenvalues $\{\lambda_i^{(a)}\}$ of matrices B_i, $i = 1, \ldots, n$. Let $Y(z)$ be the fundamental solution of this system. Then consider the equation $\partial_z Y(z) = L(z)Y(z)$, where

$$L(z) = \sum_{i=1}^{n} \frac{B_i(a_1, \ldots, a_n)}{z-a_i} dz$$

simultaneously with the following (isomonodromic) condition for the coefficients $\{B_i(a_1, \ldots, a_n), i = 1, \ldots, n\}$

$$dB_i(a_1, \ldots, a_n) = \sum_{j=1}^{n} [B_j, B_i] \, d\log(a_i - a_j)$$

called the Schlesinger equation. The last equation indicates the complete integrability condition $d\omega = \omega \wedge \omega$ for the matrix-valued 1-form $\omega = dY(z) \cdot Y(z)^{-1}$, or, in other words, it is the zero-curvature condition for the logarithmic connection $\nabla := d - \omega$ in a trivial rank N bundle on the configuration space $\mathbb{P}^1 \times \mathbb{C}^n$. Such systems were investigated originally by Schlesinger [25] and later algebraic aspects were considered by Flashka and Newell [10], Jimbo, Miwa (see [19]). Geometric aspects of the isomonodromic systems were thoroughly investigated by Röhrl [23], Bolibruch [1], [4], Hitchin [18], Arinkin and Lysenko [2] in various aspects. Actually we develop the ideas of Drinfeld (see [7], [8]) to study the isomonodromy problem from the point of view of geometric representation theory; in particular, we generalize the results of [2] to the case of arbitrary number of singularities.

In this paper, we describe the fundamental matrix of our Fuchsian system of rank $N = 2$ in terms of horizontal sections of a certain rank 2 bundle $\mathcal{L}$ with respect to the logarithmic $\mathfrak{sl}(2)$-connection on the Riemann sphere $\mathbb{P}^1$

$$\nabla \colon \mathcal{L} \to \mathcal{L} \otimes \Omega^1_{\mathbb{P}^1}(a_1 + \cdots + a_n)$$

with $\mathrm{Res}_{a_i} \nabla = B_i$. We associate to the $\Omega^1(\sum a_i)$-valued operator $L(z)$ an algebraic map from the rational curve $\mathbb{P}^1$ to the group $G = \mathrm{SL}(2)$ with a module $\mathfrak{M} = a_1 + \cdots + a_n$ (see [26]). In fact, we identify the space of logarithmic $\mathfrak{sl}(2)$-connections ∇ with the space of maps $(\mathbb{P}^1 \setminus S) \to \mathscr{F} = G/B$ (see [19]). Precisely, we construct a distinguished complete eigen-flag for ∇ at $x \in \mathbb{P}^1$ and in this way we obtain a G-representation induced from the stabilizer of the eigen-flag.

Consider the co-adjoint representation of the group $\mathrm{SL}(2) \to \mathrm{End}(\mathfrak{sl}(2)^*)$, $X \mapsto \mathrm{ad}^*_X$; here we assume that the coefficients B_i lie in (co)adjoint $\mathfrak{sl}(2)$-orbits $\mathscr{O}_i$. Fixing the eigenvalues of the residues B_i, we fix the appropriate $\mathfrak{sl}(2)$-orbits. Every $\mathfrak{sl}(2, \mathbb{C})$-orbit is a 2-dimensional non-compact variety with a natural symplectic form which in the co-adjoint representation is $\omega_\xi(X, Y) = -\langle \mathrm{ad}^*_X, Y\rangle$ for any $\xi \in \mathfrak{sl}(2)^*$. The symplectic quotient of the direct product of $\mathrm{SL}(2, \mathbb{C})$-orbits is a symplectic variety of the following dimension

$$\dim\left(\prod_{i=1}^{n}\mathscr{O}_i/\!/\operatorname{SL}(2,\,\mathbb{C})\right) = n\cdot\dim\mathscr{O}_i - 2\cdot\dim\operatorname{SL}(2,\,\mathbb{C}) = 2(n-3).$$

Identify the symplectic quotient with the initial data space and present it as an open subset of the coarse moduli space $\mathcal{M}_n(2)$ of the data

$$(\mathcal{L},\,\nabla;\;\;\phi\colon\,\det\mathcal{L}\simeq\mathcal{O}_{\mathbb{P}^1};\;\;\lambda_1,\,\dots,\,\lambda_n),$$

consisting of a rank 2 bundle $\mathcal{L}$ on $\mathbb{P}^1$ with fixed determinant and a connection $\nabla\colon\mathcal{L}\to\mathcal{L}\otimes\Omega^1_{\mathbb{P}^1}(a_1+\cdots+a_n)$ such that the eigenvalues of the residues of the connection $\operatorname{Res}_{a_i}\nabla = B_i$ at a_i, $i=1,\,\dots,\,n$, are $\{\lambda_i,\;-\lambda_i\}$. Note that in this work we consider only coarse moduli spaces and mean this even when the word “coarse” is omitted.

Let us emphasize here that our construction naturally entails the identification of the phase space of a Fuchsian system of differential equations (initial data space) with the phase space of the isomonodromic deformation of the system. In this way, one may understand this paper as a geometric presentation of the *isomonodromic method* of study of Fuchsian differential equations.

There is a natural and important problem of choosing convenient dynamical variables for our system and describing the coordinates $\{x_i,\,p_i\}$, $i=1,\,\dots,\,n-3$, on the phase space $\mathcal{M}_n$. The analytical part of the problem was done by Flashka and McLaughlin [9] for the Toda system; the investigation of geometric aspects of separation of variables was initiated by Drinfeld [5], [7], and Krichever (see [20]). Besides Gaudin (see [13]) and Sklyanin (see [27]) constructed separation of variables for various quantum integrable systems. Summarizing these ideas and methods let us emphasize that the separation of variables and related geometry of moduli spaces are closely connected with the (quantum) inverse scattering method and thus they provide a profound development in representation theory of (quantized) Poisson–Lie groups (see [14]). In his remarkable thesis [6], Drinfeld introduced original geometric objects that he used to prove the Langlands conjecture; they are elliptic modules and the Frobenius–Hecke sheaves. In our paper, we follow the geometric ideas of Drinfeld and apply them to investigate the isomonodromy deformation of Fuschsian systems of rank two. We pay a special attention to the bounds of application for the procedure of separation of variables. In particular, we construct the Drinfeld compactification (see [2], [8]) of the phase space of the system and investigate the cases in which the classical procedure of separation of variables does not work. Such analysis gives a hope to explain, analyze, and classify the Bolibruch counterexamples for the Riemann–Hilbert correspondence (see [1], [4]); this will be discussed elsewhere.

The paper is organized in the following way. In the first part (Section 3), we construct the geometric Darboux coordinates on $\mathcal{M}_n(2)$ and compare the result with calculations from [9], [13], [27]. Let us outline the points where we work in greater generality with respect to the classical works [9], [13], [27].

First of all we omit the assumption of triviality of the bundle $\mathcal{L}$ though fixing its determinant by a horizontal isomorphism $\phi\colon\,\det\mathcal{L}\simeq\mathcal{O}$. We consider the moduli space $\mathcal{M}_n(2)$ of pairs $(\mathcal{L},\,\nabla)$ equipped with ϕ and with fixed eigenvalues of the residues of the connection. We construct a parametrization of the moduli space in the sense of Drinfeld, and in this way we give a geometric interpretation of the classical explicit formulas from [9], [13], [27]. For these purposes we have to impose a notion of *stability* for our configurations. We discuss it and investigate the Schlesinger system for (semi)stable configurations.

Second, note that the recipe for the separation of variables was given for an arbitrary rank N of a trivial bundle $\mathcal{L}$ and in this paper we restrict ourselves to the case of rank 2. We consider the stratification of the moduli space $\mathcal{M}_n(N)$ by locally closed strata $\mathscr{M}^k(N)$, $k \geq 0$, corresponding to the bundles $\mathcal{L} \simeq \mathcal{O}(k) \oplus \mathcal{O}(-k)$ and obtain the classical formulas on the largest stratum $\mathscr{M}^0(N)$. Our construction is also valid for any rank N of $\mathcal{L}$ on the stratum $\mathscr{M}^0(N)$. One should be careful in the case $N > 2$ for nontrivial bundles, because on the strata $\mathscr{M}^k(N)$ for $k \neq 0$, $N > 2$, one encounters Bolibruch's counterexamples to the Riemann–Hilbert correspondence; these are so-called B-representations of the monodromy (see [1], [4]). In this paper we restrict ourselves to the case $N = 2$ and investigate the coarse moduli space $\mathcal{M}_n(2)$; the B-representations of the monodromy we are going to discuss elsewhere.

Third, the classical procedure is given under the assumption that the coordinates $\{x_1, \ldots, x_{n-3}\}$ on $\mathcal{M}_n(2)$ are distinct and do not coincide with the singularities $\{a_1, \ldots, a_n\}$. It is the most generic configuration of the coordinates corresponding to the complements of the diagonals $\{x_i = x_j\}$ and the divisors $\{x_i = a_j\}$. In this work we describe the behavior of the system on the divisors $\{x_i = a_j\}$ and show what obstacles arise at the diagonals. There are methods of resolution of diagonals and this problem is completely solved (see [12]); to demonstrate these methods we give the resolution in the simplest case when only two coordinates coincide. From the above discussion it follows that we give the geometric description of $\mathcal{M}_n(2)$ except for a certain locus of codimension two.

The second part of the paper (Section 4) contains the Drinfeld compactification of the phase space $\mathcal{M}_n(2)$. Below, we present two naïve recipes to complete the initial data space of the isomonodromic deformation. They are very simple and explicit; they are however both generalized by our construction of the compactification and illustrate it.

Consider the cotangent bundle Ω on $\mathbb{P}^1$ and denote by $\mathrm{Tot}(\mathbb{P}^1, \Omega(\sum a_i))$ the total space of the bundle $\Omega(a_1 + \cdots + a_n)$. Our construction of the Darboux coordinates provides the description of the initial data space $\mathcal{M}_n(2)$ in terms of the $(n-3)$rd symmetric power of the non-compact surface $K_n := \mathrm{Tot}(\mathbb{P}^1, \Omega(\sum a_i))$ (see for example [15]). Precisely, consider the compact surface $\overline{K}_n = \mathbb{P}(\mathcal{O} \oplus \Omega(\sum a_i)) = s_\infty \sqcup K_n$ for the infinite section s_∞ and let $F_i := \Omega(\sum a_i)|_{a_i} \subset \overline{K}_n$. The fibres F_i, $i = 1, \ldots, n$, are trivialized by the residue map $R\colon F_i \xrightarrow{\sim} \mathbb{C}$. Let us blow-up the surface $\overline{K}_n$ at $2n$ points $R^{-1}(\lambda_i^\pm) \in F_i$ for $\{\lambda_i^\pm\} = \{\lambda_1, 1-\lambda_1, \lambda_2, -\lambda_2, \ldots, \lambda_n, -\lambda_n\}$ and consider the non-compact surface

$$K_n' := (\mathcal{B}l_{R^{-1}(\pm\lambda_i)} \overline{K}_n) \setminus (s_\infty \cup \widetilde{F}_1 \cup \cdots \cup \widetilde{F}_n),$$

where $\widetilde{F}_i$ are the proper pre-images of the fibers F_i, $i = 1, \ldots, n$. There is a map

$\mathcal{M}_n(2) \to (K'_n)^{(n-3)} := (K'_n)^{n-3}/\mathfrak{S}_{n-3}$, which is an isomorphism at the generic point, and we thoroughly describe points where it is not an isomorphism.

There is no ordering on the set of variables $\{x_i, p_i\}$, $i = 1, \ldots, n-3$, and hence we have to consider either the quotient $(K'_n)^{(n-3)} := (K'_n)^{n-3}/\mathfrak{S}_{n-3}$, or the $(n-3)!$-covering

$$\widetilde{\mathcal{M}_n(2)} \simeq (K'_n)^{n-3}.$$

On the covering $\widetilde{\mathcal{M}_n(2)}$ we have a natural symplectic form

$$\varpi = \sum_{i=1}^{n-3} dx_i \wedge dp_i,$$

and it equips $\widetilde{\mathcal{M}_n(2)}$ with the structure of a symplectic variety. It is natural to complete $\mathcal{M}_n(2)$ with a pole-divisor of the symplectic form ϖ. We present the compactifying divisor as the pole-divisor of ϖ in 5.2.

Another natural way to regard the compactification problem is the following one. Consider an algebraic curve C on K_n defined by the equation $R(z, \lambda) = 0$ for

$$R(z, \lambda) := (\det L(z) - \lambda \cdot \mathrm{Id}).$$

This curve is usually called *spectral*; the genus of C is $n-3$ which equals half of the dimension of the phase space. However, C is not preserved by the isomonodromic deformation, and this fact implies the natural completion of the phase space with a limit cycle of the spectral curve C.

It is significant, that it is possible to perform a natural compactification of the initial data space $\mathcal{M}_n(2)$ in terms of a degenerate model of the curve C. Consider the surface $\overline{K}_n$, trivialized fibers F_i, $i = 1, \ldots, n$, and the zero section s_0 on it; let $\{F_i, s_0\}$ be the basis in the homology group $H_2(\overline{K}_n, \mathbb{Z})$. The intersection numbers are

$$F_i \cdot F_j = 0, \quad s_0 \cdot s_0 = \deg \Omega(a_1 + \cdots + a_n) = n-2, \quad F_i \cdot s_0 = 1, \quad C \cdot F_i = 2;$$

besides, the intersection number of the curve C with the exceptional cycle s_∞ is zero (the intersection number $s_\infty \cdot s_\infty = 2-n$). The topological class of C is preserved by the isomonodromy deformation. In this way, we compactify the initial data space $\mathcal{M}_n(2)$ with the divisor D such that its factors $\Theta_{(i)} \subset \overline{K'_n}$ preserve the topological invariant, and $\Theta_{(i)} \cdot s_\infty = 0$; for $n = 4$ this immediately implies $\Theta = 2s_\infty + F_1 + \cdots + F_4$. In the case $n > 4$ the above argument is not so explicit, and we obtain the same result in 4.2 using the FH-sheaves approach to the compactification problem.

In fact, on each surface $\overline{K'_n}$ we have a compactifying divisor

$$D_{(i)} := (\Theta_{(i)})^{\mathrm{red}} = s_\infty + \widetilde{F}_1 + \cdots + \widetilde{F}_n,$$

and the whole compactification of the phase space is the following.

$$\overline{\widetilde{\mathcal{M}_n(2)}} = \{(K'_n)_{(1)} \sqcup \Theta_{(1)}\} \times \cdots \times \{(K'_n)_{(n-3)} \sqcup \Theta_{(n-3)}\}.$$

However, we present the compactifying divisor D as a union of the symmetric product of the components $\Theta_{(i)}$, with $(\Theta_{(i)})^{(r)} \times (K'_n)^{(n-3-r)}$, $r = 1, \ldots, n-3$, and the diagonal components on $\mathcal{M}_n(2)$ defined by $\{x_i = x_j\}$. In this way, we describe the moduli space $\mathcal{M}_n(2)$ on the complement to a certain locus of codimension two.

Besides, in the fourth section, we emphasize the important role of the complete self-intersection of the compactifying divisor: $\mathbf{\Theta}_n := D \cdot D$, whose dimension is exactly $n-3$. We describe the dynamics of the isomonodromic system in terms of the cycle $\mathbf{\Theta}_n$. Finally, we explain the role of apparent singularities of the Fuchsian systems originally introduced in [11] (see also [1] and [4]). Precisely, we identify the cycle $\mathbf{\Theta}_n$ with the moduli space of the collections

$$\big(\widetilde{\mathcal{L}}_{\Theta_n},\, \nabla_{\Theta_n};\ \phi'\colon \det \widetilde{\mathcal{L}}_{\Theta_n} \xrightarrow{\sim} \mathcal{O}(-a_1 - \cdots - a_{n-2});\ (\widetilde{\lambda}_i^+,\, \widetilde{\lambda}_i^-),\, i = 1,\, \ldots,\, n\big)$$

for some $a \in S$, where $\widetilde{\mathcal{L}}_{\Theta_n}$ is the rank 2 bundle of degree $2-n$ with the logarithmic connection ∇_{Θ_n} such that the eigenvalues of $\operatorname{Res}_{a_i} \nabla_{\Theta_n}$ are $(\widetilde{\lambda}_1^+,\, \widetilde{\lambda}_1^-) := (\lambda_i,\, 1-\lambda_i)$ at a_i, $i = 1,\, \ldots,\, n-2$, and $(\widetilde{\lambda}_i^+,\, \widetilde{\lambda}_i^-) := (\lambda_i,\, -\lambda_i)$ at $a_i = a_{n-1},\, a_n$. We present the dynamical variables $\{x_i,\, p_i\}$, $i = 1,\, \ldots,\, n-3$, of the isomonodromic deformation as parameters of the Hecke correspondence between $\mathbf{\Theta}_n$ and the moduli space $\mathcal{M}'_n(2) \simeq \mathcal{M}_n(2)$ of the collections

$$\big(\widetilde{\mathcal{L}},\, \widetilde{\nabla} := \nabla|_{\widetilde{\mathcal{L}}};\ \widetilde{\phi}\colon \det \widetilde{\mathcal{L}} \simeq \mathcal{O}(-a_1);\ (\lambda_1^+,\, \lambda_1^-),\, \ldots,\, (\lambda_n^+,\, \lambda_n^-)\big),$$

where $\widetilde{\mathcal{L}}$ is a rank 2 bundle on $\mathbb{P}^1$ with a fixed horizontal isomorphism $\widetilde{\phi}\colon \det \mathcal{L} \simeq \mathcal{O}(-a_1)$ and with a connection $\widetilde{\nabla}$ having singularities at $\{a_1,\, \ldots,\, a_n\}$; the eigenvalues of $\operatorname{Res}_{a_i} \widetilde{\nabla}$ are $(\lambda_1^+,\, \lambda_1^-) := (\lambda_1,\, 1-\lambda_1)$ at a_1, and $(\lambda_i^+,\, \lambda_i^-) := (\lambda_i,\, -\lambda_i)$ at a_i, $i = 2,\, \ldots,\, n$. In terms of the connections,

$$\widetilde{\nabla} = \nabla_{\Theta_n}(p_1,\, \ldots,\, p_{n-3}) - \sum_{i=1}^{n-3} \boldsymbol{P}_{p_i} \frac{dz}{z - x_i},$$

where $\boldsymbol{P}_{p_i}$ are the projectors on the invariant one-dimensional subspaces $p_i \subset \widetilde{\mathcal{L}}_{\Theta_n}|_{x_i}$, $i = 1,\, \ldots,\, n-3$. The terms $\boldsymbol{P}_{p_i} \dfrac{dz}{z - x_i}$ do not change the monodromy of the connection, and the points $x_1,\, \ldots,\, x_{n-3}$ are called *apparent singularities* of the connection $\widetilde{\nabla}$.

2. Modificaitons of Logarithmic $\mathfrak{sl}(2)$-Connections

Originally, modifications appeared in the works of Birkhoff (see [3]) as transformations of the gauge type in the Riemann–Hilbert problem for singular integral equations. Later on, Hecke (see [17]) used an analogous arithmetic correspondences between modular curves. In 1930ies Weil analyzed various interplays between number and function global fields, and in [29], described the classes of matrix divisors on a curve in terms of generalized abelian functions. In [5] Drinfeld presented a construction of elliptic module which generalized a set of classical algebraic ideas; then in [6] the Frobenius–Hecke sheaves, (or, "shtukas") were introduced. These new concepts entailed a profound understanding of the interplay between the geometric representation theory and the differential equations. Let us note that in our setting the discussed Hecke correspondences are symplectic (singular) gauge transformations (see [21]).

For our purposes it will be convenient to modify the original definition from [6] and to introduce the following

Definition. A Frobenius–Hecke sheaf (FH-sheaf) of level K (for an integer K) on $\mathbb{P}^1$ is a flag of locally free sheaves $\mathcal{F}_0 \subset \mathcal{F}$ of the same rank on $\mathbb{P}^1$ such that the codimension of the support $\operatorname{supp}(\mathcal{F}/\mathcal{F}_0) \subset \mathbb{P}^1$ equals one, and $(\mathcal{F}/\mathcal{F}_0)$ has a K-dimensional space of sections. For a generic FH-sheaf all the points of $\operatorname{supp}(\mathcal{F}/\mathcal{F}_0)$ are distinct, that is, $\mathcal{F}/\mathcal{F}_0$ is isomorphic to a sum of sky-scraper sheaves $\bigoplus \delta_{x_i}$ and each sky-scraper sheaf δ_{x_i} has a one-dimensional space of sections.

Between the moduli spaces of FH-sheaves $(\mathcal{F}_1' \subset \mathcal{F}_1)$ and $(\mathcal{F}_2' \subset \mathcal{F}_2)$ of different levels K_1 and K_2 there are correspondences, called the Hecke correspondences. These correspondences are performed by modifications (see [8]) of the locally free sheaves $\mathcal{F}_i'$, $\mathcal{F}_i$; upper modifications reduce the level and lower ones increase it.

Given a rank 2 bundle $\mathcal{L}$ on $\mathbb{P}^1$ with a connection ∇, let $x \in \mathbb{P}^1$. Denote by V a fiber $\mathcal{L}|_x$ and let $l \subset V$ be a one-dimensional subspace. Identify $\mathcal{L}$ with its sheaf sections and consider the following locally trivial sheaves:

$$(x,\, l)^{\mathrm{low}}(\mathcal{L}) := \{s \in \mathcal{L}\colon s(x) \in l\}, \qquad (x,\, l)^{\mathrm{up}}(\mathcal{L}) := (x,\, l)^{\mathrm{low}}(\mathcal{L}) \otimes \mathcal{O}(x),$$

which are called the lower and the upper modification, respectively. Denote the lower modification by $\widetilde{\mathcal{L}} := (x,\, l)^{\mathrm{low}}(\mathcal{L})$ and consider the natural map $\widetilde{\mathcal{L}}|_x \to \mathcal{L}|_x$; evidently, its image is l. Put $\widetilde{l} := \ker(\widetilde{\mathcal{L}}|_x \to \mathcal{L}|_x)$ then $(x,\, \widetilde{l})^{\mathrm{up}}\widetilde{\mathcal{L}} = \mathcal{L}$. The lower and the upper modifications provide the following exact sequences:

$$0 \to (x,\, l)^{\mathrm{low}}(\mathcal{L}) \to \mathcal{L} \to \delta_x \otimes \mathcal{L}_x/l \to 0,$$
$$0 \to \mathcal{L} \to (x,\, l)^{\mathrm{up}}\mathcal{L} \to \delta_x \otimes l \otimes \mathcal{T}_x \to 0,$$

respectively. Here δ_x is a sky-scraper sheaf supported at x and $\mathcal{T}_x$ is the localization of the tangent bundle at x.

Roughly speaking, given a local decomposition $V = l \oplus \widetilde{l}$ of $\mathcal{L} \simeq V \otimes \mathcal{O}$, we have

$$(x,\, l)^{\mathrm{low}}(\mathcal{L}) = l \otimes \mathcal{O} \oplus \widetilde{l} \otimes \mathcal{O}(-x), \qquad (x,\, l)^{\mathrm{up}}(\mathcal{L}) = l \otimes \mathcal{O}(x) \oplus \widetilde{l} \otimes \mathcal{O}.$$

In other words, we change our bundle rescalling the basis of sections in the neighborhood of a point x; if the local basis is $\{s_1(z),\, s_2(z)\}$ with $l \otimes \mathcal{O} \simeq \langle s_1(z)\rangle$, and $\widetilde{l} \otimes \mathcal{O} \simeq \langle s_2(z)\rangle$, then the basis of the lower modification $(x,\, l)^{\mathrm{low}}$ of the bundle is generated by the sections $\{s_1(z),\, (z-x)s_2(z)\}$, and of the upper one $(x,\, l)^{\mathrm{up}}$ by $\{(z-x)^{-1}s_1(z),\, s_2(z)\}$. Consequently, in the punctured neighborhood we may represent the action of the modifications by the following gluing matrices:

$$(x,\, l)^{\mathrm{low}} = \begin{pmatrix} 1 & 0 \\ 0 & (z-x) \end{pmatrix}, \qquad (x,\, l)^{\mathrm{up}} = \begin{pmatrix} (z-x)^{-1} & 0 \\ 0 & 1 \end{pmatrix}.$$

Matrix presentation of the modifications is supposed to be quite obvious, and further on we use it freely.

Now discuss the action of the modifications on an $\mathfrak{sl}(2)$-connection with logarithmic singularities on the projective line $\mathbb{P}^1$.

Definition [26]**.** A *modulus* $\mathfrak{M}$ supported at S on an algebraic curve X is a finite set $S = \{a_1,\, \dots,\, a_n\} \subset X$ equipped with a function assigning a positive integer n_i

to every point $a_i \in S$. Sometimes we identify $\mathfrak{M}$ with the effective divisor $\sum n_i \cdot a_i$. In the present work, we consider the module

$$\mathfrak{M} = \sum_{i=1}^{n} a_i.$$

Let us look how the modifications change the connection. Suppose we start from some logarithmic (Fuchsian) $\mathfrak{sl}(2)$-connection ∇ on $\mathcal{L}$ and

$$\nabla\colon \mathcal{L} \to \mathcal{L} \otimes \Omega^1(\mathfrak{M});$$

this means that ∇ has *simple poles* at the support S of $\mathfrak{M}$. Denote the eigensubspaces of $\operatorname{Res}_{a_i} \nabla$ by $\ell_i^{\pm} := \ker(\operatorname{Res}_{a_i} \nabla \mp \lambda_i)$ and consider the modifications of our pair $(\mathcal{L}, \nabla)$ in these subspaces. Note that we modify the pairs $(\mathcal{L}, \nabla)$ in $(\operatorname{Res}_x \nabla)$-invariant subspaces of $V \subseteq \mathcal{L}|_x$; otherwise, we increase the order of the pole of the connection. Indeed, using the matrix presentation write down the action of the modification of the bundle on the non-invariant subspace at $z = 0$:

$$\begin{pmatrix} 1 & 0 \\ 0 & z \end{pmatrix} \left[d + \begin{pmatrix} \lambda/z & \varepsilon/z \\ 0 & -\lambda/z \end{pmatrix} \right] \begin{pmatrix} 1 & 0 \\ 0 & 1/z \end{pmatrix} = d + \begin{pmatrix} \lambda/z & \varepsilon/z^2 \\ 0 & -(\lambda+1)/z \end{pmatrix},$$

where z is a local parameter. Here, because of the ε in the right upper corner, the second component of the modification is not ∇-invariant.

Besides, note that the lower and upper modifications at any point $x \in \mathbb{P}^1$ change the determinant:

$$\det(x,\, l)^{\mathrm{low}} \mathcal{L} = \det \mathcal{L} \otimes \mathcal{O}(-x), \qquad \det(x,\, l)^{\mathrm{up}} \mathcal{L} = \det \mathcal{L} \otimes \mathcal{O}(x).$$

Let us illustrate the techniques that we will use in the next sections. Consider the lower modification $\widetilde{\mathcal{L}}$ with the connection

$$\nabla'\colon \widetilde{\mathcal{L}} \xrightarrow{\nabla|_{\widetilde{\mathcal{L}}}} \mathcal{L} \otimes \Omega(\mathfrak{M}) \xrightarrow{\mathrm{pr}} \widetilde{\mathcal{L}} \otimes \Omega(\mathfrak{M})$$

on $\widetilde{\mathcal{L}}$, then on the determinant bundle we get the connection

$$\operatorname{tr} \nabla' = \operatorname{tr} \nabla + \frac{dz}{z - x}.$$

Perform a pair of the lower and upper modifications at points a_i and a_j respectively to get the bundle $\mathcal{L}''$ with the same determinant

$$\det \mathcal{L}'' = \det \mathcal{L} \otimes \mathcal{O}(a_j - a_i) \simeq \det \mathcal{L};$$

to do this we have to fix a set of compatible isomorphisms $\mathcal{O} \simeq \mathcal{O}(a_i - a_j)$, such that

$$\mathcal{O} \simeq \mathcal{O}(a_i - a_j) \otimes \mathcal{O}(a_j - a_k) \simeq \mathcal{O}(a_i - a_k).$$

Nevertheless, if we start from an $\mathfrak{sl}(2)$-connection ∇, then after such a procedure we get the connection

$$\nabla'' = \nabla + \boldsymbol{P}_{l_i} \frac{dz}{z - a_i} - \boldsymbol{P}_{\widetilde{l}_j} \frac{dz}{z - a_j},$$

where $\boldsymbol{P}_*$ are the projectors on appropriate $\operatorname{Res}\nabla$-invariant subspaces; it is a $\mathfrak{gl}(2)$-connection. In order to get $\mathfrak{sl}(2)$-connection, we have to add the suitable 1-form

$$\widetilde{\nabla}'' = \nabla'' + \frac{1}{2}\left(\mathbf{1}_2\frac{dz}{z-a_j} - \mathbf{1}_2\frac{dz}{z-a_i}\right),$$

where $\mathbf{1}_2$ denotes the identity 2×2 matrix.

For two points $a_i,\, a_j \in S$ consider the modified SL(2)-bundle

$$\mathcal{L}'' = (a_j,\, l_j^+)^{\mathrm{up}} \circ (a_i,\, l_i^-)^{\mathrm{low}}\mathcal{L}$$

with modified logarithmic connection ∇'' defined above. This provides nontrivial transformations of the coarse moduli space $\mathcal{M}_n$ of rank 2 bundles with fixed horizontal isomorphism ϕ and logarithmic connection with fixed eigenvalues of residues on $\mathbb{P}^1$; in other words, we have the Hecke correspondence on $\mathcal{M}_n$as follows.

Proposition [22]**.** *The modified pair* $(\mathcal{L}'',\, \widetilde{\nabla}'')$ *is an element of the coarse moduli space* $\mathcal{M}_n$*. The eigenvalues of* $\operatorname{Res}_a \widetilde{\nabla}''$*,* $a \in S$*, are*

$$\left\{\lambda_1,\, \dots,\, \lambda_i + \frac{1}{2},\, \dots,\, \lambda_j - \frac{1}{2},\, \dots,\, \lambda_n\right\}$$

for the case of a pair of modifications at distinct points $a_i,\, a_j \in s$*; for a pair of modifications xs at one point* $a_k \in S$*, the eigenvalues are*

$$\{\lambda_1,\, \dots,\, \lambda_k + 1,\, \dots,\, \lambda_n\}.$$

In this way, we have birational isomorphisms between moduli spaces with different parameters, or between different initial data spaces; the group structure is isomorphic to the affine Weyl group $W(\widehat{C}_n)$. For precise description of the discrete symmetries of our system and their presentation as relations between the solutions, see [21].

3. Separation of Variables

In this section, following [2], we describe our initial data and construct étale coordinates on the open subset of $\mathcal{M}_n(2)$. We generalize the result of Arinkin's and Lysenko's work [2] and present calculations for an arbitrary number n of singularities; we use the ideas from [2], in particular, two linear-algebraic lemmas. Fix a collection $\lambda_1,\, \dots,\, \lambda_n$ of complex numbers and a module $\mathfrak{M}$ with the support S at distinct points $a_1,\, \dots,\, a_n$ on $\mathbb{P}^1$. The group of projective automorphisms of the Riemann sphere being three-dimensional, it is natural to restrict ourselves to the case of $n \geq 3$. Suppose $\mathcal{L}$ be a rank 2 bundle on $\mathbb{P}^1$ equipped with a fixed horizontal isomorphism ϕ: $\det\mathcal{L} \simeq \mathcal{O}$ and a connection ∇ with singularities at $\mathfrak{M} = \sum a_i$; the eigenvalues of $\operatorname{Res}_{a_i}\nabla$ are $(\lambda_i,\, -\lambda_i)$, $i = 1,\, \dots,\, n$.

3.1. Stable bundles. Let us discuss the definition of *stability* of our data. We consider the moduli space of vector bundles of rank 2 and we permanently control the pair $(\mathcal{L},\, \nabla)$ to be indecomposable in order to provide the stability. For these purposes we put the following eigenvalue condition

$$\sum \epsilon_i\lambda_i \notin \mathbb{Z}, \quad (\epsilon_1,\, \dots,\, \epsilon_n) \in (\mathbb{Z}/2\mathbb{Z})^n,$$

which guarantees the irreducibility of the pair "bundle $\mathcal{L}$ with the connection ∇" and implies the stability of this pair. Indeed, given a ∇-invariant rank 1 sub-bundle $\mathcal{L}_1 \subset \mathcal{L}$ equipped with a connection $\nabla_1 := \nabla|_{\mathcal{L}_1}$, then $(\mathcal{L}_1)|_{a_i} \subset \mathcal{L}|_{a_i}$ is an eigenspace of $\operatorname{Res}_{a_i} \nabla$, and $\operatorname{Res}_{a_i} \nabla_1$ is an eigenvalue of $\operatorname{Res}_{a_i} \nabla$. In this way we get $\operatorname{Res}_{a_i} \nabla_1 = \pm\lambda_i$, but, the other hand, $\sum \operatorname{Res}_{a_i} \nabla_1 = -\deg \mathcal{L}_1 \in \mathbb{Z}$ contradicts our eigenvalue condition.

Moreover, our bundle $\mathcal{L}$ with the trivial determinant is in general nontrivial and may have a structure $\mathcal{O}(k) \oplus \mathcal{O}(-k)$. The value of k depends on n and it is defined by the stability of the construction in the following way. Let $\mathcal{L}_0 := \mathcal{O}(k)$ be a sub-bundle, then by irreducibility we have a non-zero map

$$\nabla_0 \colon \mathcal{L}_0 \to (\mathcal{L}/\mathcal{L}_0) \otimes \Omega(\mathfrak{M}),$$

which implies

$$\deg \mathcal{L}_0 \le \deg(\mathcal{L}/\mathcal{L}_0) + \deg \Omega(\mathfrak{M}) = 0 - \deg \mathcal{L}_0 + n - 2, \quad \text{hence,} \quad k \le \frac{n-2}{2}.$$

We consider the moduli space of pairs $(\mathcal{L}, \nabla)$ and look after the automorphism group of the pair. We demand $\operatorname{Aut}(\mathcal{L}, \nabla) = \mathbb{C}^*$, and we assume that there are no ∇-invariant sub-bundles $\mathcal{L}_0 \subset \mathcal{L}$.

3.2. The map $(\mathcal{L}, \nabla) \mapsto (\mathcal{L}_0 \subset \mathcal{L}, \nabla)$. We shall act in the following way. Suppose that we can choose a *distinguished* sub-bundle $\mathcal{L}_0 \subset \mathcal{L}$. Then we will describe properties of a (semi)stable element $(\mathcal{L}, \nabla) \in \mathcal{M}_n(2)$ looking at its restriction on the (non-invariant) *distinguished* sub-bundle. We have seen that for $(\mathcal{L}, \nabla) \in \mathcal{M}_n$ the structure of our bundle $\mathcal{L}$ can be $\mathcal{O}(k) \oplus \mathcal{O}(-k)$ for some k but, for example, if $k = 0$ and $\mathcal{L} \simeq \mathcal{O} \oplus \mathcal{O}$ then there is no way to choose the distinguished sub-bundle. The fact is that a bundle of an odd degree always has a distinguished sub-bundle, and, it is in this way that we have to modify our bundle.

Take a point from S, say, a_1, and consider the bundle $\widetilde{\mathcal{L}} := (a_1, l_1^+)^{\mathrm{low}}\mathcal{L}$. The natural embedding $\widetilde{\mathcal{L}} \subset \mathcal{L}$ provides an isomorphism

$$\mathcal{M}_n(2) \simeq \mathcal{M}'_n(2) := \text{moduli space of } \big(\widetilde{\mathcal{L}}, \widetilde{\nabla} := \nabla|_{\widetilde{\mathcal{L}}};\ \widetilde{\phi}\colon \det \widetilde{\mathcal{L}} \simeq \mathcal{O}(-a_1); \\ (\lambda_1^+, \lambda_1^-), \dots, (\lambda_n^+, -\lambda_n^-)\big).$$

Here $\widetilde{\mathcal{L}}$ is a rank 2 bundle on $\mathbb{P}^1$ with a fixed horizontal isomorphism $\widetilde{\phi}\colon \det \mathcal{L} \simeq \mathcal{O}(-a_1)$ and a logarithmic connection $\widetilde{\nabla}$ with singularities at $\{a_1, \dots, a_n\}$. The eigenvalues of $\operatorname{Res}_{a_i} \widetilde{\nabla}$ are $(\lambda_1^+, \lambda_1^-) := (\lambda_1, 1-\lambda_1)$ at a_1, and $(\lambda_i^+, \lambda_i^-) := (\lambda_i, -\lambda_i)$ at a_i, $i = 2, \dots, n$. The dimension of the vector space of embeddings $\mathcal{L}/\mathcal{L}_0 \simeq \mathcal{O}(-k) \hookrightarrow \mathcal{L}$ for $k > 0$ equals

$$\dim \operatorname{Hom}(\mathcal{O}(-k), \mathcal{O}(k)) = 2k+1 = 3, \ \dots, \ 2 \cdot \left[\frac{n-2}{2}\right] + 1.$$

Thus, we can choose a sub-bundle $\mathcal{O}(-k)$ passing through at least $2k+1$ of n subspaces $l_i^+ := \ker(\operatorname{Res}_{x_i} - \lambda_i)$, and then at least one subspace lies neither in $\mathcal{L}_0$, nor in our chosen $\mathcal{O}(-k)$, as we assume the bundle $(\mathcal{L}; \phi; l_i, i = 1, \dots, n)$ to be irreducible. Thus, we get the distinguished sub-bundle $\widetilde{\mathcal{L}}_0 \subset \widetilde{\mathcal{L}}$ with possible values of degree $\deg \widetilde{\mathcal{L}}_0 := k' = 0, \dots, [\frac{n-2}{2}] - 1 = [\frac{n-4}{2}]$. For example, in both cases $n = 4$

and $n = 5$, the structure of $\mathcal{L}$ can be only $\mathcal{O} \oplus \mathcal{O}$ and $\mathcal{O}(1) \oplus \mathcal{O}(-1)$; nevertheless, for $n = 4$ the modified bundle is always $\widetilde{\mathcal{L}} \simeq \mathcal{O} \oplus \mathcal{O}(-1)$, and for $n = 5$ it can be either $\mathcal{O} \oplus \mathcal{O}(-1)$, or $\mathcal{O}(1) \oplus \mathcal{O}(-2)$, since the direction l_1^+ of the lower modification can lie in $\mathcal{L}_0 \simeq \mathcal{O}(1)$.

3.3. $\mathcal{M}'_n(2)$ as a moduli space of FH-sheaves. The algebraic variety $\mathcal{M}_n \simeq \mathcal{M}'_n$ is non-compact and consists of locally closed strata

$$\mathcal{M}'_n \supset \mathscr{M}^{k'} := \text{moduli space of } \big(\mathcal{O}(k') \subset \widetilde{\mathcal{L}};\ \widetilde{\nabla};\ \widetilde{\phi}\colon \det \widetilde{\mathcal{L}} \simeq \mathcal{O}(-a_1); \\ (\lambda_1^+, \lambda_1^-), \dots, (\lambda_n^+, \lambda_n^-)\big)$$

indexed by k'. The maximal value of k' depends on the parity of n; if n is even, then $k' = \frac{n-4}{2}$, and if n is odd, then $k' = \frac{n-3}{2}$.

Fix an isomorphism $\widetilde{\mathcal{L}}_0 \simeq \mathcal{O}(y_1 + \dots + y_{k'})$ for some $y_1, \dots, y_{k'} \in \mathbb{P}^1$, and choose a connection ∇_0 with respect to this isomorphism with k' simple poles precisely at $y_1, \dots, y_{k'}$, such that

$$\nabla_0 \colon \widetilde{\mathcal{L}}_0 \to \widetilde{\mathcal{L}}_0 \otimes \Omega(y_1 + \dots + y_{k'}), \quad \operatorname{Res}_{y_i} \nabla_0 = 1.$$

Fixing the connection ∇_0 we get a distinguished trivialization (section) $\mathcal{O} \hookrightarrow \widetilde{\mathcal{L}}_0$ of our sub-bundle.

Restrict the connection on the sub-bundle $\widetilde{\mathcal{L}}_0$ and consider the map

$$B := \widetilde{\nabla}|_{\widetilde{\mathcal{L}}_0} - \nabla_0 \colon\ \widetilde{\mathcal{L}}_0 \to \widetilde{\mathcal{L}} \otimes \Omega(\mathfrak{M}).$$

In this way we obtain the maps

$$f_{k'} \colon \mathscr{M}^{k'} \to M_1 := \text{moduli space of } (\widetilde{\mathcal{L}}_0 \simeq \mathcal{O}(k') \subset \widetilde{\mathcal{L}}, B),$$

where $\widetilde{\mathcal{L}}/\widetilde{\mathcal{L}}_0 \simeq \mathcal{O}(-k'-1)$ and $B \colon \mathcal{T}(-\mathfrak{M}) \hookrightarrow \widetilde{\mathcal{L}}$ for $\mathcal{T}(-\mathfrak{M}) := \Omega(\mathfrak{M})^{-1}$.

Using the maps $f_{k'}$ we construct the maps from our moduli space $\mathcal{M}'_n$ to the moduli space of the so-called Drinfeld FH-sheaves (see [6]):

$$\big\{\mathcal{O} \oplus \mathcal{T}(-\mathfrak{M}) \subset \widetilde{\mathcal{L}} \colon \widetilde{\mathcal{L}}/(\mathcal{O} \oplus \mathcal{T}(-\mathfrak{M})) \simeq \Delta_{n-3}\big\},$$

where $\dim \Gamma(\mathbb{P}^1, \Delta_{n-3}) = n - 3$ and $\operatorname{supp}(\Delta_{n-3})$ is in codimension one.

To present the strata of $\mathcal{M}'_n(2)$ as moduli spaces we have to reconstruct the element $(\widetilde{\mathcal{L}}, \widetilde{\nabla}) \in \mathcal{M}'_n(2)$ from the FH-sheaf $A = (\mathcal{O} \oplus \mathcal{T}(-\mathfrak{M}) \subset \widetilde{\mathcal{L}})$.

Proposition. *Let A be an FH-sheaf of level $n-3$ and let R_i be an operator $\widetilde{\mathcal{L}}|_{a_i} \to \widetilde{\mathcal{L}}|_{a_i}$ with eigenvalues $\lambda_i^\pm$. Then, on the stratum $\mathscr{M}^0$, there is a unique connection $\widetilde{\nabla}$ such that in the above notation*

(i) *$\widetilde{\nabla}|_{\widetilde{\mathcal{L}}_0} = d + B$ for the unique connection $d \colon \widetilde{\mathcal{L}}_0 \to \widetilde{\mathcal{L}}_0 \otimes \Omega$;*
(ii) *$\operatorname{Res}_{a_i} \nabla = R_i$;*
(iii) *$(\widetilde{\mathcal{L}}, \widetilde{\nabla}) \in \mathcal{M}'_n(2)$.*

In this way, we identify the generic stratum $\mathscr{M}^0$ with the moduli space of certain FH-sheaves. On the other strata the connection $\widetilde{\nabla}$ is not unique, and in the following two subsections we prove the analogous proposition for these strata. In Section 3.4 we give a simple construction from linear algebra, and in Section 3.5 we calculate the affine space of connections $\widetilde{\nabla}$ that satisfy conditions (i)–(iii).

3.4. A construction from the linear algebra. In terms of the linear algebra our description of stable pairs $(\widetilde{\mathcal{L}}, \widetilde{\nabla})$ is nothing but a reconstruction of the operator $L(z)$ such that $(\widetilde{\mathcal{L}}, \partial_z - L(z)) \in \widetilde{\mathcal{M}'_n}$ from the first row B of the operator L and the eigenvalues of the residues. Let $V_0 \subset V \simeq \mathbb{C}^2$ be a complete flag of vector spaces, and let $R_0 \in \mathrm{Hom}(V_0, V)$.

Lemma A [2]**.** *Let* $\lambda^+ \neq \lambda^- \in \mathbb{C}$ *and put* $\mathfrak{R} := \{R \in \mathrm{End}(V)$ such that $R|_{V_0} = R_0$ and the eigenvalues of R are $\lambda^+, \lambda^-\}$, $\mathfrak{L} := \{(l^+ \neq l^-)\colon l^\pm \subset V, \dim l^\pm = 1$ *with* $(R_0 - \lambda^\mp)(V_0) \subset l^\pm\}$. *Then, the map*

$$F\colon \mathfrak{R} \to \mathfrak{L}, \quad R \mapsto (\ker(R - \lambda^+) = \mathrm{im}(R - \lambda^-), \ker(R - \lambda^-))$$

is bijective.

Proof. Clearly, F is injective, so let us check the surjectivity. For $(l^+, l^-) \in \mathfrak{L}$ denote the corresponding projectors $P_\pm\colon V \to V/l^\pm \simeq l^\mp$; $P_+ + P_- = \mathrm{Id}$. The condition $(R_0 - \lambda^\mp)(V_0) \subset l^\pm$ implies $P^\mp(R_0 - \lambda^\mp)(V_0) = 0$, or, $P^-(R_0 - \lambda^-)(V_0) + P^+(R_0 - \lambda^+)(V_0) = 0$; hence, $R_0 = (\lambda^+ P^+ + \lambda^- P^-)|_{V_0}$ and for $R := (\lambda^+ P^+ + \lambda^- P^-) \in \mathfrak{R}$ we have $F(R) = (l^+, l^-)$. □

One can make similar calculations for the case $l^+ = l^-$ and proof analogous statement.

Lemma B. *Let* $\lambda := \lambda^+ = \lambda^- \in \mathbb{C}$ *and put* $\mathfrak{R} := \{R \in \mathrm{End}(V)$ such that $R|_{V_0} = R_0$ and R has only one eigenvalue $\lambda\}$, $\mathfrak{L} := \{(l \neq l')\colon l, l' \subset V, \dim l, l' = 1$ with $(R_0 - \lambda)(V_0) \subset l$ and $(R_0 - \lambda)(l') \subset V_0\}$. *Then, the map*

$$F\colon \mathfrak{R} \to \mathfrak{L}, \quad R \mapsto (\ker(R - \lambda), \mathrm{im}(R - \lambda))$$

is bijective. □

3.5. Calculation of the affine space of connections. Let us remark that the connection $\widetilde{\nabla}$ that satisfies conditions (i)–(iii) exists locally on $\mathbb{P}^1$. For an open subset $U \subset \mathbb{P}^1$ by $\mathcal{C}(U)$ denote the set of all local connections $\widetilde{\nabla} = \nabla_0 - L(z)$ on U. If we have $\widetilde{\nabla}, \widetilde{\nabla}' \in \mathcal{C}(U)$, then $E := \widetilde{\nabla} - \widetilde{\nabla}'$ is an element of $H^0(U, \mathcal{H}om(\widetilde{\mathcal{L}}, \widetilde{\mathcal{L}} \otimes \Omega)) \simeq \mathcal{H}om(\widetilde{\mathcal{L}}/\widetilde{\mathcal{L}}_0, \widetilde{\mathcal{L}}_0 \otimes \Omega)$ such that $E|_{\widetilde{\mathcal{L}}_0} = 0$ and $\mathrm{tr}\, E = 0$; denote by $\mathcal{E}(U)$ the set of such local homomorphisms. Clearly, $\mathcal{C}$ is an $\mathcal{E}$-torsor and the obstruction to the existence of global connections lies in $H^1(\mathbb{P}^1, \mathcal{E}(\mathfrak{M}))$, which by the Serre duality is dual to $H^0(\{E \in \mathrm{End}\, \mathcal{L}\colon \mathrm{tr}\, E = 0, E(a_i)(l_i^+) \subset l_i^+\}) = \{0\}$. In this way, a global connection always exists but it is not unique.

Let us recall the notion of a torsor and explain the above argument for the existence of a global connection. Actually one may replace the notion of G-torsor by a notion of principal G-bundle; the formal definition of a torsor is as follows.

Definition [16]**.** Let $\mathcal{G}$ be a group sheaf on a scheme X. A sheaf $\mathcal{T}$ on X is a $\mathcal{G}$-torsor if

$$\mathcal{G} \times \mathcal{T} \to \mathcal{T}$$

is a morphism of sheaves such that for any open subset $U \subset X$ the restriction morphism

$$\mathcal{G}(U) \times \mathcal{T}(U) \to \mathcal{T}(U)$$

is the left group action.

In our case $\mathcal{E}$ is a group sheaf of $\mathfrak{sl}(2)^*$-valued 1-forms with fixed eigenvalues of the residues at $\operatorname{supp}(\mathfrak{M})$. In other words, $\mathcal{E}$ is a group sheaf $\operatorname{End}(\mathcal{L}) \otimes \Omega(\mathfrak{M})$ with fixed eigenvalues, and we consider the co-adjoint representation $\operatorname{End}(\mathcal{L}) \to \mathfrak{sl}(2)^*$. For a collection of open subsets $\{U_i\}$, $\mathbb{P}^1 = \bigcup U_i$, denote by g_{ij} the gluing 1-cocycle of our bundle $\mathcal{L}$. Thus, for any section $L \in \Gamma(\mathbb{P}^1, \mathcal{E})$ and for its restrictions $L_i \in \mathcal{E}(U_i)$ we have

$$L_i = (g_{ij})^{-1} \cdot L_j \cdot g_{ij}.$$

However, the sheaf $\mathcal{C}$ of local connections is a $\mathcal{E}$-torsor; for $\nabla_i \in \mathcal{C}(U_i)$ we have $\nabla_i = (g_{ij})^{-1} \cdot \nabla_j \cdot g_{ij} + h_{ij}$, where $h \in H^1(\mathbb{P}^1, \mathcal{E})$. Evidently, h is an obstruction to recover ∇ from the operator L. As soon as the torsor $\mathcal{C}$ is trivial and

$$\nabla_i = (g_{ij})^{-1} \cdot L_j \cdot g_{ij} + (g_{ij})^{-1} d(g_{ij}),$$

we construct a global connection ∇ that satisfies the conditions (i)–(iii).

In our construction we recover the row $(L_{21}, -L_{11})$, and L_{21} is an element of $\mathcal{H}om(\widetilde{\mathcal{L}}/\widetilde{\mathcal{L}}_0, \widetilde{\mathcal{L}}_0 \otimes \Omega) \simeq \mathcal{E}$. Let us describe what happens on each stratum under the assumption that the FH-sheaf $A = (\mathcal{O} \oplus \mathcal{T}(-\mathfrak{M}) \subset \widetilde{\mathcal{L}})$ is generic.

On the stratum $\mathscr{M}^0$ we have the diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \mathcal{O} \oplus \mathcal{T}(-\mathfrak{M}) & \xrightarrow{A} & \widetilde{\mathcal{L}} & \longrightarrow & \bigoplus_{i=1}^{n-3} \delta_{x_i} \otimes p_i \otimes \mathcal{T}_{x_i} & \longrightarrow & 0 \\
 & & \Vert & & \Vert & & & & \\
0 & \longrightarrow & \mathcal{O} \oplus \mathcal{O}(2-n) & \xrightarrow{A} & \mathcal{O} \oplus \mathcal{O}(-1). & & & &
\end{array}$$

For all x_i we have $\operatorname{im} A(x_i) \not\subseteq \widetilde{\mathcal{L}}_0 \simeq \mathcal{O}$, hence, all $p_i < \infty$, and the map

$$\mathscr{M}^0 \to \underbrace{K'_n \times \cdots \times K'_n}_{n-3}$$

is an isomorphism at a generic point (modulo the assumption that all x_i are distinct). The sheaf $\mathcal{E} \simeq \mathcal{H}om(\widetilde{\mathcal{L}}/\widetilde{\mathcal{L}}_0, \widetilde{\mathcal{L}}_0 \otimes \Omega)$ is of degree -1, hence, any $\mathcal{E}$-torsor is trivial and we have the unique connection recovered by our procedure.

On the stratum $\mathscr{M}^1$ we have

$$A := \operatorname{Id} \oplus B \colon\ \mathcal{O} \oplus (\mathcal{T}(-\mathfrak{M})) \to \widetilde{\mathcal{L}} \simeq \mathcal{O}(y_1) \oplus \mathcal{O}(-2),$$

and, if $x_i = y_1$ for some i, then we make the upper modification at x_i in the infinite direction, and $p_i = \infty$. Note that the case $p_i = \infty$ corresponds to the point at infinity of $\overline{K}_{n'} := \mathbb{P}(\mathcal{O} \oplus \Omega(\mathfrak{M}))$ and it means that the modification in $(\mathcal{O} \oplus \mathcal{T}(-\mathfrak{M}))|_{x_i}$ is performed in the direction of $\mathcal{O}|_{x_i} \subset (\mathcal{O} \oplus \mathcal{T}(-\mathfrak{M}))|_{x_i}$. In this way, we have a map

$$\mathscr{M}^1 \to \overline{K}_{n'} \times \underbrace{K'_n \times \cdots \times K'_n}_{n-4}$$

The sheaf $\mathcal{E} = \mathcal{H}om(\widetilde{\mathcal{L}}/\widetilde{\mathcal{L}}_0, \widetilde{\mathcal{L}}_0 \otimes \Omega)$ is isomorphic to $\mathcal{H}om(\mathcal{O}(-2), \mathcal{O}(1) \otimes \Omega) \simeq \mathcal{O}(1)$, and on this stratum the affine space of connections is 2-dimensional.

On the stratum $\mathscr{M}^{k'}$ we have

$$A := \operatorname{Id} \oplus B \colon\ \mathcal{O} \oplus (\mathcal{T}(-\mathfrak{M})) \to \widetilde{\mathcal{L}} \simeq \mathcal{O}(y_1 + \cdots + y_{k'}) \oplus \mathcal{O}(-k' - 1),$$

hence,

$$\mathscr{M}^{k'} \to \underbrace{\overline{K}_{n'} \times \cdots \times \overline{K}_{n'}}_{k'} \times \underbrace{\overline{K}_{n'} \times \cdots \times \overline{K}_{n'}}_{n-3-k'}.$$

Besides, $\mathcal{E} \simeq \mathcal{H}om(\mathcal{O}(-k'-1),\, \mathcal{O}(k') \otimes \Omega) \simeq \mathcal{O}(2k'-1)$, and on this stratum the affine space of the connections parameterized by L_{21} is $2k'$-dimensional.

3.6. Étale coordinates on $\mathcal{M}'_n(2)$ at the generic point. We have constructed

$$L|_{\widetilde{\mathcal{L}}_0} = B\colon\; \mathcal{T}(-\mathfrak{M}) \hookrightarrow \widetilde{\mathcal{L}}$$

and $\mathrm{Id}\colon \mathcal{O} \hookrightarrow \widetilde{\mathcal{L}}$, hence, we get

$$A := \mathrm{Id} \oplus B\colon\; \mathcal{O} \oplus \mathcal{T}(-\mathfrak{M}) \to \widetilde{\mathcal{L}}.$$

Moreover, in the generic situation we have a decomposition

$$A = A_1 \circ \cdots \circ A_{n-3}, \qquad A_i = (x_i,\, p_i)^{\mathrm{up}},\;\; i = 1,\, \ldots,\, n-3,$$

which implies $\det A(x_i) = 0$, $i = 1,\, \ldots,\, n-3$, hence, in the neighborhood of a point x_i,

$$A(x_i) = \begin{pmatrix} B_{11} & B_{12} \\ 1 & 0 \end{pmatrix}, \qquad \text{and} \qquad B_{11}(x_i) = p_i,\;\; B_{12}(x_i) = 0,\;\; i = 1,\, \ldots,\, n-3.$$

By Lemmas A and B we recover the operator $L(z)$ from the following data; $L(z)|_{\mathcal{L}_0} = A(z)$, $\mathrm{Res}_{a_i} L(z)$ has the eigenvalues λ_i^+, λ_i^-, and the trace

$$\mathrm{tr}\, L(z) = \frac{1}{z - a_1}.$$

The $n-3$ zeroes of B_{12} are exactly x_i, $i = 1,\, \ldots,\, n-3$, étale coordinates on $\mathcal{M}'_n$. One can compare this calculation with the analogous result in [9], [13], [27].

In this way, we are given an exact sequence

$$0 \to \mathcal{O} \oplus \mathcal{T}(-\mathfrak{M}) \xrightarrow{A} \widetilde{\mathcal{L}} \to \delta_{x_i} \otimes p_i \otimes \mathcal{T}_{x_i} \to 0, \qquad i = 1,\, \ldots,\, n-3,$$

where $A_1 \circ \cdots \circ A_{n-3} = A\colon \mathcal{O} \oplus \mathcal{T}(-\mathfrak{M}) \to \widetilde{\mathcal{L}}$ is a composition of the upper modifications $(x_i,\, p_i)^{\mathrm{up}}$. The directions of the modifications $p_i \subset (\mathcal{O} \oplus \mathcal{T}(-\mathfrak{M}))|_{x_i}$ are one-dimensional subspaces and they are parameterized by the surface $\mathrm{Tot}(\mathbb{P}^1,\, \Omega(\mathfrak{M}))$. So, we want to construct maps $\mathcal{M}'_n(2) \to \mathrm{Tot}(\mathbb{P}^1,\, \Omega(\mathfrak{M}))$ and parameterize $\mathcal{M}'_n(2)$ by $\{x_i,\, p_i\}$, $i = 1,\, \ldots,\, n-3$. In fact $\{x_i,\, p_i\}$, $i = 1,\, \ldots,\, n-3$, are étale coordinates on an open subset of $\mathcal{M}'_n(2)$.

There is no ordering on our array of A_i, $i = 1,\, \ldots,\, n-3$, and we have the action of the symmetric group $\mathfrak{S}_{n-3}$ on our construction of $\mathcal{M}'_n$; a change of order of the upper modifications $A_i = (x_i,\, p_i)^{\mathrm{up}}$, $i = 1,\, \ldots,\, n-3$, induces a nontrivial automorphism of $\mathrm{Tot}(\mathbb{P}^1,\, \Omega(\mathfrak{M}))^{n-3}$. In this way, there is no map from $\mathcal{M}'_n$ to $\mathrm{Tot}(\mathbb{P}^1,\, \Omega(\mathfrak{M}))^{n-3}$, but there is one to the factor

$$\mathrm{Tot}(\mathbb{P}^1,\, \Omega(\mathfrak{M}))^{(n-3)} := \underbrace{\mathrm{Tot}(\mathbb{P}^1,\, \Omega(\mathfrak{M})) \times \cdots \times \mathrm{Tot}(\mathbb{P}^1,\, \Omega(\mathfrak{M}))}_{n-3} / \mathfrak{S}_{n-3}.$$

We may also consider the $(n-3)!$-branched covering $\widetilde{\mathcal{M}}'_n$ of $\mathcal{M}'_n$ and investigate the interplay between $\widetilde{\mathcal{M}}'_n$ and $\mathrm{Tot}(\mathbb{P}^1,\, \Omega(\mathfrak{M}))^{n-3}$.

3.7. Description of the fibers $F_i = \Omega(\mathfrak{M})|_{a_i}$. Let us analyze the behavior of the map A when x_i tends to $a \in S$. At a singular point a we have two conditions on the eigenvalues of the residue $L_a := \operatorname{Res}_a \nabla$:

$$\operatorname{tr} L_a = 0 \quad \text{and} \quad \det L_a = \lambda_a^+ \cdot \lambda_a^-, \quad a \in S.$$

We reconstruct the operator

$$L(z)|_{x_i \to a} = \begin{pmatrix} L_{11} & L_{12} \\ L_{21} & -L_{11} \end{pmatrix}$$

and obtain

$$L_{11} = B_{11} = p_i\, dz, \quad \operatorname{Res} L_{12} \to 0, \quad \operatorname{Res} L_{21} = \frac{\det L_a - p_i^2}{\operatorname{Res} L_{12}}.$$

We see that $\operatorname{Res} L_{21}$ can have a finite value only when $p_i \to \lambda_a^\pm$ and we have to calculate the limit of L_{21} by the l'Hôpital rule considering the next terms of expansions of $\det L_a - p_i^2$ and $\operatorname{Res} L_{12}$. Geometrically that means that we make a blow-up (a σ-process) at this point.

Consider $K_n := \operatorname{Tot}(\mathbb{P}^1, \mathcal{O} \oplus \Omega(\mathfrak{M}))$ with the fibers $F_a \subset K_n$ at $a \in \mathbb{P}^1$. Since $\operatorname{Res}_a \colon \Omega(\mathfrak{M})|_a \simeq \mathbb{C}$, we have $R_a \colon F_a \xrightarrow{\sim} \mathbb{C}$; blow up K_n at $2n$ points $R_a^{-1}(\lambda_a^\pm)$ and get

$$K_n' := (\mathcal{B}l_{R_a^{-1}(\lambda_a^\pm)} K_n) \setminus \bigsqcup \widetilde{F}_a,$$

where $\widetilde{F}_a$ are the pre-images of the fibers $F_a \subset K_n$ after the blow-up. Finally, we have a map

$$\widetilde{\mathcal{M}}_n' \to \underbrace{K_n' \times \cdots \times K_n'}_{n-3}.$$

For $n = 4$ this map is an isomorphism but, in general, as we have seen in Section 3.5, this map is neither injective, nor surjective; nevertheless, it is an isomorphism at the generic point of $\widetilde{\mathcal{M}}_n'$.

3.8. Description of diagonals $\Delta_{(i,j)} = \{x_i = x_j\}$. We have described the operator $L(z)$ and the moduli space $\mathcal{M}_n'(2)$ by means of the FH-sheaf

$$A = (\mathcal{O} \oplus \mathcal{T}(-\mathfrak{M}) \subset \widetilde{\mathcal{L}})$$

of level $n-3$. We have considered the generic situation and have assumed all the points x_i to be distinct. Thus we ve got the following exact sequence

$$0 \to \mathcal{O} \oplus \mathcal{T}(-\mathfrak{M}) \xrightarrow{A} \widetilde{\mathcal{L}} \to \delta_{x_i} \otimes p_i \otimes \mathcal{T}_{x_i} \to 0, \qquad i = 1, \ldots, n-3,$$

and have defined the "geometric" Darboux coordinates on $\mathcal{M}_n'(2)$. Let us look what happens if two points x_i and x_j coincide. Denote $x_i = x_j = x$, and let z be a local parameter. In the neighborhood of x we have the double upper modification:

$$0 \to \mathcal{O} \oplus \mathcal{T}(-\mathfrak{M})|_x \xrightarrow{A} \widetilde{\mathcal{L}}|_x \to \Delta_2 \to 0.$$

Here Δ_2 is a sheaf with support at x and a two-dimensional space of sections; denote $U := \Gamma(\mathbb{P}^1, \Delta_2)$. Note that in this case we have no way to define a coordinate p; nevertheless, let us parameterize the double modifications.

Naturally, a localization of our rank 2 bundle $\mathcal{L}$ at the point x is a four-dimensional vector space $V := \langle e_1, e_2\rangle \otimes \mathbb{C}(z)/(z^2) \simeq \langle a, b, c, d\rangle$; we restrict ourselves to terms z^2 because we discuss only double modifications. There is a nilpotent operator $\otimes z$ on V, and we have to choose a two-dimensional subspace $U \subset V$ invariant under $\otimes z$. We see that double modifications are parameterized by the degenerated quadric $Q := \{a \cdot b = c \cdot d\} \subset \mathbb{P}^3$, and we have to blow up the point $[0 : 0 : 0 : 0]$ that corresponds to the subspace $U_0 := \langle e_1 \otimes z, e_2 \otimes z\rangle \subset V$ vanishing under our nilpotent operator $\otimes z\colon U_0 \mapsto \{0\}$. In this way we get

$$\mathrm{Diag}(K_n \times K_n) \simeq \mathcal{B}l_0 Q \simeq \overline{K}_0 := \mathbb{P}(\mathcal{O} \oplus \Omega^1_{\mathbb{P}^1}) \simeq \mathbb{P}(\mathcal{O} \oplus \mathcal{O}(-2));$$

thus we have the following map:

$$\mathcal{M}'_n(2) \to \overline{K}_0 \times \underbrace{K'_n \times \cdots \times K'_n}_{n-4}.$$

For distinct and thorough presentation of the general theory and methods of resolution of such diagonals see [12].

4. Compactification and Dynamics of the System

We have found the étale coordinates $\{x_i, p_i\}$, $i = 1, \ldots, n-3$, on the open subset of the initial data space $\mathcal{M}_n(2)$ and now we investigate a compactification of $\mathcal{M}_n(2)$ in terms of these variables. On the open subset of the moduli space, $\mathcal{M}_n(2)$ is isomorphic to the symmetric power of the surface K'_n; each factor is $(K'_n)_{(i)} \simeq \mathcal{B}l_{\lambda_i^{\pm}} \mathbb{P}(\mathcal{O} \oplus \Omega(\mathfrak{M})) \setminus \Theta_{(i)}$, $i = 1, \ldots, n-3$. In the same way, the factors of the compactifying divisor D are the components

$$(\Theta_{(i)})^{\mathrm{red}} = s_\infty + \widetilde{F}_1 + \cdots + \widetilde{F}_n \subset \mathcal{B}l_{\lambda_i^{\pm}} \mathbb{P}(\mathcal{O} \oplus \Omega(\mathfrak{M})),$$

where s_∞ is the infinite section $\mathbb{P}(\mathcal{O} \oplus \Omega^1(\mathfrak{M})) \setminus \mathrm{Tot}(\Omega^1(\mathfrak{M}))$ and $\widetilde{F}_i$ are the preimages of the fibres $F_i := \Omega^1(\mathfrak{M})|_{a_i} \subset \mathrm{Tot}(\Omega^1(\mathfrak{M}))$ at singular points $a_1, \ldots, a_n$. In this way the compactifying divisor is

$$D = \mathbf{\Theta}_n + \sum_{r=1}^{n-3} (\Theta_{(i)})^r \times (K'_n)^{n-3-r},$$

where $\mathbf{\Theta}_n = D \cdot D$ is the complete self-intersection cycle, and evidently $\mathbf{\Theta}_n = \Theta^{(n-3)}$.

In this section, we present the Drinfeld compactification (see [2], [8]) in terms of FH-sheaves ([6]). We describe our compactification in terms of FH-sheaves and this construction gives the description of the compactifying set as a moduli space of certain FH-sheaves. In Section 4.3 we give a geometric presentation of the isomonodromic deformation dynamics in terms of $\mathbf{\Theta}_n$.

4.1. FH-sheaves approach to the Drinfeld compactification. Let us describe our compactificaiton in terms of FH-sheaves. It is convenient to recall the construction of the moduli space $\mathcal{M}_n(2)$ as a subset in the moduli space of FH-sheaves; it is as follows. Note that all moduli spaces considered here are coarse moduli spaces, and we do not discuss here the interplay between the corresponding algebraic stacks.

First, we present an isomorphism $\mathcal{M}_n(2) \xrightarrow{\sim} \mathcal{M}'_n(2)$, where $\mathcal{M}'_n(2)$ is the moduli space of rank 2 bundles $\widetilde{\mathcal{L}}$ with the horizontal isomorphism $\widetilde{\phi}\colon \det\widetilde{\mathcal{L}} \simeq \mathcal{O}(-a_1)$. This bundle is equipped with a logarithmic connection $\widetilde{\nabla}$ with fixed eigenvalues $\{\lambda_i^+, \lambda_i^-\}$ of the residues $\operatorname{Res}_{a_i}\nabla$. This isomorphism is given by the lower modification $\widetilde{\mathcal{L}} := (a_1, l_1^+)^{\mathrm{low}}$ in the direction $l_1^+ := \ker(\operatorname{Res}_{a_1}\nabla - \lambda_1) \subset \widetilde{\mathcal{L}}|_{a_1}$, and the eigenvalues of the residues of the connection are

$$\lambda_i^+ = \lambda_i,\ \ i = 1, \dots, n, \qquad \lambda_1^- = 1 - \lambda_1, \quad \lambda_i^- = -\lambda_i,\ \ i > 1.$$

The upper modification $(a_1, l_1^-)^{\mathrm{up}}$ defines the inverse isomorphism.

Second, the pair $(\widetilde{\mathcal{L}}, \nabla)$ is irreducible and contains a distinguished sub-sheaf $\widetilde{\mathcal{L}}_0 \subset \widetilde{\mathcal{L}}$ of degree $k' = 0, \dots, \left[\frac{n-4}{2}\right]$. We fix a set of distinct points $y_1, \dots, y_{k'} \in \mathbb{P}^1$ such that

$$\widetilde{\mathcal{L}}_0 \xrightarrow{\sim} \mathcal{O}(y_1 + \dots + y_{k'})$$

and define a connection

$$\nabla_0\colon \widetilde{\mathcal{L}}_0 \to \widetilde{\mathcal{L}}_0 \otimes \Omega(y_1 + \dots + y_{k'});$$

fixing ∇_0 we define a distinguished section $\mathcal{O} \subseteq \widetilde{\mathcal{L}}_0$.

Denote by M_1 the coarse moduli space of triples

$$(\widetilde{\mathcal{L}}_0 \subset \widetilde{\mathcal{L}}, A, \widetilde{\phi}),$$

where

$$(\widetilde{\mathcal{L}}/\widetilde{\mathcal{L}}_0 \simeq \mathcal{O}(-k'-1)), \quad k' = 0, \dots, \left[\frac{n-4}{2}\right],$$

and $A \in \operatorname{Hom}(\widetilde{\mathcal{L}}_0, \widetilde{\mathcal{L}} \otimes \Omega(\mathfrak{M}))$ such that $\operatorname{im}(A) \not\subseteq \widetilde{\mathcal{L}}_0 \otimes \Omega(\mathfrak{M})$. We have a map $\mathcal{M}'_n(2) \to M_1$ defined by

$$(\widetilde{\mathcal{L}}, \nabla, \widetilde{\phi}) \mapsto \big(\widetilde{\mathcal{L}}_0 \subset \widetilde{\mathcal{L}}, A := \nabla|_{\widetilde{\mathcal{L}}_0} - \nabla_0, \widetilde{\phi}\big).$$

Note that on the open subset the moduli space M_1 is isomorphic to the $(n-3)$-th symmetric power of the non-compact surface $\operatorname{Tot}(\mathbb{P}^1, \Omega(\mathfrak{M}))$ and the condition $\operatorname{im}(A) \subset \widetilde{\mathcal{L}}_0 \otimes \Omega(\mathfrak{M})$ defines the infinite section $s_\infty \subset \operatorname{Tot}(\mathbb{P}^1, \Omega(\mathfrak{M}))$.

Third, identify the moduli space $\mathcal{M}'_n(2)$ with the coarse moduli space of collections

$$\big(\widetilde{\mathcal{L}}_0 \subset \widetilde{\mathcal{L}}, A, \widetilde{\phi}; l_1^+, l_1^-, \dots, l_n^+, l_n^-\big),$$

such that

(i) $(\widetilde{\mathcal{L}}_0 \subset \widetilde{\mathcal{L}}, A, \widetilde{\phi})$ is a point of the moduli space M_1;

(ii) $l_i^\pm \subset \widetilde{\mathcal{L}}|_{a_i}$ is the one-dimensional subspace defined by

$$(\operatorname{Res}_{a_i} A - \lambda^\mp)(\widetilde{\mathcal{L}}_0|_{a_i}) \subset l_i^\pm;$$

(iii) $l_i^+ \neq l_i^-$.

In the previous section, it was shown that on the open subset we may identify the $(n-3)$!-covering $\widetilde{\mathcal{M}_n(2)}$ with the $(n-3)$-th power of the surface K'_n. The surface $K'_n \simeq \mathcal{B}l_{\lambda_i^\pm}\operatorname{Tot}(\mathbb{P}^1, \Omega(\mathfrak{M}))$ is obtained by blowing up $K_n := \operatorname{Tot}(\mathbb{P}^1, \Omega(\mathfrak{M}))$ at $2n$ points $(a_i, \lambda_i^\pm)$.

Denote by M_2 the coarse moduli space of $(\widetilde{\mathcal{L}}_0 \subset \widetilde{\mathcal{L}},\, A,\, \widetilde{\phi};\, l_1^+,\, l_1^-,\, \dots,\, l_n^+,\, l_n^-)$ such that only the conditions (i), (ii) are satisfied, and (iii) is hold for all a_i except for some $a \in S$. It is the condition $l_i^+ = l_i^-$ that defines the union of pre-images of the fibers $F_i := \Omega^1(\mathfrak{M})|_{a_i} \subset \mathrm{Tot}(\Omega^1(\mathfrak{M}))$ and the infinite section s_∞. Thus M_2 is a divisor on $\mathcal{M}'_n(2)$; on the open subset of the $(n-3)!$-covering, $\widetilde{M_2}$ is isomorphic to

$$(s_\infty + F_1 + \cdots + F_n) \times \underbrace{\mathcal{B}l_{\lambda^\pm}\mathrm{Tot}(\mathbb{P}^1,\, \Omega(\mathfrak{M})) \times \cdots \times \mathcal{B}l_{\lambda^\pm}\mathrm{Tot}(\mathbb{P}^1,\, \Omega(\mathfrak{M}))}_{n-4}.$$

Naturally, M_2 completes our moduli space $\mathcal{M}'_n(2)$, and we identify M_2 with the compactifying divisor D. Moreover, we state that it is the Drinfeld compactification in the sense of [2] and [8]. Indeed, following the Drinfeld idea we present the moduli space $\mathcal{M}'_n(2)$ as the moduli space of FH-sheaves with some restricting conditions and then we complete the moduli space by removing these restrictions.

Denote by M'_2 the coarse moduli space of $(\widetilde{\mathcal{L}}_0 \subset \widetilde{\mathcal{L}},\, A,\, \widetilde{\phi};\, l_1^+,\, l_1^-,\, \dots,\, l_n^+,\, l_n^-)$ such that only the conditions (i), (ii) are satisfied; the condition (iii) does not hold for all $a_i \in S$, $i = 1, \dots, n-3$. Identify M'_2 with the complete self-intersection locus of the compactifying divisor D and denote it $\mathbf{\Theta}_n$. In fact,

$$\mathbf{\Theta}_n \simeq (s_\infty + \widetilde{F}_1 + \cdots + \widetilde{F}_n)^{(n-3)}.$$

4.2. Geometry of the compactifying divisor and its self-intersection. As we have seen the divisor D (and its complete self-intersection $\mathbf{\Theta}_n$) may be identified with the coarse moduli space of $(\widetilde{\mathcal{L}}_0 \subset \widetilde{\mathcal{L}},\, A,\, \widetilde{\phi})$ with $A \in \mathrm{Hom}(\widetilde{\mathcal{L}}_0,\, \widetilde{\mathcal{L}} \otimes \Omega(\mathfrak{M}))$, satisfied the following two conditions:

(1) $\mathrm{im}(A) \subset \widetilde{\mathcal{L}}_0 \otimes \Omega(\mathfrak{M})$;
(2) $l_a^- := (\mathrm{Res}_a A - \lambda^+)(\widetilde{\mathcal{L}}_0|_a) = l_i^+ := (\mathrm{Res}_a A - \lambda^-)(\widetilde{\mathcal{L}}_0|_a)$ for some (and for all) $a \in S$.

Condition (2) implies $l_a^+ = l_a^- = (\widetilde{\mathcal{L}}_0|_a)$, and for $a = a_i$ it defines the fiber F_i. Altogether, conditions (2) imply (1), and the first condition means that all the subspaces l_i^+ and l_i^-, for $i = 1, \dots, n$, coincide with $\widetilde{\mathcal{L}}_0|_{a_i}$ and define the (blow-up of the) intersection of all fibers F_i, $i = 1, \dots, n$. In this way the conditions (1) and (2) give us components

$$\Theta_{(i)} := (s_\infty + s_\infty + F_1 + \cdots + F_n) \subset \overline{K}_n.$$

Consider the Hecke correspondence between our moduli space $\mathbf{\Theta}_n$ of FH-sheaves $(\mathcal{O} \oplus \mathcal{T}(-\mathfrak{M}) \subset \widetilde{\mathcal{L}})$ of level $n-3$ and the moduli space $\mathbf{\Theta}'_n$ of FH-sheaves $(\mathcal{O} \oplus \mathcal{T}(-\mathfrak{M}) \subset \widetilde{\mathcal{L}}')$ of level zero. In other words, let us perform $n-3$ lower modifications of our bundle $\widetilde{\mathcal{L}}$ of degree -1 at distinct points $a \in \{a_1, \dots, a_n\}$ in the direction $l_a^+ = l_a^-$; for example, let us make $n-3$ lower modifications at points $a_2, \dots, a_{n-2}$. Thus, after such $n-3$ lower modifications, we get the bundle $\mathcal{O} \oplus \mathcal{T}(-\mathfrak{M})$ of degree $2-n$ for the chosen directions $l_a^+ = l_a^-$ lie in $\widetilde{\mathcal{L}}_0|_{a_i}$).

It is more convenient to investigate the complete self-intersection locus $\mathbf{\Theta}_n$ of the compactifying divisor $D = \overline{\mathcal{M}_n(2)} \setminus \mathcal{M}_n(2)$. In fact, it is isomorphic to the coarse moduli space of collections

$$\big(\widetilde{\mathcal{L}}_{\Theta_n},\, \nabla_{\Theta_n},\, \phi'\big),$$

with the fixed eigenvalues of residues of the connection, where $\widetilde{\mathcal{L}}_{\Theta_n}$ is a bundle of degree $2-n$ on $\mathbb{P}^1$ with the horizontal isomorphism $\phi'\colon \det\widetilde{\mathcal{L}}' \xrightarrow{\sim} \mathcal{O}(-a_1 - \dots - a_{n-2})$, and the connection ∇_{Θ_n} has the following eigenvalues of the residues. For $a_i = a_1, \dots, a_{n-2}$ the residues $\operatorname{Res}\nabla_{\Theta_n}$ have eigenvalues $(\lambda_i, 1-\lambda_i)$ and for $a_i = a_{n-1}, a_n$ the eigenvalues are $(\lambda_i, -\lambda_i)$.

The connection ∇_{Θ_n} exists but it is not unique. Let us calculate the dimension of the appropriate affine space. Given an open subset $U \subset \mathbb{P}^1$, denote by $\mathcal{C}(U)$ the set of all local connections $\nabla_{\Theta_n} = \nabla_0 - L(z)$ on U. For two connections $\nabla'_{\Theta_n}, \nabla''_{\Theta_n} \in \mathcal{C}(U)$ their difference $E' := \nabla''_{\Theta_n} - \nabla'_{\Theta_n}$ is an element of $H^0(U, \mathcal{H}om(\widetilde{\mathcal{L}}_{\Theta_n}, \widetilde{\mathcal{L}}_{\Theta_n} \otimes \Omega)) \simeq \mathcal{H}om(\widetilde{\mathcal{L}}_{\Theta_n}/\mathcal{O}, \mathcal{O}\otimes\Omega)$, such that $E'|_{\mathcal{O}} = 0$ and $\operatorname{tr} E' = 0$; let $\mathcal{E}_{\Theta_n}(U)$ be the set of such morphisms E'. Hence, $\mathcal{C}$ has a natural structure of $\mathcal{E}_{\Theta_n}$-torsor, and the obstruction to the existence of a global connection lies in the group $H^1(\mathbb{P}^1, \mathcal{E}_{\Theta_n}(\mathfrak{M}))$, which is dual to $H^0(\{E' \in \operatorname{End}(\mathcal{L}_{\Theta_n})\colon \operatorname{tr} E' = 0,\ E'(a_i)(l_i^+) \subset l_i^+\}) = \{0\}$ by the Serre duality. We define our global connection by reconstructing the row $(L_{21}, -L_{11})$ of the operator $L(z)$; the connection is parameterized by the element L_{21} that lies in $\mathcal{H}om(\widetilde{\mathcal{L}}_{\Theta_n}/\mathcal{O}, \mathcal{O}\otimes\Omega) \simeq \mathcal{E}_{\Theta_n}$. In this way

$$\mathcal{E}_{\Theta_n} \simeq \Omega_{\mathbb{P}^1}^{\otimes 2}(\mathfrak{M}) \simeq \mathcal{O}(n-4),$$

and the dimension of the affine space of the connection ∇_{Θ_n} on the bundle $\widetilde{\mathcal{L}}_{\Theta_n} \simeq \mathcal{O} \oplus \mathcal{T}(-\mathfrak{M})$ equals $n-3$.

At last, just note that one can interpret the divisor D as a moduli space of certain FH-sheaves of level zero considering the appropriate Hecke correspondence, one should however be careful making $n-3$ lower modifications at the distinguished point $a \in S$ and considering certain diagonal components in the sense of 3.8.

4.3. Dynamics of the $\mathfrak{sl}(2)$ Schlesinger system. In the final part of the section, let us present the étale coordinates $\{x_i, p_i\}$, $i = 1, \dots, n-3$, as parameters of the Hecke correspondence between the coarse moduli spaces $\mathbf{\Theta}_n$ and $\mathcal{M}'_n(2)$. Precisely, consider the space of sections of the sheaf $\mathcal{E}_{\Theta_n} \simeq \mathcal{H}om(\widetilde{\mathcal{L}}_{\Theta_n}/\mathcal{O}, \mathcal{O}\otimes\Omega^1_{\mathbb{P}^1})$ on the moduli space $\mathbf{\Theta}'_n$ of collections

$$\big(\widetilde{\mathcal{L}}_{\Theta_n}, \nabla_{\Theta_n};\, \phi'\colon \det\widetilde{\mathcal{L}}_{\Theta_n} \xrightarrow{\sim} \mathcal{O}(-a_1 - \dots - a_{n-2});\, (\widetilde{\lambda}_i^+, \widetilde{\lambda}_i^-),\ i = 1, \dots, n\big)$$

for $\widetilde{\lambda}_i^+ := \lambda_i$ and $\widetilde{\lambda}_i^- = 1 - \lambda_i$, for $a_i \neq a_{n-1}, a_n$; the rest $\widetilde{\lambda}_i^- = -\lambda_i$ for $a_i = a_{n-1}, a_n$. Note here that the configuration $(\widetilde{\mathcal{L}}_{\Theta_n}; l_1^+, \dots, l_n^+)$ is semi-stable in our notation, since we have

$$\operatorname{Aut}(\widetilde{\mathcal{L}}_{\Theta_n}) \simeq \begin{matrix} \operatorname{End}(\mathcal{O}) & \oplus\ \operatorname{Hom}(\mathcal{T}(-\mathfrak{M}),\ \mathcal{O}) \\ \oplus & \oplus \\ \operatorname{Hom}(\mathcal{O},\ \mathcal{T}(-\mathfrak{M}))\ \oplus & \operatorname{End}(\mathcal{T}(-\mathfrak{M})) \end{matrix} \simeq \mathcal{O}\oplus\mathcal{O}\oplus\mathcal{O}(n-2),$$

hence, $\operatorname{Aut}(\widetilde{\mathcal{L}}_{\Theta_n}; l_1^+, \dots, l_n^+) \simeq \mathbb{C}^*$.

As we have seen, the space of sections of the sheaf $\mathcal{E}_{\Theta_n}$ on $\mathbf{\Theta}_n$ has dimension $(n-3)$, hence,

$$\dim\Gamma(\mathbf{\Theta}_n, \mathcal{E}_{\Theta_n}) + \dim\mathbf{\Theta}_n = 2\cdot(n-3),$$

that is, exactly the dimension of the moduli space $\mathcal{M}'_n(2)$. Take a collection of distinct points $\{x_1, \dots, x_{n-3}\} \subset \mathbb{P}^1$ and a collection of one-dimensional subspaces

$p_i \subset \widetilde{\mathcal{L}}_{\Theta_n}|_{x_i}$, $i = 1, \ldots, n-3$, and perform the modifications

$$A := (x_1, p_1)^{\mathrm{up}} \circ \cdots \circ (x_{n-3}, p_{n-3})^{\mathrm{up}} \colon\ \widetilde{\mathcal{L}}_{\Theta_n} \to \widetilde{\mathcal{L}},$$

where $\widetilde{\mathcal{L}}$ is a rank 2 bundle of degree -1 on $\mathbb{P}^1$. As we have already seen, this gives us a map from $\mathcal{M}'_n(2)$ to the symmetric product $(\mathrm{Tot}(\mathbb{P}^1, \Omega(\mathfrak{M})))^{(n-3)}$ at the generic point.

Besides, choose the unique connection $\nabla_{\Theta_n}(p_1, \ldots, p_{n-3}) \in \mathcal{E}_{\Theta_n}$ such that the subspaces $p_1, \ldots, p_{n-3}$ are invariant for it. The modification of the connection is

$$A\colon\ \nabla_{\Theta_n}(p_1, \ldots, p_{n-3}) \to \widetilde{\nabla} = \nabla_{\Theta_n}(p_1, \ldots, p_{n-3}) - \sum_{i=1}^{n-3} \boldsymbol{P}_{p_i} \frac{dz}{z - x_i},$$

where $\boldsymbol{P}_{p_i}$ are the projectors on the invariant one-dimensional subspaces $p_1, \ldots, p_{n-3}$. Note that this correspondence is isomonodromic and the terms $\boldsymbol{P}_{p_i} \dfrac{dz}{z - x_i}$ do not change the monodromy of the connection; the points $x_1, \ldots, x_{n-3}$ are called *apparent singularities* of the connection $\widetilde{\nabla}$. Originally, apparent singularities were introduced in [11] by L. Fuchs; more detailed approach to Fuchsian equations and systems one can find in the books [1], [4].

In this way, we interpret the Hecke correspondence between the moduli spaces $\boldsymbol{\Theta}_n$ and $\mathcal{M}'_n(2)$ as the deformation of the most denerate locus $\boldsymbol{\Theta}_n$ of D in the fibred space $\mathrm{Tot}(\boldsymbol{\Theta}_n, \mathcal{E}_{\Theta_n})$ performed by modifications of the connection ∇_{Θ_n} with apparent singularities $\boldsymbol{P}_{p_i} \dfrac{dz}{z - x_i}$. In the case when $x_i \in S$, the dynamics of the Schlesinger system is discrete and presented by the lattice C_n; for calculations, see the proposition in Section 2; for applications to the relations between special functions — solutions of the Fuchsian equations, —, see [22].

5. An Example: the Painlevé-VI System

Now, we illustrate our constructions of the étale coordinates on the initial data space and its compactification in the simplest example of the $\mathfrak{sl}(2)$-Schlesinger system with four marked points, called the sixth Painlevé system. In this section we suppose that $\mathcal{L}$ is a rank 2 vector bundle on $\mathbb{P}^1$ with $\det \mathcal{L} \simeq \mathcal{O}$, and a logarithmic connection ∇ with eigenvalues $(\lambda_i, -\lambda_i)$ of the residues at four singularities a_i, $i = 1, \ldots, 4$. So, we have a module $\mathfrak{M} = \sum a_i$, and, modulo projective transformations of $\mathbb{P}^1$ by the three-dimensional group $\mathrm{PGL}(2, \mathbb{C})$, we can suppose $\mathfrak{M} = 0 + 1 + t + \infty$, where $t := r(a_1, a_2, a_3, a_4)$ is the cross-ratio; however, $\nabla\colon \mathcal{L} \to \mathcal{L} \otimes \Omega^1(\mathfrak{M})$.

Following the ideas of previous sections, we shall investigate the geometry of the moduli space $\mathcal{M}_4$ of such pairs $(\mathcal{L}, \nabla)$; its biggest cell is isomorphic to the symplectic quotient $\mathscr{O}_1 \times \cdots \times \mathscr{O}_4 /\!/ \mathrm{SL}(2, \mathbb{C})$. We identify it with the phase space of the Schlesinger system with four points on $\mathbb{P}^1$, called the sixth Painlevé equation. We define suitable coordinates using the geometric construction of the Schlesinger system from [2]; then we construct a natural compactification of the phase space also considered in [2]. At the end, we discuss the geometric realization of the dynamics and the interplay with the apparent singularities, which is original.

First, consider the configuration space of the Painlevé-VI system. It is the moduli space of so-called quasi-parabolic bundles $\mathcal{N}_4$. Precisely, $\mathcal{N}_4$ is the moduli space of the collections

$$\big(\mathcal{L};\ \phi\colon \det\mathcal{L} \simeq \mathcal{O};\ l_1,\ \dots,\ l_4\big),$$

where $\mathcal{L}$ is a rank 2 bundle with a horizontal isomorphism ϕ, and $l_i \subset \mathcal{L}|_{a_i}$ are one-dimensional subspaces; there is a canonical surjection $\pi\colon \mathcal{M}_4 \twoheadrightarrow \mathcal{N}_4$ defined by

$$\big(\mathcal{L},\ \nabla;\ \lambda_1,\ \dots,\ \lambda_n\big) \mapsto \big(\mathcal{L};\ l_i^+ := \ker(\operatorname{Res}_{a_i}\nabla - \lambda_i),\ i = 1,\ \dots,\ 4\big).$$

In fact, the configuration space $\mathcal{N}_4(2)$ is parameterized by the x coordinate. As we have seen above, each pair x_i, p_i naturally parameterizes the non-trivial bundle $\operatorname{Tot}(\mathbb{P}^1,\ \Omega(\mathfrak{M}))$; in this way, it is interesting to calculate the map π.

5.1. Geometry of $\mathcal{N}_4(2)$. Let us describe the configuration space of four eigenvectors in the two-dimensional vector space or the configurations of four points l_1, l_2, l_3, l_4 in $\mathbb{P}^1$. The invariant of the configuration is the cross-ratio

$$r(l_1,\ l_2,\ l_3,\ l_4) := \frac{l_1 - l_3}{l_1 - l_4}\cdot\frac{l_2 - l_4}{l_2 - l_3};$$

naturally, it is a coordinate on $\mathcal{N}_4(2)$. Since we have the action of the projective group $\mathrm{PGL}(2,\ \mathbb{C})$, we can suppose

$$l_1 = X, \quad l_2 = 1, \quad l_3 = 0, \quad l_4 = \infty, \quad \text{hence,} \quad r(l_1,\ l_2,\ l_3,\ l_4) = X;$$

let us calculate the behavior of $X = r(l_1,\ l_2,\ l_3,\ l_4)$ under the action of the permutational factor-group

$$0 \to (\mathbb{Z}/2\mathbb{Z})^2 \to \mathfrak{S}_4 \to \mathfrak{S}_3 \to 1.$$

The possible values of the cross-ratio are $1 - X$, X^{-1}, $1 - X^{-1}$. For example the value

$$1 - X = 1 - \frac{l_1 - l_3}{l_1 - l_4}\cdot\frac{l_2 - l_4}{l_2 - l_3} = \frac{l_4 - l_3}{l_4 - l_1}\cdot\frac{l_2 - l_1}{l_2 - l_3}$$

corresponds to two different permutations: $(14) := l_1 \leftrightarrow l_4$ and $(23) := l_2 \leftrightarrow l_3$. Thus, it corresponds to two different quasi-parabolic bundles: one with $\{l_4 = l_1 \neq l_2 \neq l_3 \neq l_1\}$ and another with $\{l_3 = l_2 \neq l_1 \neq l_4 \neq l_2\}$. In this way, if two of the four points on the Riemann sphere try to glue, then two other glue too: $X \to \infty$ if and only if $1 \to 0$. Moreover, for every value $X = 0$, $X = 1$, $X = \infty$, there are two different configurations of quasi-parabolic bundles. Note that the configuration of the quasi-parabolic bundle for the value $X = r(l_1,\ l_2,\ l_3,\ l_4) = t = r(a_1,\ a_2,\ a_3,\ a_4)$ corresponds to the nontrivial bundle $\mathcal{L} \simeq \mathcal{O}(1) \oplus \mathcal{O}(-1)$.

Choose a basis in the two-dimensional fiber of our bundle: $\mathcal{L}|_{a_i} := \langle l_2,\ l_3\rangle$; then

$$\begin{cases} l_1 = \alpha\cdot l_2 + \beta\cdot l_3 = l_2 + l_3; \\ l_2 = 1\cdot l_2 + 0\cdot l_3; \\ l_3 = 0\cdot l_2 + 1\cdot l_3; \\ l_4 = \gamma\cdot l_2 + \delta\cdot l_3 = l_2 + r(\alpha,\ \beta,\ \gamma,\ \delta)\cdot l_3, \end{cases} \qquad X = r(\alpha,\ \beta,\ \gamma,\ \delta);$$

consider the action of pairs of modifications on our bundle (see Section 2):

$$(a_2,\, l_2)^{\mathrm{up}}\colon \mathcal{L}\to\mathcal{L}',\quad \langle l_2,\, l_3\rangle \to \left\langle l_2' := \frac{l_2}{X-a_2},\, l_3\right\rangle,$$
$$(a_3,\, l_3)^{\mathrm{low}}\colon \mathcal{L}'\to\widetilde{\mathcal{L}},\quad \langle l_2',\, l_3\rangle \to \left\langle \widetilde{l}_2 := \frac{X-a_3}{X-a_2}\cdot l_2,\, l_3\right\rangle.$$

We have the modified eigenvectors

$$\begin{cases} \widetilde{l}_1 = \left(\dfrac{X-a_3}{x-a_2}\cdot l_2 + l_3\right)_{X=a_1} = l_2 + l_3; \\ l_2 = 1\cdot l_2 + 0\cdot l_3; \\ l_3 = 0\cdot l_2 + 1\cdot l_3; \\ \widetilde{l}_4 = \left(\dfrac{X-a_3}{X-a_2}\cdot l_2 + r(\alpha,\,\beta,\,\gamma,\,\delta)\cdot l_3\right)_{X=a_4} = r(a_1,\,a_2,\,a_3,\,a_4)\cdot l_2 + r(\alpha,\,\beta,\,\gamma,\,\delta)\cdot l_3; \end{cases}$$

if $r(\alpha,\,\beta,\,\gamma,\,\delta) \to t = r(a_1,\,a_2,\,a_3,\,a_4)$, then $\widetilde{l}_4 \to \widetilde{l}_1$. An analogous calculation with the pair of modifications $(a_1,\, l_1)^{\mathrm{up}}(a_4,\, l_4)^{\mathrm{low}}$ shows that the case $\widetilde{l}_2 \to \widetilde{l}_3$ gives the same value $x = t$, hence, this value corresponds to two different nontrivial quasi-parabolic bundles, and finally we have the following

Statement [2]**.** *$\mathcal{N}_4$ is isomorphic to two copies of $\mathbb{P}^1$ glued outside* $\{0,\,1,\,t,\,\infty\}$.

The action of pairs of modifications on $\mathcal{N}_4$ is evident, and it presents the affine $\widehat{D}_4$ lattice.

5.2. Geometry of $\mathcal{M}_4(2)$. Describe the geometry of the moduli space of collections

$$\big(\mathcal{L},\,\nabla;\;\; \phi\colon \det\mathcal{L}\simeq\mathcal{O};\;\; \lambda_1,\,\lambda_2,\,\lambda_3,\,\lambda_4\big),$$

where $\mathcal{L}$ is a rank 2 vector bundle with fixed holomorphic structure ϕ on the determinant, and ∇ is a logarithmic connection with fixed eigenvalues of the residues at the points of the support S of the module $\mathfrak{M} = 0 + 1 + t + \infty$ on $\mathbb{P}^1$. Put the eigenvalue condition

$$\sum \epsilon_i\lambda_i \notin \mathbb{Z},\qquad (\epsilon_1,\,\epsilon_2,\,\epsilon_3,\,\epsilon_4)\in(\mathbb{Z}/2\mathbb{Z})^4,$$

which provides the irreducibility of our pair $(\mathcal{L},\,\nabla)$. Our notion of stability (see 3.1) of our pair $(\mathcal{L},\,\nabla)$ implies that neither of the eigenvectors $l_i^+ := \ker(\mathrm{Res}_{x_i}\nabla - \lambda_i)$ may lie in the sub-bundle $\mathcal{L}_0 \simeq \mathcal{O}(1)$. Modify our bundle, say, at $(\infty,\, l_\infty^+)^{\mathrm{low}}$, we necessarily get the bundle $\widetilde{\mathcal{L}} \simeq \mathcal{O}\oplus\mathcal{O}(-\infty)$; this modification presents an isomorphism

$$\mathcal{M}_4 \simeq \mathcal{M}_4' := \text{moduli space of } \big(\widetilde{\mathcal{L}},\,\widetilde{\nabla};\;\; \widetilde{\phi}\colon \det\widetilde{\mathcal{L}}\simeq\mathcal{O}(-\infty);$$
$$(\lambda_1,\,-\lambda_1),\,\ldots,\,(\lambda_\infty,\,1-\lambda_\infty)\big).$$

In this way, we get a uniquely defined sub-bundle

$$\widetilde{\mathcal{L}} \supset \widetilde{\mathcal{L}}_0 \simeq \mathcal{O}$$

with the standard connection d. Restrict our connection to the sub-bundle, and consider the operator

$$A(z) := \mathrm{Id} \oplus (\nabla|_{\widetilde{\mathcal{L}}_0} - \partial_z)\colon\ \mathcal{O} \oplus \widetilde{\mathcal{L}}_0 \to \widetilde{\mathcal{L}} \otimes \Omega^1(\mathfrak{M}).$$

Our pair is irreducible, $\mathrm{im}(\nabla|_{\widetilde{\mathcal{L}}_0} - \partial_z)(\widetilde{\mathcal{L}}_0) \nsubseteq \widetilde{\mathcal{L}}_0$, hence,

$$A(z) := \mathrm{Id} \oplus (\nabla|_{\widetilde{\mathcal{L}}_0} - \partial_z)\colon\ \mathcal{O} \oplus \mathcal{T}(-\mathfrak{M}) \to \widetilde{\mathcal{L}}.$$

The determinant $\det A(z)$ has a simple pole at some point x and, moreover, $A(z) = (x,\, p)^{\mathrm{up}}$; the variables x and p are the canonical coordinates on the two-dimensional phase space $\mathcal{M}_4$ of our Schlesinger system. The surface $\mathcal{M}_4$ is noncompact and has a structure of a fibred space over $\mathcal{N}_4$; we calculate this projection at the end of this paragraph. Note that in our case the cohomological calculations are very simple: $\mathcal{E} \simeq \mathcal{O}(-2)^* \otimes \mathcal{O}(-1) \otimes \Omega \simeq \mathcal{O}(-1)$ and $H^1(\mathcal{E}) = 0$, hence, $\mathcal{M}_4 \simeq K'_4$.

Let us perform some explicit calculations to illustrate the behavior of our system. First of all, we have $\nabla = \mathbf{1} \cdot \partial_z - L(z)\colon \widetilde{\mathcal{L}} \to \widetilde{\mathcal{L}} \otimes \Omega^1(\mathfrak{M})$, where

$$L(z) = \begin{pmatrix} \omega & \eta \\ \varrho & -\omega \end{pmatrix}.$$

The following conditions hold:

$$A(z) = (x,\, p)^{\mathrm{up}} = \mathrm{Id} \oplus (\nabla|_{\widetilde{\mathcal{L}}_0} - \partial_z) = \begin{pmatrix} \omega & \eta \\ 1 & 0 \end{pmatrix}\colon\ \mathcal{O} \oplus \mathcal{O}(-2) \to \mathcal{O} \oplus \mathcal{O}(-\infty).$$

This implies $\det A(x) = 0$, hence, $\eta(x) = 0$, and $\omega(x) = p \cdot dz$. Moreover, the following conditions on the eigenvalues of the residues hold:

$$-(\mathrm{Res}_{x_\alpha} \omega)^2 - (\mathrm{Res}_{x_\alpha} \eta)(\mathrm{Res}_{x_\alpha} \varrho) = -\lambda_\alpha^2, \quad \alpha = 0,\, 1,\, t,$$

and

$$-(\mathrm{Res}_\infty \omega)^2 - (\mathrm{Res}_\infty \eta)(\mathrm{Res}_\infty \varrho) = \lambda_\infty(1 - \lambda_\infty).$$

Thus, the 1-form

$$\varrho(z) = \left(\varrho_\infty + \frac{\varrho_0}{z} + \frac{\varrho_1}{z-1} + \frac{\varrho_t}{z-t}\right) dz.$$

The element $\eta(z) \in \mathrm{Hom}(\mathcal{T}(-0-1-t-\infty),\, \mathcal{O}(-\infty)) = \Omega^1(0+1+t)$, hence, first,

$$\eta(z) = c\frac{dz}{z^2} + \left(\frac{\mu_0}{z} + \frac{\mu_1}{z-1} + \frac{\mu_t}{z-t}\right) dz, \qquad -\mu_t = \mu_0 + \mu_1;$$

and second, we can define 1-form ω modulo the addition of $\eta \cdot s$, where $s \in \Gamma(\mathbb{P}^1,\, \mathcal{O}(-\infty))$ and $\omega(z) = \left(\frac{\nu}{z} - \frac{\nu}{z-1}\right) dz$. Finally, we perform the procedure of symplectic reduction $\mathscr{O} \times \mathscr{O} \times \mathscr{O} \times \mathscr{O} /\!/ \mathrm{SL}(2,\, \mathbb{C})$, and choose the canonical variables x and p. Express all the matrix elements through the canonical ones:

$$-\mu_0 \cdot \frac{t}{x} = \mu_1 \cdot \frac{t-1}{x-1}, \quad \nu = -px(x-1),$$

$$\varrho_0 = \frac{\lambda_0^2 - \nu^2}{\mu_0}, \quad \varrho_1 = \frac{\lambda_1^2 - \nu^2}{\mu_1}, \quad \varrho_t = -\frac{\lambda_t^2}{\mu_0 + \mu_1}.$$

At the infinity in our bundle $\mathcal{O}\oplus\mathcal{O}(-\infty)$, we have to reglue our connection $\widetilde{\nabla}$ to get

$$\begin{pmatrix}1 & 0\\ 0 & z^{-1}\end{pmatrix}\left\{\begin{pmatrix}\partial_z & 0\\ 0 & \partial_z\end{pmatrix}+\begin{pmatrix}0 & \varrho_\infty\\ c\cdot z^{-2} & 0\end{pmatrix}\right\}\begin{pmatrix}1 & 0\\ 0 & z\end{pmatrix}=\begin{pmatrix}0 & \varrho_\infty\cdot z^{-1}\\ c\cdot z^{-1} & z\end{pmatrix},$$

and then calculate the behavior of $\eta(z)$ in the following way:

$$\eta(z)=\left(\frac{\mu_0}{z}+\frac{\mu_1}{z}\cdot\frac{1}{1-z^{-1}}+\frac{\mu_t}{z}\cdot\frac{1}{1-\frac{t}{z}}\right)dz=\frac{1}{z}\underbrace{(\mu_0+\mu_1+\mu_t)}_{=0}+\frac{1}{z^2}(\mu_1+t\cdot\mu_t);$$

therefore we have

$$c=-(\mu_0\cdot t+\mu_1(t-1))\quad\text{and}\quad \varrho_\infty=\frac{\lambda_\infty(1-\lambda_\infty)}{\mu_0\cdot t+\mu_1(t-1)}.$$

Set $L_i:=\operatorname{Res}_{a_i}L$ and calculate the Hamiltonian of the Painlevé VI system in our coordinates x and p:

$$H=\frac{\operatorname{tr}L_tL_0}{t}+\frac{\operatorname{tr}L_tL_1}{t-1}=\frac{1}{t(t-1)}\Bigg[p^2x(x-1)(x-t)+\frac{\lambda_0^2}{x}(x-t)+\frac{\lambda_1^2}{x-1}(x-t)+\frac{\lambda_t^2}{x-t}(t(x-t)+x(t-1))\Bigg];$$

one can check that it is equivalent to the “standard” one (see [22]).

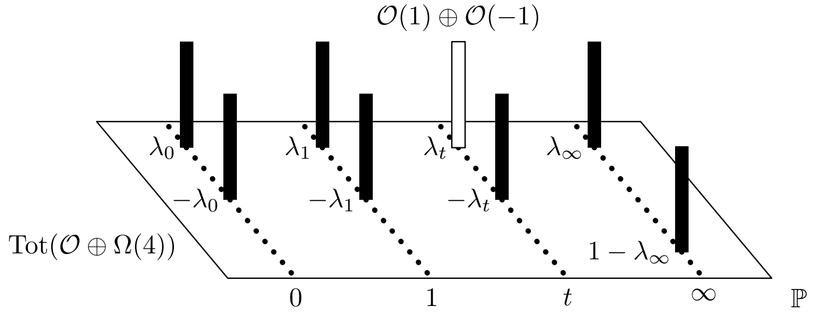
At last, calculate the map $\pi\colon\mathcal{M}_4\twoheadrightarrow\mathcal{N}_4$ in coordinates x and p. The eigenvectors l_i^+ of Res_{a_i} are

$$l_0^+=\left(\frac{\lambda_0+\nu}{\mu_0},1\right),\qquad l_1^+=\left(\frac{\lambda_1-\nu}{\mu_1},1\right),$$
$$l_t^+=\left(-\frac{\lambda_t}{\mu_0+\mu_1},1\right),\quad l_\infty^+=\left(\frac{\lambda_\infty-1}{\mu_0t+\mu_1(t-1)},1\right);$$

and $\mathcal{N}_4$ is parameterized by the coordinate

$$X=\frac{\dfrac{\lambda_0}{x}-p(x-1)-\dfrac{\lambda_t}{x-t}(t-1)}{\dfrac{\lambda_0}{x}-p(x-1)-(\lambda_\infty-1)\dfrac{1}{t}}\cdot\frac{\dfrac{\lambda_1}{x-1}+px+(\lambda_\infty-1)\dfrac{1}{t-1}}{\dfrac{\lambda_1}{x-1}+px+\dfrac{\lambda_t}{x-t}t}.$$

5.3. Geometry of the Painlevé-VI system. As we have seen, the moduli space $\mathcal{M}_4'(2)$ is the non-compact surface



The exceptional divisor at a point (t, λ_t) corresponds to the collection $\big(\widetilde{\mathcal{L}}, \widetilde{\nabla}; \widetilde{\phi}\colon \det\widetilde{\mathcal{L}} \simeq \mathcal{O}(-\infty); (\lambda_1, -\lambda_1), \dots, (\lambda_\infty, 1-\lambda_\infty)\big)$ with a nontrivial bundle $\widetilde{\mathcal{L}} \simeq \mathcal{O}(1) \oplus \mathcal{O}(-1)$.

In this way we have the following presentation of the initial data space

$$\mathcal{M}_4(2) \simeq K_4' := \big(\mathcal{B}l_{R^{-1}(\lambda_i^\pm)}\mathrm{Tot}(\mathbb{P}^1, \mathcal{O}(2))\big) \setminus \bigsqcup \widetilde{F}_i, \quad i = 1, \dots, 4,$$

it is isomorphic to the moduli space of stable FH-sheaves

$$(\mathcal{O} \oplus \mathcal{T}(-0-1-t-\infty) \subset \mathcal{O} \oplus \mathcal{O}(-\infty))$$

of level 1. In other words, the coordinates (x, p) on the initial data space present it as the moduli space of exact sequences

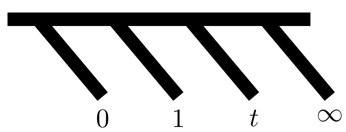
$$0 \to \mathcal{O} \oplus \mathcal{T}(-4) \to \mathcal{O} \oplus \mathcal{O}(-\infty) \to \delta_x \otimes p \otimes \mathcal{T}_x \to 0$$

such that $p < \infty$, and if $x = a \in S$, then $p = \lambda_a^\pm$.

Consider the natural symplectic form $\varpi = dx \wedge dp$ on $\mathbb{P}(\mathcal{O} \oplus \Omega(4))$, and let us look at its behavior when $x \in S$. At singular points of the connection, the dynamics is discrete and performed by the lattice $\widehat{F}_4$. We blow-up eight points $(x, p) = (a, \lambda_a^\pm)$, $a \in S$, on the surface $\mathbb{P}(\mathcal{O} \oplus \Omega(4))$; locally this procedure is performed by $p = s \cdot x$ for s a coordinate on the exceptional divisor. Then, remove four fibers $\widetilde{F}_a := \{a, p\} \subset \mathbb{P}(\mathcal{O} \oplus \Omega(4))$, and in this way at $x = a$ we have two exceptional curves with

$$ds = \frac{dp}{x} - s \cdot \frac{dx}{x}.$$

The compactifying set is exactly the divisor of poles of the symplectic form $\varpi = dx \wedge dp$, and it performs the degeneration of an elliptic curve C. The divisor is

$$D = (2 \cdot s_\infty + \widetilde{F}_0 + \widetilde{F}_1 + \widetilde{F}_t + \widetilde{F}_\infty)^{\mathrm{red}} =$$ 

,

it is defined by the conditions $p = \infty$ and $l_a^+ = l_a^-$, $a = 0, 1, t, \infty$. Let $\widetilde{\mathcal{L}}$ be the bundle corresponding to a point on the compactifying divisor, and perform the lower modification, say, at $a = 0$ in the direction

$$l_0^+ \subset \mathcal{O}|_{z=0} \subset (\mathcal{O} \oplus \mathcal{O}(-1))|_{z=0}.$$

We get the bundle $\widetilde{\mathcal{L}}_D \simeq \mathcal{O} \oplus \mathcal{T}(-4)$, and we have an isomorphism of D with the moduli space of collections

$$\big(\widetilde{\mathcal{L}}_D, \nabla_D, \phi', (\widetilde{\lambda}_i^+, \widetilde{\lambda}_i^-)\big),$$

where $\widetilde{\mathcal{L}}_D$ is a bundle of degree -2 on $\mathbb{P}^1$ with the horizontal isomorphism $\phi'\colon \det\widetilde{\mathcal{L}}' \xrightarrow{\sim} \mathcal{O}(-0-\infty)$ and the connection ∇_D with the following eigenvalues of residues $(\widetilde{\lambda}_0^+, \widetilde{\lambda}_0^-) = (\lambda_0, 1-\lambda_0)$,

$$(\widetilde{\lambda}_1^+, \widetilde{\lambda}_1^-) = (\lambda_1, -\lambda_1), \quad (\widetilde{\lambda}_t^+, \widetilde{\lambda}_t^-) = (\lambda_t, -\lambda_t), \quad (\widetilde{\lambda}_\infty^+, \widetilde{\lambda}_\infty^-) = (\lambda_\infty, 1-\lambda_\infty).$$

Finally, we have the following diagram

$$\mathcal{O} \oplus \mathcal{T}(-4) \xrightarrow{(x,p)^{\mathrm{up}}} \mathcal{O} \oplus \mathcal{O}(-1) \leftrightarrows \begin{bmatrix} \mathcal{O} \oplus \mathcal{O}, \\ \mathcal{O}(1) \oplus \mathcal{O}(-1). \end{bmatrix}$$

The right two arrows $\leftrightarrows$ denote the action of discrete $\widehat{F}_4$-symmetries (see [3], [21]), and the left arrow $\xrightarrow{(x,p)^{\mathrm{up}}}$ in terms of the connections is

$$(x,\, p)^{\mathrm{up}} \colon \widetilde{\nabla} = \nabla_D(p) - \boldsymbol{P}_p \frac{dz}{z - x}.$$

Note here that the connection ∇_D is not uniquely defined. Such connections on the bundle $\mathcal{O} \oplus \mathcal{O}(-2)$ form a one-dimensional affine space, and we choose uniquely the connection $\nabla_D(p)$ for which the direction p is proper; otherwise, as it was shown, we can get the quadratic pole of $\widetilde{\nabla}$ at $z = x$.

The term $\boldsymbol{P}_p \frac{dz}{z - x}$ does not change the monodromy of connections and the simple pole at $z = x$ is called an apparent singular point for the corresponding Fuchsian system. In this way, we present the isomonodromic system Painlevé-VI as the deformation of the moduli space D by the Hecke correspondence $(x,\, p)^{\mathrm{up}}$.

For the interpretation of the Painlevé-VI system as a deformation of the compactifying divisor in terms of the Kodaira–Spencer theory, see [24] and [28].

Acknowledgements. I am deeply grateful to A. Levin for numerous fruitful and stimulating discussions, in particular, for teaching me the FH-sheaves technique and for discussions of D. Arinkin's and S. Lysenko's papers. I am thankful to M. Finkelberg, S. Khoroshkin, D. Lebedev, S. Morel, A. Zotov for useful discussions. I appreciate the interest to this work of I. Krichever, V. Roubtsov, and M. Olshanetsky. Finally, I would like to thank and express my gratitude to the administration of the Independent University of Moscow, École Normale Supérieure de Paris, and Département de mathématiques de l'Université d'Angers for support, hospitality, and excellent working conditions.

References

[1] D. V. Anosov and A. A. Bolibruch, *The Riemann-Hilbert problem*, Aspects of Mathematics, E22, Friedr. Vieweg & Sohn, Braunschweig, 1994. MR 1276272

[2] D. Arinkin and S. Lysenko, *On the moduli of* SL(2)*-bundles with connections on* $\mathbf{P}^1 \setminus \{x_1, \ldots, x_4\}$, Internat. Math. Res. Notices (1997), no. 19, 983–999. MR 1488348

[3] G. D. Birkhoff, *Collected mathematical papers (in three volumes). Vol. I: Boundary value problems and associated Sturmian problems, differential equations, difference equations, dynamics (partial)*, Dover Publications Inc., New York, 1968. MR 0235971

[4] A. A. Bolibrukh, *Hilbert's twenty-first problem for linear Fuchsian systems*, Trudy Mat. Inst. Steklov. **206** (1994) (Russian). MR 1395110. English translation: Proc. Steklov Inst. Math. **1995**, no. 5 (206).

[5] V. G. Drinfeld, *Elliptic modules*, Mat. Sb. (N.S.) **94(136)** (1974), 594–627, 656 (Russian). MR 0384707. English translation: Math. USSR-Sb. **23** (1974), no. 4, 561–592 (1976).

[6] V. G. Drinfeld, *Elliptic modules and their applications to the Langlands and to the Peterson conjectures for* GL(2) *over functional field*, Master's thesis, Moscow Univ., 1977.

[7] V. G. Drinfeld, *Commutative subrings of certain noncommutative rings*, Funktsional. Anal. i Prilozhen. **11** (1977), no. 1, 11–14, 96 (Russian). MR 0476732. English translation: Functional Anal. Appl. **11** (1977), no. 1, 9–12.

[8] V. G. Drinfeld, *Two-dimensional l-adic representations of the fundamental group of a curve over a finite field and automorphic forms on* GL(2), Amer. J. Math. **105** (1983), no. 1, 85–114. MR 692107

[9] H. Flaschka and D. W. McLaughlin, *Canonically conjugate variables for the Korteweg-de Vries equation and the Toda lattice with periodic boundary conditions*, Progr. Theoret. Phys. **55** (1976), no. 2, 438–456. MR 0403368

[10] H. Flaschka and A. C. Newell, *Monodromy- and spectrum-preserving deformations. I*, Comm. Math. Phys. **76** (1980), no. 1, 65–116. MR 588248

[11] L. Fuchs, *Zur theorie der linearen Differentialgleichungen mit ver`anderlichen Koeffizienten*, J. für Math **68** (1868), 354–385.

[12] W. Fulton and R. MacPherson, *A compactification of configuration spaces*, Ann. of Math. (2) **139** (1994), no. 1, 183–225. MR 1259368

[13] M. Gaudin, *La fonction d'onde de Bethe*, Collection du Commissariat à l'Énergie Atomique: Série Scientifique. [Collection of the Atomic Energy Commission: Science Series], Masson, Paris, 1983. MR 693905

[14] A. Gerasimov, S. Kharchev, D. Lebedev, and S. Oblezin, *On a class of representations of the Yangian and moduli space of monopoles*, Comm. Math. Phys. (2005). to appear.

[15] A. Gorsky, N. Nekrasov, and V. Rubtsov, *Hilbert schemes, separated variables, and D-branes*, Comm. Math. Phys. **222** (2001), no. 2, 299–318. MR 1859601

[16] A. Grothendieck and M. Demazure (eds.), *Schémas en groupes. I: Propriétés générales des schémas en groupes*, Séminaire de Géométrie Algébrique du Bois Marie 1962/64 (SGA 3). Lecture Notes in Mathematics, Vol. 151, Springer-Verlag, Berlin, 1970. MR 0274458

[17] E. Hecke, *Mathematische Werke*, Herausgegeben im Auftrage der Akademie der Wissenschaften zu Göttingen, Vandenhoeck & Ruprecht, Göttingen, 1959. MR 0104550

[18] N. J. Hitchin, *Twistor spaces, Einstein metrics and isomonodromic deformations*, J. Differential Geom. **42** (1995), no. 1, 30–112. MR 1350695

[19] M. Jimbo and T. Miwa, *Monodromy preserving deformation of linear ordinary differential equations with rational coefficients. II*, Phys. D **2** (1981), no. 3, 407–448. MR 625446

[20] I. M. Krichever, *Integration of nonlinear equations by the methods of algebraic geometry*, Funkcional. Anal. i Priložen. **11** (1977), no. 1, 15–31, 96 (Russian). MR 0494262. English translation: Functional Anal. Appl. **11** (1977), no. 1, 12–26.

[21] A. M. Levin, M. A. Olshanetsky, and A. Zotov, *Hitchin systems—symplectic Hecke correspondence and two-dimensional version*, Comm. Math. Phys. **236** (2003), no. 1, 93–133. MR 1977883

[22] S. V. Oblezin, *Discrete symmetries of systems of isomonodromic deformations of second-order differential equations of Fuchsian type*, Funktsional. Anal. i Prilozhen. **38** (2004), no. 2, 38–54, 95. MR 2086626

[23] H. Röhrl, *Das Riemann-Hilbertsche Problem der Theorie der linearen Differentialgleichungen*, Math. Ann. **133** (1957), 1–25. MR 0086958

[24] M.-H. Saito, T. Takebe, and H. Terajima, *Deformation of Okamoto-Painlevé pairs and Painlevé equations*, J. Algebraic Geom. **11** (2002), no. 2, 311–362. MR 1874117

[25] L. Schlesinger, *Uber eine Klasse von Differentialsystemen beliebiger Ordnung mit festen kritischen Punkten*, J. Reine Angew. Math. **141** (1912), 96–145.

[26] J.-P. Serre, *Groupes algébriques et corps de classes*, Publications de l'institut de mathématique de l'université de Nancago, VII. Hermann, Paris, 1959. MR 0103191

[27] E. K. Sklyanin, *Separation of variables—new trends*, Progr. Theoret. Phys. Suppl. (1995), no. 118, 35–60. MR 1356514

[28] H. Terajima, *Okamoto–Painlevé pairs and Painlevé equations*, Master's thesis, Kobe Univ., 2001.

[29] A. Weil, *Généralisation des fonctions abéliennes*, J. Math. Pures Appl. (9) **17** (1938), 47–87.

Institute of Theoretical and Experimental Physics, 117259, Moscow, Bol. Cheremushkinskaya, 25

E-mail address: oblezin@mccme.ru, sergey.oblezin@itep.ru