Critical path analysis in the network with fuzzy interval numbers as activity times

V.Sireesha , K.Srinivasa Rao, N.Ravi Shankar,  S. Suresh Babu
Department of Applied Mathematics, GIS, GITAM University, Visakhapatnam, INDIA

Abstract
In most critical path method (CPM) problems, the activity times are not known exactly. By assuming that the activity times are known statistically, one proceeds to find the critical path by using the random numbers as activity times. However, in many practical cases the activity times are known imprecisely, rather than uncertainty, and hence in such cases statistics may not be useful. In such cases, the imprecision may be represented using fuzzy sets. In this paper, a new graphical method of solving an fuzzy interval CPM problem has been proposed. We have considered fuzzy interval to represent the activity times which is more realistic in nature. In this paper, we introduce an effective new graphical method to compute project characteristics such as total float, earliest and latest fuzzy interval times of activities in fuzzy project network. The proposed method in this paper is very much effective in determining the fuzzy interval time of completing the fuzzy project and also finding the critical path of fuzzy project network. Numerical example is provided to explain the proposed procedure in detail; the results have shown that the procedure is very useful and flexible in finding fuzzy critical path.

Keywords : Fuzzy interval times ; Critical path ; Float time ; Fuzzy earliest time ; Fuzzy latest time

1. Introduction

Project management concerns the scheduling and control of activities in such a way that the project can be completed in a minimum time [1]. An activity network is a connected graph without cycles with nonnegative weights and with a unique initial node and terminal node. In the activity on node graph, the nodes represent activities and edges represent precedence relations. In the activity on edge graph, the edges represent the activities and the nodes represent the sequence. A path through a project network is one of the ways from the initial node to the terminal node. The length of a path is the sum of the durations of the activities on the path. The project duration equals the length of the longest path through the project network. The longest path is called the critical path in the network. The CPM and Program Evaluation and Review Technique (PERT) are standard methods for scheduling of activities. In real world applications some activity times must be forecasted subjectively; for example, we have to use human judgment instead of stochastic assumptions to determine activity times. An alternative way to deal with imprecise data is to employ the concept of fuzziness [2], whereby the vague activity times can be represented by fuzzy numbers. Fuzzy numbers are used to describe uncertain activity durations, reflecting vagueness, imprecision and subjectivity in the estimation of them.

There have been several attempts in the literature to apply fuzzy numbers to the critical path method since the late 1970s and it has led to the development of fuzzy CPM [3-10]. In particular, problems of determining possible values of latest starting times and floats in networks with imprecise activity durations which are represented by fuzzy or interval numbers have attracted many researchers [11-15]. Many researchers have studied the possible critical paths and activities in networks with duration time intervals [16-21].

In this paper, a simple method that is simple to value and improves difficulty for the problem of computing latest starting times of activities in networks using fuzzy intervals with fuzzy activity durations is developed. Total float time of each activity can be found by this method without using the forward pass and backward pass computations. A new fuzzy operator is introduced to find latest fuzzy time intervals. In Section 2, some basic definitions related to fuzzy numbers (fuzzy intervals) are presented. In Section 3, a new approach to compute the fuzzy latest times and float times of activities in a fuzzy project network with fuzzy interval times is presented. In section 4, two examples are presented for fuzzy critical path analysis using the networks with dummy activities and without dummy activities. Section 5 provides the advantages of the proposed method over existing methods.
2. Preliminaries

In this section some basic definitions, arithmetic operations of fuzzy intervals and comparison between fuzzy intervals are reviewed.

2.1 Basic definitions

In this subsection, some basic definitions are reviewed.

A fuzzy set can be mathematically constructed by assigning to each possible individual in the universe of discourse a value representing its grade of membership in the fuzzy set [22,23]. This grade corresponds to the individual’s similarity to the concept represented by the fuzzy set. The fuzzy number \( \tilde{A} \) is a fuzzy set whose membership function \( \mu_{\tilde{A}}(x) \) satisfies the following conditions [24]:

(i) \( \mu_{\tilde{A}}(x) \) is piecewise continuous;
(ii) \( \mu_{\tilde{A}}(x) \) is a convex fuzzy subset;
(iii) \( \mu_{\tilde{A}}(x) \) is the normality of a fuzzy subset, implying that for at least one element \( x_0 \) the membership grade must be 1, i.e. \( \mu_{\tilde{A}}(x_0) = 1 \).

Definition 1: A fuzzy number with membership function in the form

\[
\mu_{\tilde{A}}(x) = \begin{cases} 
\frac{x-a}{b-a}, & a \leq x < b, \\
\frac{c-x}{c-b}, & b \leq x \leq c, \\
0, & \text{otherwise}
\end{cases}
\]

is called a triangular fuzzy number \( \tilde{A} = (a, b, c) \).

Definition 2: A fuzzy number with membership function in the form

\[
\mu_{\tilde{A}}(x) = \begin{cases} 
\frac{x-a}{b-a}, & a \leq x < b, \\
1, & b \leq x < c, \\
\frac{d-x}{d-c}, & c \leq x \leq d, \\
0, & \text{otherwise}
\end{cases}
\]

is called a trapezoidal fuzzy number \( \tilde{A} = (a, b, c, d) \).

2.2 Arithmetic operations of fuzzy intervals

In this subsection addition and subtraction operations between two fuzzy intervals are reviewed.
Given two positive fuzzy intervals $\tilde{A} = [a_1, b_1]$ and $\tilde{B} = [a_2, b_2]$, fuzzy interval addition and subtraction can be performed as follows:

$$
\tilde{A} \oplus \tilde{B} = [a_1 + a_2, b_1 + b_2]
$$

$$
\tilde{A} \otimes \tilde{B} = [a_1 - b_2, b_1 - a_2]
$$

### 2.3 Comparison between fuzzy intervals

**Definition 3**: A closed interval $P = [a_L, a_R] = \{ a \in \mathbb{R} | a_L \leq a \leq a_R, \mathbb{R} \text{ is a set of real numbers} \}$ is called an interval number. The numbers $a_L$ and $a_R$ are called respectively the lower and upper limit of the interval number $P$.

A real number $P$ may also be regarded as a special interval number $[P,P]$ called degenerated interval number. Interval number $P$ is alternatively represent as $P = <\text{mean}(P), \text{radius}(P)>$ where mean $(P)$ and radius $(P)$ are the midpoint and half width of the interval number $P$ i.e

$$
\text{Mean}(P) = \frac{a_L + a_R}{2}
$$

$$
\text{Radius}(P) = \frac{a_R - a_L}{2}
$$

To get the maximum or the minimum interval number from a given set of interval number it is necessary to compare interval number. An overview on comparison between interval numbers [15] is presented in this section.

Comparison of two intervals numbers is very important in interval arithmetic’s. Let $P = [a_L, a_R]$ and $Q = [b_L, b_R]$ be two fuzzy intervals. These two intervals $P$ and $Q$ may be of the following types:

- **Type I**: Both the intervals are disjoint
- **Type II**: One interval is contained in the other
- **Type III**: The intervals are partially overlapping

**Definition 4**: For $\text{mean}(P) \leq \text{mean}(Q)$, the acceptability index of the premise $P < Q$ is defined by

$$
A(P < Q) = \frac{m(Q) - m(P)}{r(P) + r(Q)}
$$

where $r(P) + r(Q) \neq 0$ and we judge whether $P$ is smaller than $Q$.

$$
P \lor Q = \begin{cases} 
Q, & \text{if } A(P < Q) > 0 \\
P, & \text{if } A(P < Q) = 0, r(P) < r(Q) \text{ and Decision maker is pessimistic} \\
Q, & \text{if } A(P < Q) = 0, r(P) < r(Q) \text{ and Decision maker is optimistic}
\end{cases}
$$

where $P \lor Q$ represents the maximum among the interval numbers $P$ and $Q$.

Similarly, if

$$
P \land Q = \begin{cases} 
P, & \text{if } A(P < Q) > 0 \\
Q, & \text{if } A(P < Q) = 0, r(P) < r(Q) \text{ and Decision maker is pessimistic} \\
P, & \text{if } A(P < Q) = 0, r(P) < r(Q) \text{ and Decision maker is optimistic}
\end{cases}
$$

where $P \land Q$ represents the minimum among the interval numbers $P$ and $Q$.

### 2. Criticality in the network with interval activity times

The purpose of critical path method (CPM) is to aid in the planning and control of large, complex projects. A network $N = (V, A, T)$ is given where $V = \{ v_1, v_2, ..., v_n \}$ is the set of vertices of the graph representing the
project, $\mathcal{A} \subseteq \mathcal{V} \times \mathcal{V}$ is the set of arcs connecting the vertices, and function $T$ maps the set of activities $\mathcal{A}$ into the set of interval times with non-negative ends.

**Definition 5**: A path $p$ in a network is interval critical in $\mathcal{N}$ if there exists a set of times in an interval, such that $p$ is a critical path in the usual sense, after replacing the interval times with the exact values.

**Definition 6**: An activity is interval critical in the network $\mathcal{N}$ if there exists a set of times, such that the activity is a critical in the usual sense in the network $\mathcal{N}$, after replacing the interval times with the exact values.

3. Proposed Fuzzy critical path method

3.1 New approach to find total float of each activity in a fuzzy project network

**Step 1**: Construct Fuzzy project network with fuzzy interval times as fuzzy activity times.

**Step 2**: Find all possible paths in fuzzy project network using rooted tree representation.

**Step 3**: Add all fuzzy interval activity times in each path using addition of fuzzy intervals which gives fuzzy path length $\mathcal{F}_P$ in fuzzy interval.

**Step 4**: Find maximum fuzzy interval time among $\mathcal{F}_P$ using comparison of fuzzy intervals. Let the maximum of $\mathcal{F}_P$ be $\mathcal{M}_P$.

**Step 5**: Find $\mathcal{M}_P \oplus \mathcal{F}_P$ for each activity.

**Step 6**: Rank the paths using comparison of fuzzy intervals for $\mathcal{M}_P \oplus \mathcal{F}_P$.

**Step 7**: Assign the total float of each activity by the following:
- Choose the path $P_1$, assign the path value of $P_1$ as the total float of each activity in that path.
- Choose path $P_2$, assign the path value of $P_2$ as the total float of each activity in the path discarding the activities already assigned.
- Continue the process until all the activities assigned the float time.

**Step 8**: Find the Fuzzy Total float of each path by adding the total float of each activity in the path.

**Step 10**: Rank the fuzzy total float of each path based on comparison of fuzzy intervals.

The path with least rank is fuzzy critical path of the fuzzy project network.

3.2 Computing earliest times of activities

The following procedure describes the proposed method in a step-by-step manner.

**Step 1**: The computations begin from the start node and move towards the end node. Let $[0,0]$ be the starting fuzzy interval time for the fuzzy project network.

**Step 2**: Earliest starting time of each activity $(i,j)$ $\mathcal{E}_S_{ij} = \mathcal{E}_i$ is the earliest possible time when an activity can begin assuming that all the predecessors are also started at their earliest starting time.
**Step 3**: Calculate the earliest finish time of each activity \((i,j)\) \(\tilde{EF}_{ij}\), 
\[
\tilde{EF}_{ij} = \tilde{ES}_{ij} \oplus \tilde{t}_{ij}
\]
where \(\tilde{t}_{ij}\) is the fuzzy interval time of the activity \((i,j)\).

### 3.3 Computing modified latest times of activities

For two fuzzy intervals \(\tilde{A} = [a_1, a_2]\), \(\tilde{B} = [b_1, b_2]\).

We define the operation \(\ast\) as 
\[
\tilde{A} \ast \tilde{B} = [a_1 + b_1, a_2 + b_2]
\]

Find the latest starting time of each activity \((i,j)\) \(\tilde{LS}_{ij}\),
\[
\tilde{LS}_{ij} = \tilde{T}_{ij} \ast \tilde{ES}_{ij}
\]
where \(\tilde{T}_{ij}\) is the fuzzy total time of the activity \((i,j)\).

Calculate the latest finish time of each activity \((i,j)\) \(\tilde{LF}_{ij}\),
\[
\tilde{LF}_{ij} = \tilde{LS}_{ij} \ast \tilde{t}_{ij}
\]

### 4. Numerical Examples

**Example 1**: Critical path analysis of fuzzy project network with dummy activities

Fig. 1 represents the fuzzy project network with dummy activities. Activities with fuzzy interval times are presented in Table I. A rooted tree is drawn by taking initial node as a root and it is represented in Fig. 2.

![Fuzzy project network with dummy activities](image-url)

**Table I**: Activities and their Fuzzy interval durations

<table>
<thead>
<tr>
<th>Activity</th>
<th>Interval Durations</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>[a1, a2]</td>
</tr>
<tr>
<td>2</td>
<td>[b1, b2]</td>
</tr>
<tr>
<td>3</td>
<td>[c1, c2]</td>
</tr>
<tr>
<td>4</td>
<td>[d1, d2]</td>
</tr>
<tr>
<td>5</td>
<td>[e1, e2]</td>
</tr>
<tr>
<td>6</td>
<td>[f1, f2]</td>
</tr>
<tr>
<td>7</td>
<td>[g1, g2]</td>
</tr>
<tr>
<td>8</td>
<td>[h1, h2]</td>
</tr>
</tbody>
</table>
Fig. 2. A rooted tree representation of fuzzy project network with dummy activities.

All paths of the fuzzy project network of Fig. 1 using rooted tree representation are:

(i) 1-2-3-6-7-8
(ii) 1-2-4-5-7-8
(iii) 1-2-4-6-7-8
(iv) 1-2-4-7-8
(v) 1-2-5-7-8
(vi) 1-3-6-7-8
Completion time intervals of the paths (i) to (vi) are [15,23], [15,23], [14,22], [14,22], [16,24], [14,22] respectively. Maximum completion time interval using section 2 of this paper is [16,24]. Using steps of the proposed critical path analysis, we obtain total float of each path from (i) to (vi) as [-32,43], [-38,42], [-36,44], [-29,35], [-32,32], [-28,36] respectively. As per the proposed analysis, we calculate the total float interval time of each activity and presented in Table II. Likewise, using section 3.2 and section 3.3, we compute the earliest and latest interval times and presented in Table II.

### Table-II : Computation results of Earliest and Latest times

<table>
<thead>
<tr>
<th>Activity</th>
<th>Activity time $\bar{t}_{ij}$</th>
<th>Total float $\bar{T}_{ij}$</th>
<th>Earliest start time $\bar{ES}$</th>
<th>Earliest finish time $\bar{EF}$</th>
<th>Latest start time $\bar{LS}$</th>
<th>Latest finish time $\bar{LF}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1-2</td>
<td>[4,6]</td>
<td>[-8,8]</td>
<td>[0,0]</td>
<td>[4,6], [-8,8]</td>
<td>[-2,2]</td>
<td></td>
</tr>
<tr>
<td>1-3</td>
<td>[3,5]</td>
<td>[-6,10]</td>
<td>[0,0]</td>
<td>[3,5], [-6,10]</td>
<td>[-1,13]</td>
<td></td>
</tr>
<tr>
<td>2-3</td>
<td>[0,0]</td>
<td>[-7,9]</td>
<td>[4,6]</td>
<td>[4,6], [-1,13]</td>
<td>[-1,13]</td>
<td></td>
</tr>
<tr>
<td>2-4</td>
<td>[2,4]</td>
<td>[-7,9]</td>
<td>[4,6]</td>
<td>[6,10], [-1,13]</td>
<td>[3,15]</td>
<td></td>
</tr>
<tr>
<td>2-5</td>
<td>[3,5]</td>
<td>[-8,8]</td>
<td>[4,6]</td>
<td>[7,11], [-2,12]</td>
<td>[3,15]</td>
<td></td>
</tr>
<tr>
<td>3-6</td>
<td>[3,5]</td>
<td>[-7,9]</td>
<td>[4,6]</td>
<td>[7,11], [-1,13]</td>
<td>[4,16]</td>
<td></td>
</tr>
<tr>
<td>4-5</td>
<td>[0,0]</td>
<td>[-7,9]</td>
<td>[6,10]</td>
<td>[6,10], [3,15]</td>
<td>[3,15]</td>
<td></td>
</tr>
<tr>
<td>4-6</td>
<td>[0,0]</td>
<td>[-6,10]</td>
<td>[6,10]</td>
<td>[6,10], [4,16]</td>
<td>[4,16]</td>
<td></td>
</tr>
<tr>
<td>4-7</td>
<td>[4,6]</td>
<td>[-6,10]</td>
<td>[6,10]</td>
<td>[10,16], [4,16]</td>
<td>[10,20]</td>
<td></td>
</tr>
<tr>
<td>5-7</td>
<td>[5,7]</td>
<td>[-8,8]</td>
<td>[7,11]</td>
<td>[12,18], [3,15]</td>
<td>[10,20]</td>
<td></td>
</tr>
<tr>
<td>6-7</td>
<td>[4,6]</td>
<td>[-7,9]</td>
<td>[7,11]</td>
<td>[11,17], [10,20]</td>
<td>[16,24]</td>
<td></td>
</tr>
<tr>
<td>7-8</td>
<td>[4,6]</td>
<td>[-8,8]</td>
<td>[12,18]</td>
<td>[16,24], [16,24]</td>
<td>[22,28]</td>
<td></td>
</tr>
</tbody>
</table>

Example 2 : Critical path analysis of fuzzy project network without dummy activities

Fig.3 represents the fuzzy project network without dummy activities. Activities with fuzzy interval times are presented in Table III. A rooted tree is drawn by taking initial node as a root and final node as leaf and it is represented in Fig.4.

![Fuzzy project network without dummy activities](image-url)
Table III: Activities and their Fuzzy interval times

<table>
<thead>
<tr>
<th>Activity</th>
<th>Fuzzy activity time</th>
</tr>
</thead>
<tbody>
<tr>
<td>1-2</td>
<td>[9,10]</td>
</tr>
<tr>
<td>1-3</td>
<td>[3,15]</td>
</tr>
<tr>
<td>1-4</td>
<td>[8,9]</td>
</tr>
<tr>
<td>1-5</td>
<td>[6,9]</td>
</tr>
<tr>
<td>2-3</td>
<td>[4,5]</td>
</tr>
<tr>
<td>2-4</td>
<td>[10,15]</td>
</tr>
<tr>
<td>2-5</td>
<td>[1,2]</td>
</tr>
<tr>
<td>3-4</td>
<td>[10,11]</td>
</tr>
<tr>
<td>4-5</td>
<td>[2,3]</td>
</tr>
</tbody>
</table>

Fig. 4. A rooted tree representation of fuzzy project network without dummy activities

All paths of the fuzzy project network of Fig. 3 using rooted tree representation are

(i) 1-2-3-4-5
(ii) 1-2-4-5
(iii) 1-2-5
(iv) 1-3-4-5
(v) 1-4-5
(vi) 1-5
Completion time intervals of the paths (i) to (vi) are $[25,29]$, $[21,28]$, $[10,12]$, $[15,29]$, $[10,12]$, $[6,9]$ respectively. Maximum completion time interval is calculated using section 2 of this paper and the interval is $[25,29]$. Using steps of the proposed critical path interval analysis, we obtain total float of each path from (i) to (vi) as $[-4,4]$, $[-3,8]$, $[13,19]$, $[-4,14]$, $[13,19]$, $[16,23]$ respectively. As per the proposed analysis, we calculate the total float interval time of each activity and presented in Table IV. Similarly, using section 3.2 and section 3.3, we compute the earliest and latest interval times and presented in Table IV.

Table IV: Computation results of Earliest and Latest fuzzy interval time

<table>
<thead>
<tr>
<th>Activity</th>
<th>Activity time $\tilde{t}_{ij}$</th>
<th>Total float $\tilde{T}_{ij}$</th>
<th>Earliest start time $\tilde{ES}$</th>
<th>Earliest finish time $\tilde{EF}$</th>
<th>Latest start time $\tilde{LS}$</th>
<th>Latest finish time $\tilde{LF}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1-2</td>
<td>$[9,10]$</td>
<td>$[-4,4]$</td>
<td>$0.0$</td>
<td>$[9,10]$</td>
<td>$[-4,4]$</td>
<td>$[6,13]$</td>
</tr>
<tr>
<td>1-3</td>
<td>$[3,15]$</td>
<td>$[-4,14]$</td>
<td>$0.0$</td>
<td>$[3,15]$</td>
<td>$[-4,14]$</td>
<td>$[11,17]$</td>
</tr>
<tr>
<td>1-4</td>
<td>$[8,9]$</td>
<td>$[13,19]$</td>
<td>$0.0$</td>
<td>$[8,9]$</td>
<td>$[13,19]$</td>
<td>$[22,27]$</td>
</tr>
<tr>
<td>1-5</td>
<td>$[6,9]$</td>
<td>$[16,23]$</td>
<td>$0.0$</td>
<td>$[6,9]$</td>
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<td>$[25,29]$</td>
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<td>2-3</td>
<td>$[4,5]$</td>
<td>$[-4,4]$</td>
<td>$[9,10]$</td>
<td>$[13,15]$</td>
<td>$[6,13]$</td>
<td>$[11,17]$</td>
</tr>
<tr>
<td>2-4</td>
<td>$[10,15]$</td>
<td>$[-3,3]$</td>
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<td>$[19,25]$</td>
<td>$[7,17]$</td>
<td>$[22,27]$</td>
</tr>
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<td>2-5</td>
<td>$[1,2]$</td>
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<td>$[9,10]$</td>
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<td>$[25,29]$</td>
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<tr>
<td>3-4</td>
<td>$[10,11]$</td>
<td>$[-4,4]$</td>
<td>$[13,15]$</td>
<td>$[23,26]$</td>
<td>$[11,17]$</td>
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<td>$[2,3]$</td>
<td>$[-4,4]$</td>
<td>$[23,26]$</td>
<td>$[25,29]$</td>
<td>$[22,27]$</td>
<td>$[25,29]$</td>
</tr>
</tbody>
</table>

5. Advantages of the proposed method over existing methods

Main advantages of the proposed method:

(i) It shows an alternative way of finding and explaining the calculations of Fuzzy CPM, latest times and slack times using fuzzy intervals as activity times.

(ii) The proposed method reduces the complexity of model development and computations of solving problem.

(iii) It incorporates the decision-maker’s risk attitude into the problem of fuzzy network.

(iv) It provides easy implementation to decision maker.

Conclusions

A new method to find the fuzzy latest times and float times of activities in a fuzzy project network with fuzzy intervals as activity times has been proposed. Network diagram is represented as a rooted tree to obtain the paths from root to leaf. Total float of each activity has been obtained using new approach of fuzzy interval times. Total float each activity plays an important role in project management. New fuzzy operator is defined to find latest fuzzy interval times. The advantages of the new method over the existing method has been explained.

References


Zielinski P, On computing the latest starting times and floats of activities in a network with imprecise durations, Fuzzy sets and Systems 150 (2005) 53-76.

L.A. Zadeh, Fuzzy sets, Information and Control 8 (1965) 338-353.
