Revisiting the Fradkin-Vilkovisky Theorem

Jan Govaerts$^{a,c)}$ and Frederik G. Scholtz$^{b,c)}$

$^{a)}$*Institute of Nuclear Physics*
Department of Physics, Catholic University of Louvain
2, Chemin du Cyclotron, B-1348 Louvain-la-Neuve, Belgium
`jan.govaerts@fynu.ucl.ac.be`

$^{b)}$*Institute of Theoretical Physics*
Department of Physics, University of Stellenbosch
Stellenbosch 7600, South Africa
`fgs@sun.ac.za`

$^{c)}$*Stellenbosch Institute for Advanced Study (STIAS)*
Private Bag X1, 7602 Stellenbosch, South Africa
`http://www.stias.ac.za`

Abstract

The status of the usual statement of the Fradkin-Vilkovisky theorem, claiming complete independence of the Batalin-Fradkin-Vilkovisky path integral on the gauge fixing "fermion" even within a nonperturbative context, is critically reassessed. Basic, but subtle reasons why this statement cannot apply as such in a nonperturbative quantisation of gauge invariant theories are clearly identified. A criterion for admissibility within a general class of gauge fixing conditions is provided for a large ensemble of simple gauge invariant systems. This criterion confirms the conclusions of previous counter-examples to the usual statement of the Fradkin-Vilkovisky theorem.

1   Introduction

Among available approaches towards the quantisation of locally gauge invariant systems, the general BRST quantisation methods are certainly the most popular and widely used. Within the BRST-BFV Hamiltonian setting[1], one result stands out as being most relevant, namely the so-called Fradkin-Vilkovisky (FV) theorem according to which, in its statement as usually given[1, 2], the BRST invariant BFV path integral (BFV-PI) representation of transition amplitudes is totally independent of the choice of gauge fixing conditions, the latter thus being made to one's best convenience. However in this form, such a claim has been disputed on different grounds[3, 4, 5, 6, 7], while general classes of explicit counter-examples have been presented[3, 4, 5, 8] within simple gauge invariant systems.

Indeed, all these examples agree with the following facts, which are to be considered as defining the actual content of the FV theorem[3, 5]. Given the gauge invariance properties built into the formalism, the BFV-PI is, by construction, manifestly BRST and gauge invariant. Consequently, whatever the choice of gauge fixing conditions being implemented, the BFV-PI always reduces to some integral over the space of gauge orbits of the original gauge invariant system. In particular, any two sets of gauge fixing conditions which are gauge transforms of one another lead to the same final result for the BFV-PI. Nevertheless, which "covering" (an integration domain with some measure) of the space of gauge orbits is thereby selected, depends directly on the gauge equivalence class of gauge fixing conditions to which the specific choice of gauge fixing functions belongs. In other words, the BFV-PI depends on the choice of gauge fixing conditions only through the gauge equivalence class to which these conditions belong. Nonetheless, the BFV-PI cannot be totally independent of the choice of gauge fixing conditions. Gauge invariance of the BFV-PI is a necessary condition, but it is not a sufficient one for a choice of gauge fixing conditions to be admissible. Indeed, an admissible gauge fixing is one whose gauge equivalence class defines a single covering of the space of gauge orbits, namely such that each of these orbits are included with equal nonvanishing weight in the final integration. Nonadmissibility, namely a Gribov problem[9], arises whenever either some orbits are counted with a smaller or larger weight than others (Gribov problem of Type I), or when some orbits are not included at all (Gribov problem of Type II), or both[5]. Since the identification of a general criterion to characterise admissibility of arbitrary gauge fixing conditions appears to be difficult at least[6, 7], this issue is best addressed on a case by case basis.

Notwithstanding the explicit examples confirming the more precise statement of the FV theorem as just described, the arguments purporting to establish complete independence of the BFV-PI on the choice of gauge fixing seem to be so general and transparent, being based on the nilpotency of the BRST charge and BRST invariance of the external states for which the BFV-PI is computed, that the usual FV theorem statement is most often just simply taken for granted and to be perfectly undisputable. Confronted with this contradictory situation, it is justified to reconsider the status of the FV theorem and identify the subtle reasons why the formal arguments do not apply as usually described. This is the purpose of the present note, at least within a general class of simple constrained systems to be described in Sect.2.1.

One should point out in this context that there is no reason to question the validity of the usual statement of the FV theorem within the restricted context of ordinary perturbation theory for Yang-Mills theories. Indeed, there exists explicit and independent proof of this fact[10]. Furthermore, perturbation theory amounts to considering a set of gauge orbits in the immediate vicinity of the gauge orbit belonging to the trivial gauge configuration. However, Gribov problems and nonperturbative gauge fixing issues involve the larger topological properties of the space of gauge orbits[11], and it is within this context that the relevance of the FV theorem is addressed in the present note. There is

no doubt that in the case of Yang-Mills theories, for example, such issues must play a vital role when it comes to the nonperturbative topological features of strongly interacting nonlinear dynamics.

The outline of this note is follows. After having described in Sect.2 the general class of gauge invariant systems to be considered, including their quantisation within Dirac's approach which is free of any gauge fixing procedure, Sect.3 addresses their BRST quantisation. Based on the usual plane wave representation of the Lagrange multiplier sector of the extended phase space within that context, the actual content of the FV theorem is critically reassessed within a general class of gauge fixing conditions, while subtle aspects explaining why its usual statement fails to apply are pointed out. Then in Sect.4, a regularisation procedure for the Lagrange multiplier sector is considered, which avoids the use of the non-normalisable plane wave states, by compactifying that degree of freedom into a circle. A general admissibility criterion for the classes of gauge fixing conditions considered is then identified, while further subtle reasons explaining why the usual statement of the FV theorem fails also in that context are again pointed out. No inconsistencies between the two considered approaches arise, confirming the actual and precise content of the Fradkin-Vilkovisky theorem as given above. Concluding remarks are presented in Sect.5.

2 A Simple General Class of Models

2.1 Classical formulation

Let us consider a system whose configuration space is spanned by a set of bosonic coordinates q^n, with canonically conjugate momenta denoted p_n, thus with the canonical brackets $\{q^n, p_m\} = \delta^n_m$. These phase space degrees of freedom are subjected to a single first-class constraint $\phi(q^n, p_n) = 0$, which defines a local gauge invariance for such a system. Finally, dynamics is generated from a first-class Hamiltonian $H(q^n, p_n)$, which we shall assume to have a vanishing bracket with the constraint ϕ, $\{H, \phi\} = 0$. Given that large classes of examples fall within such a description, the latter condition is only a mild restriction, which is made to ease some of the explicit evaluations to be discussed hereafter.

A well known[5] system meeting all the above requirements is that of the relativistic scalar particle, in which case the first-class constraint ϕ defines both the local generator for world-line diffeomorphisms as well as the mass-shell condition for the particle energy-momentum. Other examples in which the first-class constraint is the generator of a local internal U(1) gauge invariance may easily be imagined, such as those discussed Refs.[8, 12]. In the latter reference for instance, one has a collection of degrees of freedom $q_i^a(t)$ $(a = 1, 2; i = 1, 2, \cdots, d)$ with Lagrange function

$$L = \frac{1}{2}\left[\dot{q}_i^a - \lambda\epsilon^{ab}q_i^b\right]^2 - \frac{1}{2}\omega^2 q_i^a q_i^a \quad , \quad \epsilon^{ab} = -\epsilon^{ba} \ . \tag{1}$$

This system may be interpreted as that of d spherical harmonic oscillators in a plane subjected to the constraint that their total angular momentum vanishes at all times,

$$\phi = \epsilon^{ab} p_i^a q_i^b = 0 \quad , \quad p_i^a = \dot{q}_i^a - \lambda\epsilon^{ab} q_i^b \ . \tag{2}$$

The U(1) gauge invariance of the system is that of arbitrary time-dependent rotations in the plane acting identically on all oscillators, with $\lambda(t)$ being both the associated Lagrange multiplier and U(1) gauge degree of freedom (the time component of the gauge "field").

Returning to the general setting, all the above characteristics may be condensed into one single information, namely the first-order Hamiltonian action principle over phase space expressed as

$$S[q^n, p_n; \lambda] = \int dt\ [\dot{q}^n p_n - H - \lambda\,\phi]\,, \tag{3}$$

2

where $\lambda(t)$ is an arbitrary Lagrange multiplier associated to the first-class constraint $\phi(q^n, p_n) = 0$. The Hamiltonian equations of motion are generated from the total Hamiltonian $H_T = H + \lambda\phi$, in which the Lagrange multiplier parametrises the freedom associated to small gauge transformations throughout time evolution of the system. These small gauge transformations are generated by the first-class constraint $\phi(q^n, p_n)$.

Indeed, in their infinitesimal form, small gauge transformations are generated by the first-class constraint as

$$\delta_\epsilon q^n = \epsilon\{q^n, \phi\} \quad , \quad \delta_\epsilon p_n = \epsilon\{p_n, \phi\} \quad , \quad \delta_\epsilon\lambda = \dot\epsilon, \tag{4}$$

$\epsilon(t)$ being an arbirary function of time (the above action then changes only by a total time derivative). Related to this simple character of gauge transformations, it is readily established[3, 5] that, given a choice of boundary conditions (b.c.) for which the coordinates $q^n(t)$ are specified at the boundary of some time interval $[t_i, t_f]$ $(t_i < t_f)$, which then also requires that the gauge transformation function obeys the b.c. $\epsilon(t_{i,f}) = 0$, the space of gauge orbits is in one-to-one correspondence with Teichmüller space, *i.e.* the space of gauge orbits for the Lagrange multiplier $\lambda(t)$. In the present instance, this Teichmüller space reduces to the real line spanned by the gauge invariant modular or Teichmüller parameter

$$\gamma = \int_{t_i}^{t_f} dt\, \lambda(t). \tag{5}$$

Consequently, any admissible gauge fixing of the system is thus to induce a covering of this modular space in which each of all the possible real values for γ is accounted for with an equal weight. Indeed, any real value for γ characterises in a unique manner a possible gauge orbit of the system, while on the other hand any configuration of the system belongs to a given gauge orbit. Thus, in order to account for all possible physically distinct gauge invariant configurations of the system, all possible values for the single coordinate parameter γ on modular space must be accounted for in any given admissible gauge fixing procedure (absence of a Gribov problem of Type I), while at the same time none of these orbits may be included with a weight that differs from that of any of the other gauge orbits (absence of a Gribov problem of Type II). An admissible gauge fixing procedure must induce a covering of modular space which includes all real values for γ with a γ-independent integration measure over modular space.

2.2 Quantum formulation

As the above notation makes already implicit, in order to avoid any ambiguity in the forthcoming discussion, the configuration space manifold is assumed to be of countable discrete dimension, if not simply finite. Furthermore at the quantum level, we shall also assume that the associated Hilbert space of quantum states itself is spanned by a discrete basis of states. Depending on the system, this may require to compactify configuration space, such as for instance into a torus topology, or introduce some further interaction potential, such as a harmonic well, it being understood that such regularisation procedures may be removed at the very end of the analysis. In this manner, typical problems associated to plane wave representations of the Heisenberg algebra, $[\hat q^n, \hat p_m] = i\hbar\delta_m^n$ with $\hat q^{n\dagger} = \hat q^n$ and $\hat p_n^\dagger = \hat p_n$, are avoided from the outset. As a matter of fact, a torus regularisation procedure will be applied to the Lagrange multiplier sector when considering BRST quantisation at some later stage of our discussion.

Furthermore, for ease of expression hereafter, we shall assume to be working in a basis of Hilbert space which diagonalises the first-class constraint operator $\hat\phi$,

$$\hat\phi|k\rangle = \phi_k|k\rangle, \tag{6}$$

with in particular the integers k_0 denoting the subset of these states which is associated to a vanishing eigenvalue for the constraint with an unspecified degeneracy,

$$\hat{\phi}|k_0\rangle = 0 \quad , \quad \phi_{k_0} = 0. \tag{7}$$

The latter states $|k_0\rangle$ for all the possible values k_0 thus define a basis for the subspace of gauge invariant or physical states, which are to be annihilated by the constraint.

The examples mentioned in (1) provide explicit illustrations of such a general setting. The spectra of both the Hamiltonian and constraint eigenstates are discrete, with specific degeneracies for each class, including the physical sector of gauge invariant states. In the case of the relativistic scalar particle, the same situation arises provided one introduces a regulating harmonic potential term quadratic in the spacetime coordinates in order to render the spectrum discrete. Even for a system as simple as a topological particle on a circle, for which the Lagrange function is given by $L = N\dot{q}$ where N is some normalisation factor that needs to take on a quantised value at the quantum level, the momentum constraint operator, $\hat{\phi} = \hat{p} - N$, then also possesses a discrete spectrum, and thus falls within the general setting of systems addressed in our discussion (in this case, the first-class Hamiltonian vanishes identically, while the gauge invariance associated to the first-class constraint is that of arbitrary coordinate redefinitions of the degree of freedom $q(t)$).

Given an arbitrary choice for the Lagrange multiplier $\lambda(t)$, and since it is also assumed that quantisation preserves the gauge invariance property of the first-class Hamiltonian $\hat{H}$ (namely that even at the quantum level we still have the vanishing commutator $[\hat{H}, \hat{\phi}] = 0$, which also implies that the time-ordered exponential of the total Hamiltonian, $\hat{H}_T(t) = \hat{H} + \lambda(t)\hat{\phi}$, coincides with its ordinary exponential), time evolution of the quantum system is generated by the operator

$$\hat{U}(t_f, t_i) = e^{-\frac{i}{\hbar}\int_{t_i}^{t_f} dt[\hat{H} + \lambda(t)\hat{\phi}]}, \tag{8}$$

which propagates both gauge variant and invariant states. Propagation of physical states only is achieved by introducing the physical projection operator[13] $\boldsymbol{E}$, obtained essentially by integrating over the gauge group of all finite small gauge transformations $e^{-i/\hbar\gamma\hat{\phi}}$, which in the present case may be expressed as

$$\boldsymbol{E} = \lim_{\gamma_0 \to \infty} \int_{-\gamma_0}^{\gamma_0} \frac{d\gamma}{2\gamma_0} e^{-\frac{i}{\hbar}\gamma\hat{\phi}} = \sum_{k_0} |k_0\rangle\langle k_0| \quad , \quad \boldsymbol{E}^2 = \boldsymbol{E} \quad , \quad \boldsymbol{E}^\dagger = \boldsymbol{E}. \tag{9}$$

Consequently, the physical evolution operator is given by $\hat{U}_{\text{phys}}(t_f, t_i) = \hat{U}(t_f, t_i)\boldsymbol{E} = \boldsymbol{E}\hat{U}(t_f, t_i)\boldsymbol{E}$, for which all matrix elements in the basis $|k\rangle$ vanish, except on the physical subspace spanned by the states $|k_0\rangle$,

$\cdot$ If $k_i \neq k_0$ or $k_f \neq k_0$: $\langle k_f|\hat{U}_{\text{phys}}(t_f, t_i)|k_i\rangle = 0$;

$\cdot$ If $k_i = k_{0,i}$ and $k_f = k_{0,f}$: $\langle k_{0,f}|\hat{U}_{\text{phys}}(t_f, t_i)|k_{0,i}\rangle = \langle k_{0,f}|e^{-\frac{i}{\hbar}\Delta t\hat{H}}|k_{0,i}\rangle \quad , \quad \Delta t = t_f - t_i.$

$$\tag{10}$$

The latter are thus the matrix elements that the BFV-PI must reproduce from BRST quantisation given any admissible gauge fixing choice.

Note that one may also write,

$$\hat{U}_{\text{phys}}(t_f, t_i) = \lim_{\gamma_0 \to \infty} \int_{-\gamma_0}^{\gamma_0} \frac{d\gamma}{2\gamma_0} e^{-\frac{i}{\hbar}[\Delta t\hat{H} + \gamma\hat{\phi}]} = e^{-\frac{i}{\hbar}\Delta t\hat{H}} \lim_{\gamma_0 \to \infty} \int_{-\gamma_0}^{\gamma_0} \frac{d\gamma}{2\gamma_0} e^{-\frac{i}{\hbar}\gamma\hat{\phi}}, \tag{11}$$

which clearly reproduces the above matrix elements, and makes it explicit that one indeed has performed an admissible integration over the modular space of the system parametrised by $-\infty < \gamma < +\infty$ with a uniform integration measure[14], precisely a covering of modular space which is characteristic of an admissible gauge fixing choice.

3   BFV-BRST Formulation

3.1   BFV extended phase space

Within the BFV approach[1, 2, 5], phase space is first extended by introducing a momentum $\pi(t)$ canonically conjugate to the Lagrange multiplier $\lambda(t)$, $\{\lambda(t), \pi(t)\} = 1$. Consequently, one then has the set of first class constraints $G_a = (G_1, G_2) = (\pi, \phi) = 0$, $a = 1, 2$, such that $\{H, G_a\} = 0$. To compensate for these additional dynamical degrees of freedom, a further system of pairs of Grassmann odd canonically conjugate ghost degrees of freedom, $\eta^a(t)$ and $\mathcal{P}_a(t)$ with $\eta^{a\dagger} = \eta^a$, $\mathcal{P}_a^\dagger = -\mathcal{P}_a$ and $\{\eta^a, \mathcal{P}_b\} = -\delta_b^a$, is introduced. By convention, η^a (resp. $\mathcal{P}_a$) are of ghost number $+1$ (resp. -1). The ghost number is given by $Q_g = \mathcal{P}_a \eta^a$.

Within this setting, small local gauge transformations are traded for global BRST transformations, generated by the BRST charge Q_B, which in the present situation is simply given by

$$Q_B = \eta^a G_a = \eta^1 \pi + \eta^2 \phi, \tag{12}$$

a Grassmann odd quantity, real under complex conjugation and of ghost number $(+1)$, characterised by its nilpotency property, $\{Q_B, Q_B\} = 0$.

BRST invariant dynamics on this extended phase space is generated by the general BRST invariant Hamiltonian

$$H_{\text{eff}} = H - \{\Psi, Q_B\}, \tag{13}$$

Ψ being an *a priori* arbitrary Grassmann odd function of extended phase space, pure imaginary under complex conjugation and of ghost number (-1), known as the "gauge fixing fermion" as this is indeed the role it takes within this formalism.

In order to obtain a BRST invariant dynamics, the equations of motion generated from H_{eff} must be supplemented with BRST invariant boundary conditions. Considering BRST transformations,

$$\delta_B q^n = \{q^n, Q_B\} = \eta^2 \{q^n, \phi\} \;\;,\;\; \delta_B \lambda = \eta^1 \;\;,\;\; \delta_B \eta^1 = 0 \;\;,\;\; \delta_B \eta^2 = 0, \tag{14}$$

$$\delta_B p_n = \{p_n, Q_B\} = \eta^2 \{p_n, \phi\} \;\;,\;\; \delta_B \pi = 0 \;\;,\;\; \delta_B \mathcal{P}_1 = -\pi \;\;,\;\; \delta_B \mathcal{P}_2 = -\phi, \tag{15}$$

it appears that a choice of b.c. which is universally BRST invariant is such that

$$\pi(t_{i,f}) = 0 \;\;,\;\; \mathcal{P}_1(t_{i,f}) = 0 \;\;,\;\; \eta^2(t_{i,f}) = 0, \tag{16}$$

while the b.c. in the original "matter" sector (q^n, p_n) are those already mentioned in the discussion of Sect.2.1. Note that since $Q_B(t_{i,f}) = 0$ as well as $Q_g(t_{i,f}) = 0$, while $\dot{Q}_B = \{Q_B, H_{\text{eff}}\} = 0$ and $\dot{Q}_g = \{Q_g, H_{\text{eff}}\} = 0$ on account of the BRST invariance and vanishing ghost number of H_{eff}, these b.c. imply that any solution is indeed BRST invariant and of vanishing ghost number, $Q_B(t) = 0$ and $Q_g(t) = 0$. These are precisely the b.c. that are imposed in the construction of the BFV-PI for the BRST invariant quantised system.

Obviously, a condition which the gauge fixing function Ψ must meet is that, given the above b.c., the set of solutions to the equations of motion generated by the corresponding Hamiltonian

5

H_{eff} coincides exactly with the set of solutions obtained in the initial formulation of Sect.2.1. This requirement restricts already on the classical level the classes of gauge fixing functions Ψ that may be considered. Even the classical BRST invariant dynamics is not entirely independent of the choice of Ψ[5], a point we shall not pursue further here (having already been discussed to some extent in Ref.[5] through detailed examples), but which indicates that it cannot be so either at the quantum level.

A general class of functions that is to be used explicitly hereafter within the quantised system is of the form

$$\Psi = \mathcal{P}_1 F(\lambda) + \beta \mathcal{P}_2 \lambda, \tag{17}$$

$F(\lambda)$ being an arbitrary real function and β an arbitrary real parameter. It may readily be established that associated to this choice, the classical Hamiltonian equation of motion for $\lambda(t)$ amounts to the gauge fixing condition

$$\frac{d\lambda(t)}{dt} = F(\lambda) . \tag{18}$$

In terms of some integration constant λ_0, the solution $\lambda(t; \lambda_0)$ defines a value $\gamma(\lambda_0)$ for the Teichmüller parameter. Given a choice for $F(\lambda)$, as the value λ_0 varies over its domain of definition, $\gamma(\lambda_0)$ varies over a certain domain in modular space with a specific oriented covering or measure over that domain. It is only when the entire set of real values for the Teichmüller parameter γ is obtained with a γ-independent integration measure that the function $F(\lambda)$, namely Ψ, defines an admissible gauge fixing choice.

For instance, the case $F(\lambda) = 0$ is readily seen to meet this admissibility requirement, and to define a choice of gauge fixing which is known to be admissible for the considered class of systems[2, 3, 5, 15]. Indeed, the equation of motion for λ then simply reads $\dot{\lambda} = 0$, showing that all values of the Teichmüller parameter γ are obtained with a single multiplicity when integrating over the free integration constant $\lambda_0 = \lambda(t_0)$ at some time value $t = t_0$, the b.c. in this sector being $\pi(t_{i,f}) = 0$. One of the purposes of the present note is to identify, at the quantum level, a general criterion for the admissibility of the class of gauge fixing functions in (17).

3.2   BRST quantisation

Quantisation of the BFV formulation amounts to constructing a linear representation space for the (anti)commutation relations,

$$[\hat{q}^n, \hat{p}_m] = i\hbar \delta_m^n \quad , \quad \left[\hat{\lambda}, \hat{\pi}\right] = i\hbar \quad , \quad \left\{\hat{c}^a, \hat{b}_b\right\} = \delta_b^a, \tag{19}$$

with $\hat{\eta}^a = \hat{c}^a$ and $\hat{\mathcal{P}}_a = -i\hbar \hat{b}_a$, and equiped with an hermitean inner product $\langle \cdot | \cdot \rangle$ such that all these operators are self-adjoint. Note that $\hat{c}^{a2} = 0$ and $\hat{b}_a^2 = 0$.

Quantisation of the "matter" sector (q^n, p_n) has already been dealt with in Sect.2.2, for which we shall use the same notations and choice of basis. An abstract representation space for the ghost sector (c^a, b_a) is constructed as follows[5]: that sector of Hilbert space is spanned by a basis with $2^2 = 4$ vectors denoted $| \pm \pm \rangle$ (the first entry refering to the sector $a = 1$ and the second to the sector $a = 2$; this convention also applies to the bra-states $\langle \pm \pm |$), on which the ghost operators act as

follows,

$$\hat{c}^1|--\rangle = |+-\rangle \quad , \quad \hat{c}^1|+-\rangle = 0 \quad , \qquad \hat{c}^1|-+\rangle = |++\rangle \quad , \quad \hat{c}^1|++\rangle = 0,$$

$$\hat{c}^2|--\rangle = |-+\rangle \quad , \quad \hat{c}^2|+-\rangle = -|++\rangle \quad , \quad \hat{c}^2|-+\rangle = 0 \quad , \qquad \hat{c}^2|++\rangle = 0,$$

$$\hat{b}_1|--\rangle = 0 \quad , \qquad \hat{b}_1|+-\rangle = |--\rangle \quad , \quad \hat{b}_1|-+\rangle = 0 \quad , \qquad \hat{b}_1|++\rangle = |-+\rangle,$$

$$\hat{b}_2|--\rangle = 0 \quad , \qquad \hat{b}_2|+-\rangle = 0 \quad , \qquad \hat{b}_2|-+\rangle = |--\rangle \quad , \quad \hat{b}_2|++\rangle = -|+-\rangle.$$
$$\tag{20}$$

Their only nonvanishing inner products are

$$\langle --|++\rangle = -\langle +-|-+\rangle = \langle -+|+-\rangle = -\langle ++|--\rangle, \tag{21}$$

with any of these numbers pure imaginary, such as for instance $\langle --|++\rangle = \pm i$. Finally, the normal ordered quantum ghost number operator is defined as

$$\hat{Q}_g = \frac{1}{2}\left[\hat{c}^a \hat{b}_a - \hat{b}_a \hat{c}^a\right] \quad , \quad \hat{Q}_g^\dagger = -\hat{Q}_g. \tag{22}$$

Consequently, one has the following ghost number values for these states,

$$\hat{Q}_g|--\rangle = (-1)|--\rangle \quad , \quad \hat{Q}_g|+-\rangle = 0 \quad , \quad \hat{Q}_g|-+\rangle = 0 \quad , \quad \hat{Q}_g|++\rangle = (+1)|++\rangle. \tag{23}$$

Even though at some later stage in our discussion we shall perform a circle compactification of the Lagrange multiplier degree of freedom, let us at this point consider the usual plane wave representation of the Heisenberg algebra in the $(\hat{\lambda}, \hat{\pi})$ Lagrange multiplier sector. Eigenstates of these operators are thus defined by

$$\hat{\lambda}|\lambda\rangle = \lambda|\lambda\rangle \quad , \quad -\infty < \lambda < +\infty \quad ; \quad \hat{\pi}|\pi\rangle = \pi|\pi\rangle \quad , \quad -\infty < \pi < +\infty \quad , \tag{24}$$

with the normalisation choices,

$$\langle \lambda|\lambda'\rangle = \delta(\lambda - \lambda') \quad , \quad \langle \pi|\pi'\rangle = \delta(\pi - \pi') \quad ; \quad \mathbb{1} = \int_{-\infty}^{\infty} d\lambda|\lambda\rangle\langle\lambda| = \int_{-\infty}^{\infty} d\pi|\pi\rangle\langle\pi| \ . \tag{25}$$

Consequently, one has the wave function representations of these operators acting on any state $|\psi\rangle$,

$$\langle\lambda|\hat{\lambda}|\psi\rangle = \lambda\,\langle\lambda|\psi\rangle \quad , \quad \langle\lambda|\hat{\pi}|\psi\rangle = -i\hbar\frac{\partial}{\partial\lambda}\,\langle\lambda|\psi\rangle \quad ; \quad \langle\pi|\hat{\lambda}|\psi\rangle = i\hbar\frac{\partial}{\partial\pi}\,\langle\pi|\psi\rangle \quad , \quad \langle\pi|\hat{\pi}|\psi\rangle = \pi\,\langle\pi|\psi\rangle, \tag{26}$$

with the matrix elements for the change of basis,

$$\langle\lambda|\pi\rangle = \frac{1}{\sqrt{2\pi\hbar}}\,e^{\frac{i}{\hbar}\lambda\pi} \quad , \quad \langle\pi|\lambda\rangle = \frac{1}{\sqrt{2\pi\hbar}}\,e^{-\frac{i}{\hbar}\lambda\pi} \ . \tag{27}$$

The quantum BRST charge is given by

$$\hat{Q}_B = \hat{c}^1\hat{\pi} + \hat{c}^2\hat{\phi} \quad , \quad \hat{Q}_B^2 = 0 \quad , \quad \hat{Q}_B^\dagger = \hat{Q}_B. \tag{28}$$

Furthermore, time evolution of the quantised system is generated by the BRST invariant Hamiltonian operator

$$\hat{H}_{\text{eff}} = \hat{H} + \frac{i}{\hbar}\left\{\hat{\Psi}, \hat{Q}_B\right\}, \tag{29}$$

7

leading to the BRST invariant evolution operator

$$\hat{U}_{\text{eff}}(t_f, t_i) = e^{-\frac{i}{\hbar}\Delta t \hat{H}_{\text{eff}}}.\tag{30}$$

For the class of gauge fixing functions (17), an explicit evaluation finds

$$\hat{H}_{\text{eff}} = \hat{H} + \beta\hat{\lambda}\hat{\phi} + \frac{1}{2}\left[F(\hat{\lambda})\hat{\pi} + \hat{\pi}F(\hat{\lambda})\right] + \frac{1}{2}\left[F(\hat{\lambda})\hat{\pi} - \hat{\pi}F(\hat{\lambda})\right]\left[\hat{b}_1\hat{c}^1 - \hat{c}^1\hat{b}_1\right] + i\hbar\beta\hat{b}_2\hat{c}^1,\tag{31}$$

this operator being expressed in such a way as to make manifest its hermiticity property, $\hat{H}_{\text{eff}}^\dagger = \hat{H}_{\text{eff}}$.

Classically within the extended formulation, physical states need to meet the constraints $\pi(t) = 0$ and $\phi(t) = 0$, which implies that for the BRST quantised system, the BRST invariance conditions characterising physical states must lead to the eigenvalues $\phi_k = 0$ and $\pi = 0$, namely $k = k_0$ and $\pi = 0$. This is achieved by considering the cohomology of the BRST charge, *i.e.* by considering the states which are BRST invariant but are defined modulo a BRST transformation,

$$|\psi\rangle = |\psi_{\text{phys}}\rangle + \hat{Q}_B|\varphi\rangle \quad, \quad \hat{Q}_B|\psi\rangle = 0.\tag{32}$$

It may be shown that the general solution to this equation is given by

$$\begin{aligned}|\psi_{\text{phys}}\rangle &= \sum_{k_0} \psi_{k_0;--}(\pi = 0)\,|k_0; \pi = 0; --\rangle\\&\quad + \sum_{k_0}\left\{\psi_{k_0;+-}(\pi = 0)\,|k_0; \pi = 0; +-\rangle + \psi_{k_0;-+}(\pi = 0)\,|k_0; \pi = 0; -+\rangle\right\}\\&\quad + \sum_{k_0} \psi_{k_0;++}(\pi = 0)\,|k_0; \pi = 0; ++\rangle,\end{aligned}\tag{33}$$

while the state $|\varphi\rangle$ may be constructed from the remaining components of the BRST invariant state $|\psi\rangle$ expanded in the basis $|k; \pi; \pm\pm\rangle$, $|\psi\rangle = \sum_{k;\pm\pm}\int_{-\infty}^{\infty} d\pi\,\psi_{n;\pm\pm}(\pi)\,|n; \pi; \pm\pm\rangle$. Consequently, both the BRST cohomology classes at the smallest and largest ghost numbers, $\hat{Q}_g = -1$ and $\hat{Q}_g = +1$, are in one-to-one correspondence with the physical states $|k_0\rangle$ in Dirac's quantisation (or $|k_0; \pi = 0\rangle$ when the Lagrange multiplier sector is included), while the BRST cohomology class at zero ghost number, $\hat{Q}_g = 0$, includes two copies of the Dirac physical states, associated to each of the ghost states $|+-\rangle$ and $|-+\rangle$. Physical states are usually defined to correspond to the BRST cohomology class at zero ghost number[2].

The matrix elements of the BRST invariant evolution operator $\hat{U}_{\text{eff}}(t_f, t_i)$ between states of ghost number (-1) all vanish identically, on account of the vanishing ghost number of $\hat{H}_{\text{eff}}$ and the vanishing inner product $\langle --\,|\,--\rangle = 0$,

$$\langle k_f; \pi_f; --|\hat{U}_{\text{eff}}(t_f, t_i)|k_i; \pi_i; --\rangle = 0,\tag{34}$$

irrespective of the choice of gauge fixing function Ψ, and whether the external states of ghost number (-1) are BRST invariant or not.

However, these are not the matrix elements of $\hat{U}_{\text{eff}}(t_f, t_i)$ that ought to correspond to those in (10) and (11) which describe in Dirac's quantisation the propagation of physical states only. Indeed, the latter may be obtained only for external states which are BRST invariant and of vanishing ghost number, in direct correspondence with the choice of such b.c. in (16). Equivalently, given the action of the ghost and BRST operators, such states are spanned by the set $|k; \pi = 0; -+\rangle$, so that we now have to address the explicit evaluation of the matrix elements

$$\langle k_f; \pi_f = 0; -+|\hat{U}_{\text{eff}}(t_f, t_i)|k_i; \pi_i = 0; -+\rangle.\tag{35}$$

8

By construction, these matrix elements are clearly BRST and thus gauge invariant, and include those of the BRST cohomology class at zero ghost number associated to one of the two sets of states corresponding to Dirac's physical states. Nevertheless, these matrix elements are not totally independent of the choice of gauge fixing function Ψ, as shall now be established.

3.3 The BFV-BRST invariant propagator

Given the choice of gauge fixing function in (17) and the expression for the associated Hamiltonian $\hat{H}_{\text{eff}}$ in (31), it is clear that (35) factorizes into two contributions, whether the conditions $\pi_f = 0 = \pi_i$ required for BRST invariance of the external states are enforced or not,

$$\langle k_f; \pi_f; -+|\hat{U}_{\text{eff}}(t_f, t_i)|k_i; \pi_i; -+\rangle = \langle k_f|e^{-\frac{i}{\hbar}\Delta t \hat{H}}|k_i\rangle \times \mathcal{N}(\pi_f, \pi_i; \phi_{k_i}), \tag{36}$$

with the factor $\mathcal{N}(\pi_f, \pi_i; \phi_{k_i})$ given by[1]

$$\mathcal{N}(\pi_f, \pi_i; \phi_{k_i}) = \langle \pi_f; -+|e^{-\frac{i}{\hbar}\Delta t \left[\beta \phi_{k_i}\hat{\lambda} + \frac{1}{2}\left(F(\hat{\lambda})\hat{\pi} + \hat{\pi}F(\hat{\lambda})\right) + \frac{1}{2}\left(F(\hat{\lambda})\hat{\pi} - \hat{\pi}F(\hat{\lambda})\right)\left(\hat{b}_1\hat{c}^1 - \hat{c}^1\hat{b}_1\right) + i\hbar\beta\hat{b}_2\hat{c}^1\right]}|\pi_i; -+\rangle. \tag{37}$$

Of course, one is particularly interested in the value for $\mathcal{N}(\pi_f = 0, \pi_i = 0; \phi_{k_i})$ as function of the first-class constraint spectral value ϕ_{k_i} or ϕ_{k_f}.

As a warm-up, let us first restrict to the choice $F(\lambda) = 0$, known to be admissible. On basis of the above explicit expression for $\mathcal{N}(\pi_f, \pi_i; \phi_{k_i})$, it is clear that in this case, one has the further factorisation

$$\mathcal{N}(\pi_f, \pi_i; \phi_{k_i}) = \langle -+|e^{\beta\Delta t \hat{b}_2\hat{c}^1}|-+\rangle \langle \pi_f|e^{-\frac{i}{\hbar}\beta\Delta t \phi_{k_i}\hat{\lambda}}|\pi_i\rangle, \tag{38}$$

whose value readily reduces to

$$\mathcal{N}(\pi_f, \pi_i; \phi_{k_i}) = -\beta\Delta t \langle -+|+-\rangle \, \delta(\pi_i - \pi_f - \beta\Delta t \phi_{k_i}) \,. \tag{39}$$

Restricting then to the BRST invariant external states, one has finally (the condition $\Delta t > 0$ is implicit),

$$\langle k_f; \pi_f = 0; -+|\hat{U}_{\text{eff}}(t_f, t_i)|k_i; \pi_i = 0; -+\rangle = -\text{sgn}(\beta) \, \langle -+|+-\rangle \, \delta(\phi_{k_i})\langle k_f|e^{-\frac{i}{\hbar}\Delta t \hat{H}}|k_i\rangle \,. \tag{40}$$

Hence indeed, all these matrix elements vanish identically, unless both external states are physical, namely $k_i = k_{0,i}$ and $k_f = k_{0,f}$ or $\phi_{k_i} = 0 = \phi_{k_f}$. However, when the external states are physical, these matrix elements are singular, on account of the δ-function $\delta(\phi_{k_i})$. Clearly, this is a direct consequence of the plane wave representation of the Heisenberg algebra in the Lagrange multiplier sector $(\hat{\lambda}, \hat{\pi})$ of the BFV extended phase space. Nevertheless, up to the singular normalisation factor $(-\text{sgn}(\beta)\langle -+|+-\rangle\delta(\phi_{k_i}))$, the BFV-BRST invariant matrix elements reproduce correctly the result in (10) for the propagator of physical states within Dirac's quantisation approach. On the other hand, note also that this distribution-valued normalisation factor is not entirely independent of the choice of function Ψ even when $F(\lambda) = 0$, since it depends on the sign of the arbitrary parameter β. In spite of that dependency, an admissible gauge fixing is achieved since all of modular space is indeed recovered with a γ-independent integration measure.

Turning now to an arbitrary choice of function $F(\lambda)$, the explicit and exact evaluation of $\mathcal{N}(\pi_f, \pi_i; \phi_{k_i})$ may proceed through its discretized path integral representation. Applying the approach detailed in Ref.[5], one then establishes the general and exact result,

$$\mathcal{N}(\pi_f = 0, \pi_i = 0; \phi_{k_i}) = -\beta\langle -+|+-\rangle \int_{\mathcal{D}[F]} \frac{d\gamma}{2\pi\hbar} \, e^{-\frac{i}{\hbar}\beta\gamma\phi_{k_i}}, \tag{41}$$

---

[1]Note that in this expression one could as well replace the value ϕ_{k_i} by ϕ_{k_f}.

where the domain of integration $\mathcal{D}[F]$ in modular space is identified as follows. Given a choice for the function $F(\lambda)$ and thus the gauge fixing condition in (18), the solution $\lambda(t; \lambda_f)$ is obtained as function of the integration constant $\lambda_f = \lambda(t_f; \lambda_f)$ at the final value of the time interval $[t_i, t_f]$, thereby leading to a specific value $\gamma(\lambda_f)$ for the Teichmüller parameter. As the integration constant λ_f varies over its entire domain of definition, $-\infty < \lambda_f < +\infty$, the modular parameter $\gamma(\lambda_f)$ then defines a certain domain $\mathcal{D}[F]$ in modular space, including the orientation induced by the sign of $d\gamma(\lambda_f)/d\lambda_f$ in the case of multicoverings. This is how the choice of gauge fixing function $F(\lambda)$ determines a specific covering of modular space, namely a specific domain $\mathcal{D}[F]$ in γ together with a specific integration measure. This is precisely the manner[3, 4, 5] in which the gauge invariant BFV-PI is dependent on the choice of gauge fixing fermion function Ψ, in contradiction with the usual statement[2] of the Fradkin-Vilkovisky theorem.

An admissible choice of gauge fixing is thus associated to $\mathcal{D}[F]$ being the entire real line, in which case,

$$\mathcal{N}(\pi_f = 0, \pi_i = 0; \phi_{k_i}) = -\beta \langle -+ | +- \rangle \, \delta(\beta \phi_{k_i}) = -\mathrm{sgn}(\beta) \, \langle -+ | +- \rangle \, \delta(\phi_{k_i}) \,, \tag{42}$$

thus reproducing indeed the result (40) established for $F(\lambda) = 0$. A more general class of admissible gauge choices is given by

$$F(\lambda) = a + b\lambda, \tag{43}$$

a and b being constant parameters. On the other hand, choices such as

$$F(\lambda) = a + b\lambda + c\lambda^2 \ (c \neq 0) \quad , \quad F(\lambda) = a\lambda^3 \ (a > 0) \quad , \quad F(\lambda) = e^{-a\lambda} \ (a > 0) \,, \tag{44}$$

all define gauge fixing choices which are not admissible[3, 4, 5, 8]. For instance when $F(\lambda) = a\lambda^3$, the modular domain $\mathcal{D}[F]$ is finite and given by the interval $[-\sqrt{2\Delta t/a}, \sqrt{2\Delta t/a}]$ in modular space. The BFV-PI is thus indeed dependent on the choice of gauge fixing function, albeit in a gauge invariant manner. Nonetheless, in the limit that $a \to 0$, an admissible covering of modular space is recovered, associated to the choice $F(\lambda) = 0$.

3.4 Deconstructing the Fradkin-Vilkovisky theorem

An argument often invoked[2] in support of complete independence of the BFV-PI on the choice of gauge fixing fermion is based on the observation that for BRST invariant external states $|\psi_1\rangle$ and $|\psi_2\rangle$ such that $\hat{Q}_B|\psi_i\rangle = 0$ $(i = 1, 2)$, the matrix elements of the operator $\{\hat{\Psi}, \hat{Q}_B\}$ vanish identically,

$$\langle \psi_1 | \{\hat{\Psi}, \hat{Q}_B\} | \psi_2 \rangle = \langle \psi_1 | \left[\hat{\Psi}\hat{Q}_B + \hat{Q}_B\hat{\Psi} \right] | \psi_2 \rangle = 0, \tag{45}$$

where the last equality follows by considering the separate action of the BRST operator $\hat{Q}_B$ on the external states adjacent to it. Indeed, given nilpotency of the BRST charge, $\hat{Q}_B^2 = 0$, this argument should also extend to similar matrix elements of the evolution operator $\hat{U}_{\mathrm{eff}}(t_f, t_i)$ which includes the contribution,

$$e^{-\frac{i}{\hbar}\Delta t \frac{i}{\hbar}\{\hat{\Psi}, \hat{Q}_B\}} = \mathbb{1} \ + \ \frac{\Delta t}{\hbar^2} \left[\hat{\Psi}\hat{Q}_B + \hat{Q}_B\hat{\Psi} \right] \ + \ \frac{1}{2!}\left(\frac{\Delta t}{\hbar^2}\right)^2 \left[\hat{\Psi}\hat{Q}_B\hat{\Psi}\hat{Q}_B + \hat{Q}_B\hat{\Psi}\hat{Q}_B\hat{\Psi} \right] \ + \ \cdots \,. \tag{46}$$

In the case of the factor $\mathcal{N}(\pi_f, \pi_i; \phi_{k_i})$, this argument would appear to imply that one should have, for the states of interest,

$$\langle \pi_f = 0; -+ | e^{-\frac{i}{\hbar}\Delta t \frac{i}{\hbar}\{\hat{\Psi}, \hat{Q}_B\}} | \pi_i = 0; -+ \rangle = \langle \pi_f = 0; -+ | \pi_i = 0; -+ \rangle = \delta(0) \cdot 0 \,, \tag{47}$$

given the facts that $\langle \pi_f | \pi_i \rangle = \delta(\pi_f - \pi_i)$ and $\langle - + | - + \rangle = 0$. Even though this expression is ill-defined, it appears to be totally independent of the choice of gauge fixing fermion Ψ, in sharp contrast with its previous evaluations.

The singular character of this result follows once again from the plane wave representation of the Lagrange multiplier sector $(\hat{\lambda}, \hat{\pi})$. Consequently, matrix elements are generally distribution-valued, and cannot simply be evaluated at specific values of their arguments. Rather, they should be convolved with test functions, or else evaluated first for arbitrary values of their arguments[5]. Hence the above argument certainly cannot be claimed to be standing on a sound basis, and needs to be reconsidered carefully for the explicit evaluation of $\mathcal{N}(\pi_f, \pi_i; \phi_{k_i})$ given a specific value ϕ_{k_i} for the constraint eigenvalue but as yet unspecified values for π_f and π_i.

In order to remain faithful to the spirit of the above argument, the calculation needs to be performed in the form as given in (46), namely not by first computing the result of the anticommutator $\{\hat{\Psi}, \hat{Q}_B\}$ and only then compute its matrix elements—in effect, this is the procedure used to reach the results of Sect.3.3—, but rather by having the operators act from left to right onto the external state $|\psi_2\rangle$ for the first term inside the square brackets at each order in $\Delta t / \hbar^2$ in (46), and from right to left onto the state $\langle \psi_1 |$ for the second term. This calculation is straightforward for the specific admissible choice $F(\lambda) = 0$, in which case one obtains,

$$\mathcal{N}(\pi_f, \pi_i; \phi_{k_i}) = i\hbar\beta(\pi_i - \pi_f)\langle - + | + - \rangle \sum_{k=1}^{\infty} \frac{1}{k!} \left(\frac{\Delta t}{\hbar^2} \right)^k (-i\hbar\beta\phi_{k_i})^{k-1} \langle \pi_f | \hat{\lambda}^k | \pi_i \rangle . \qquad (48)$$

It would appear that indeed, this expression vanishes whenever one considers BRST invariant external states for which $\pi_f = 0 = \pi_i$. However, this is not the case, since the factor which is multiplied by $(\pi_i - \pi_f)$ is itself singular for the values $\pi_f = 0 = \pi_i$, being distribution-valued. Indeed, the above sum may also be expressed as

$$
\begin{aligned}
\mathcal{N}(\pi_f, \pi_i; \phi_{k_i}) = \;& -\frac{1}{\phi_{k_i}} \langle - + | + - \rangle (\pi_i - \pi_f) \langle \pi_f | \sum_{k=1}^{\infty} \frac{1}{k!} \left(\frac{\Delta t}{\hbar^2} \right)^k (-i\hbar\beta\phi_{k_i})^k \hat{\lambda}^k | \pi_i \rangle \\[2mm]
= \;& -\frac{1}{\phi_{k_i}} \langle - + | + - \rangle (\pi_i - \pi_f) \langle \pi_f | \left[e^{-\frac{i}{\hbar}\Delta t\beta\phi_{k_i}\hat{\lambda}} - 1 \right] | \pi_i \rangle \\[2mm]
= \;& -\frac{1}{\phi_{k_i}} \langle - + | + - \rangle (\pi_i - \pi_f) \left[\delta(\pi_i - \pi_f - \beta\Delta t\phi_{k_i}) - \delta(\pi_i - \pi_f) \right] \\[2mm]
= \;& -\beta\Delta t \langle - + | + - \rangle \delta(\pi_i - \pi_f - \beta\Delta t\phi_{k_i}),
\end{aligned}
\qquad (49)
$$

a result that coincides with (39). Nevertheless, the details of the calculation in the above series of relations make manifest the fact that had one set from the outset the values $\pi_f = 0 = \pi_i$, an identically vanishing result would have been obtained, rather than the correct but distribution-valued one, $\mathcal{N}(\pi_f = 0, \pi_i = 0; \phi_{k_i}) = -\beta\Delta t \langle - + | + - \rangle \delta(\beta\Delta t\phi_{k_i})$, which does vanish unless when precisely $\phi_{k_i} = 0$. On the other hand, if from the outset one considers a value $\phi_{k_i} = 0$, one finds through the above analysis,

$$
\begin{aligned}
\mathcal{N}(\pi_f, \pi_i; \phi_{k_i}) = \;& i\hbar\beta(\pi_i - \pi_f)\langle - + | + - \rangle \frac{\Delta t}{\hbar^2} \langle \pi_f | \hat{\lambda} | \pi_i \rangle \\[2mm]
= \;& i\hbar\beta(\pi_i - \pi_f)\langle - + | + - \rangle \frac{\Delta t}{\hbar^2} \left(-i\hbar\frac{\partial}{\partial\pi_i}\delta(\pi_i - \pi_f) \right) \\[2mm]
= \;& -\beta\Delta t \langle - + | + - \rangle \delta(\pi_i - \pi_f) ,
\end{aligned}
\qquad (50)
$$

once again in agreement with the general results in (39) and (49). However, performing such a calculation with $\pi_f = 0 = \pi_i$ from the outset leads back to an identically vanishing result, missing once again the correct distribution-valued result, $\mathcal{N}(\pi_f = 0, \pi_i = 0; \phi_{k_i}) = -\beta\Delta t\langle -+|+-\rangle\delta(\beta\Delta t\phi_{k_i})$.

In conclusion, these considerations establish that the argument based on (45) or (46), purportedly a confirmation that the BFV-PI is necessarily totally independent of the gauge fixing fermion Ψ, is not warranted. Being distribution-valued quantities, the relevant matrix elements have to be convolved with test functions, or equivalently, first be evaluated for whatever external states, and only at the end restricted to the BRST invariant ones. In particular, setting from the outset the values $\pi_f = 0 = \pi_i$ is ill-fated, indeed even leads to ill-defined quantities such as $0 \cdot \delta(0)$. Nevertheless, when properly computed, the end result is perfectly consistent with that established in the previous section in a totally independent manner. And in the latter approach, a general expression for $\mathcal{N}(\pi_f, \pi_i; \phi_{k_i})$ is even amenable to an exact evaluation for whatever choice of function $F(\lambda)$, through a path integral representation of the matrix elements of relevance. This exact result displays explicitly the full extent to which, in a manner totally consistent with the built-in gauge invariance properties of the BFV-PI, the gauge fixed BFV-BRST path integral is indeed dependent on the choice of gauge fixing fermion Ψ[3, 4, 5], namely only through the gauge equivalence class to which that gauge fixing choice belongs, such a gauge equivalence class being characterised by a specific covering of modular space. Being gauge invariant, the BFV-PI necessarily reduces to an integral over modular space, irrespective of the gauge fixing choice. Nevertheless, which domain and integration measure over modular space are thereby induced are function of the choice of gauge fixing conditions. The BFV-PI is not totally independent of the choice of gauge fixing fermion Ψ.

4   The Admissibility Criterion

As manifest from previous expressions, the plane wave representation of the Heisenberg algebra in the Lagrange multiplier sector $(\hat{\lambda}, \hat{\pi})$ leads to distribution-valued results for specific BFV-BRST matrix elements. Consequently, it is sometimes claimed[16] that this very fact calls into question the relevance of the counter-examples to the usual statement of the FV theorem available in the literature and described in the previous sections, while a proper handling of the ensuing singularities would show that these counter-examples are actually ill-fated, and that indeed, the BFV-PI ought to be totally independent of the choice of gauge fixing fermion Ψ.

In order to avoid having to deal with non-normalisable plane wave states, let us now regularise the Lagrange multiplier sector by compactifying the degree of freedom λ onto a circle of circumference $2L$ such that $-L \leq \lambda < L$, it being understood that any quantity of interest has to be evaluated in the decompactification limit $L \to \infty$. Furthermore, the representation of the Heisenberg algebra $[\hat{\lambda}, \hat{\pi}] = i\hbar$, which is to be used on this space with the nontrivial mapping class group $\pi_1(S_1) = \mathbb{Z}$, is that of vanishing U(1) holonomy[2][17]. Consequently, this sector of Hilbert space is now spanned by a discrete set of $\hat{\pi}$-eigenstates for all integer values m,

$$\hat{\pi}|m\rangle = \pi_m|m\rangle \quad , \quad \pi_m = \frac{\pi\hbar}{L}m \quad , \quad \langle m|m'\rangle = \delta_{m,m'} \quad , \quad \mathbb{1} = \sum_m |m\rangle\langle m|. \tag{51}$$

The configuration space wave functions are

$$\langle\lambda|m\rangle = \frac{1}{\sqrt{2L}}e^{i\frac{\pi m}{L}\lambda} \quad , \quad \langle m|\lambda\rangle = \frac{1}{\sqrt{2L}}e^{-i\frac{\pi m}{L}\lambda}, \tag{52}$$

---

[2]A representation of nonvanishing U(1) holonomy may also be used, provided the BRST charge is given a quantum correction linear in the holonomy in order to preserve its nilpotency, thereby preserving all our conclusions.

$|\lambda\rangle$ being the configuration space basis such that

$$\hat{\lambda}|\lambda\rangle = \lambda|\lambda\rangle \quad , \quad -L \leq \lambda < L \quad , \quad \langle\lambda|\lambda'\rangle = \delta_{2L}(\lambda - \lambda') \quad , \quad \mathbb{1} = \int_{-L}^{L} d\lambda|\lambda\rangle\langle\lambda|. \tag{53}$$

Given an arbitrary state $|\psi\rangle$ and its configuration space wave function $\psi(\lambda) = \langle\lambda|\psi\rangle$ which must be single-valued on the circle, one has

$$\langle\lambda|\hat{\lambda}|\psi\rangle = \lambda\,\psi(\lambda) \quad , \quad \langle\lambda|\hat{\pi}|\psi\rangle = -i\hbar\frac{d}{d\lambda}\,\psi(\lambda). \tag{54}$$

States in this sector are thus characterised by the normalisibility condition $\int_{-L}^{L} d\lambda|\psi(\lambda)|^2 < \infty$. In the above relations, $\delta_{2L}(\lambda - \lambda')$ stands for the δ-function on the circle of circumference $2L$,

$$\delta_{2L}(\lambda - \lambda') = \frac{1}{2L}\sum_m e^{i\frac{\pi m}{L}(\lambda-\lambda')}. \tag{55}$$

Given such a discretization of the Lagrange multiplier sector $(\hat{\lambda}, \hat{\pi})$, let us now address again the different points raised previously concerning the FV theorem.

BRST cohomology classes remain characterised in the same way as previously. The general solution to the BRST invariance condition $\hat{Q}_B|\psi\rangle = 0$, namely $|\psi\rangle = |\psi_{\text{phys}}\rangle + \hat{Q}_B|\varphi\rangle$, is given by

$$\begin{aligned}|\psi_{\text{phys}}\rangle \;\; = \;\; & \textstyle\sum_{k_0} \psi_{k_0;m=0;--}\,|k_0; m = 0; --\rangle \\[4pt] & + \textstyle\sum_{k_0}\{\psi_{k_0;m=0;+-}\,|k_0; m = 0; +-\rangle \;\; + \;\; \psi_{k_0;m=0;-+}\,|k_0; m = 0; -+\rangle\} \\[4pt] & + \textstyle\sum_{k_0} \psi_{k_0;m=0;++}\,|k_0; m = 0; ++\rangle,\end{aligned} \tag{56}$$

while the state $|\varphi\rangle$ may be constructed from the remaining components of the BRST invariant state $|\psi\rangle$ expanded in the basis $|k; m; \pm\pm\rangle$, $|\psi\rangle = \sum_{k;m;\pm\pm} \psi_{k;m;\pm\pm}\,|k; m; \pm\pm\rangle$. Consequently, both the BRST cohomology classes at the smallest and largest ghost numbers, $\hat{Q}_g = -1$ and $\hat{Q}_g = +1$, are in one-to-one correspondence with the physical states $|k_0\rangle$ in Dirac's quantisation (or $|k_0; m = 0\rangle$ when the Lagrange multiplier sector is included), while the BRST cohomology class at zero ghost number, $\hat{Q}_g = 0$, includes two copies of the Dirac physical states, associated to each of the ghost states $|+-\rangle$ and $|-+\rangle$. Physical states are usually defined to correspond to the BRST cohomology class at zero ghost number[2].

The matrix elements of the BRST invariant evolution operator $\hat{U}_{\text{eff}}(t_f, t_i)$ between states of ghost number (-1) all vanish identically, on account of the vanishing ghost number of $\hat{H}_{\text{eff}}$ and the vanishing inner product $\langle -- | -- \rangle = 0$,

$$\langle k_f; m_f; -- |\hat{U}_{\text{eff}}(t_f, t_i)|k_i; m_i; --\rangle = 0, \tag{57}$$

irrespective of the choice of gauge fixing function Ψ, and whether the external states of ghost number (-1) are BRST invariant or not.

However, these are not the matrix elements of $\hat{U}_{\text{eff}}(t_f, t_i)$ that ought to correspond to those in (10) and (11) which describe in Dirac's quantisation the propagation of physical states only. Indeed, the latter may be obtained only for external states which are BRST invariant and of vanishing ghost number, in direct correspondence with the choice of such b.c. in (16). Equivalently, given the action

of the ghost and BRST operators, such states are spanned by the set $|k; m = 0; -+\rangle$, so that we now have to address the explicit evaluation of the matrix elements

$$\langle k_f; m_f = 0; -+|\hat{U}_{\text{eff}}(t_f, t_i)|k_i; m_i = 0; -+\rangle, \tag{58}$$

the discretized analogues of the matrix elements in (35). As before these matrix elements are, by construction, BRST and thus gauge invariant, and include those of the BRST cohomology class at zero ghost number associated to one of the two sets of states corresponding to Dirac's physical states. Nevertheless, they are not totally independent of the choice of gauge fixing function Ψ, as shall now be established once again.

4.1 Evaluation of the BRST invariant matrix elements

In order to evaluate the matrix elements (58), rather than using a path integral approach, the operator representation of the quantised system shall be considered. Given the choice of gauge fixing function in (17) and the expression for the associated Hamiltonian $\hat{H}_{\text{eff}}$ in (31), it is clear that (58) as well as its extension for whatever values for m_f and m_i factorizes as

$$\langle k_f; m_f; -+|\hat{U}_{\text{eff}}(t_f, t_i)|k_i; m_i; -+\rangle = \langle k_f|e^{-\frac{i}{\hbar}\Delta t \hat{H}}|k_i\rangle \times \mathcal{N}_L(m_f, m_i; \phi_{k_i}), \tag{59}$$

with the factor $\mathcal{N}_L(m_f, m_i; \phi_{k_i})$ given by[3]

$$\mathcal{N}_L(m_f, m_i; \phi_{k_i}) =$$
$$= \langle m_f; -+|e^{-\frac{i}{\hbar}\Delta t\left[\beta\phi_{k_i}\hat{\lambda} + \frac{1}{2}\left(F(\hat{\lambda})\hat{\pi} + \hat{\pi}F(\hat{\lambda})\right) + \frac{1}{2}\left(F(\hat{\lambda})\hat{\pi} - \hat{\pi}F(\hat{\lambda})\right)\left(\hat{b}_1\hat{c}^1 - \hat{c}^1\hat{b}_1\right) + i\hbar\beta\hat{b}_2\hat{c}^1\right]}|m_i; -+\rangle. \tag{60}$$

The evaluation of the ghost contribution to this factor, through a direct expansion of the exponential operator and a resolution of the ensuing recurrence relations, implies a further factorization

$$\mathcal{N}_L(m_f, m_i; \phi_{k_i}) = -i\hbar\beta\langle -+|+-\rangle \times$$
$$\times \langle m_f|\sum_{n=0}^{\infty}\frac{1}{(n+1)!}\left(-\frac{i}{\hbar}\Delta t\right)^{n+1}\sum_{k=0}^{n}\left(\beta\phi_{k_i}\hat{\lambda} + \hat{\pi}F(\hat{\lambda})\right)^k\left(\beta\phi_{n_i}\hat{\lambda} + F(\hat{\lambda})\hat{\pi}\right)^{n-k}|m_i\rangle. \tag{61}$$

Consider then the quantities

$$\left[\beta\phi_{k_i}\hat{\lambda} + F(\hat{\lambda})\hat{\pi}\right]^n|m = 0\rangle = G_n(\hat{\lambda})|m = 0\rangle \quad , \quad n = 0, 1, 2, \cdots, \tag{62}$$

where the functions $G_n(\lambda)$ are defined by their relation to the l.h.s. operator acting on the state $|m = 0\rangle$. These functions obey the recurrence relations

$$G_{n+1}(\lambda) = \beta\phi_{k_i}\lambda G_n(\lambda) - i\hbar F(\lambda)\frac{dG_n(\lambda)}{d\lambda} \quad , \quad G_0(\lambda) = 1. \tag{63}$$

Introducing the variable u such that

$$\frac{d\lambda(u)}{du} = F(\lambda(u)), \tag{64}$$

given some initial value $\lambda_0 = \lambda(u_0)$, the functions $G_n(\lambda)$ are solved by

$$G_n(\lambda) = e^{-\frac{i}{\hbar}\beta\phi_{k_i}\int_{u_0}^u dv\lambda(v)}\left(-i\hbar\frac{d}{du}\right)^n e^{\frac{i}{\hbar}\beta\phi_{k_i}\int_{u_0}^u dv\lambda(v)}. \tag{65}$$

[3]Note that in this expression one could as well replace the value ϕ_{k_i} by ϕ_{k_f}.

Using the representation

$$\left[\beta\phi_{k_i}\hat{\lambda} + F(\hat{\lambda})\hat{\pi}\right]^n |m = 0\rangle = \int_{-L}^{L} \frac{d\lambda}{\sqrt{2L}}|\lambda\rangle\, G_n(\lambda),$$

(66)

it thus follows that one may write

$$\mathcal{N}_L(m_f = 0, m_i = 0; \phi_{k_i}) = -i\hbar\beta\langle -+ | +-\rangle \int_{-L}^{L} \frac{d\lambda}{2L} \sum_{n=0}^{\infty} \frac{1}{(n+1)!}\left(-\frac{i}{\hbar}\Delta t\right)^{n+1} \sum_{k=0}^{n} G_k^*(\lambda)G_{n-k}(\lambda).$$

(67)

It is of interest to first consider the choice $F(\lambda) = 0$, which is known to define an admissible gauge fixing. One then has $G_n(\lambda) = (\beta\phi_{k_i}\lambda)^n$, leading to the following values,

$$\cdot \text{ If } \phi_{k_i} = 0 : \; \mathcal{N}_L(m_f = 0, m_i = 0; \phi_{k_i}) = -\beta\Delta t\langle -+ | +-\rangle;$$

(68)

$$\cdot \text{ If } \phi_{k_i} \neq 0 : \; \mathcal{N}_L(m_f = 0, m_i = 0; \phi_{k_i}) = -\beta\Delta t\langle -+ | +-\rangle \times \frac{\sin\left(\beta\Delta t L\phi_{k_i}/\hbar\right)}{\left(\beta\Delta t L\phi_{k_i}/\hbar\right)}.$$

Consequently, in the limit $L \to \infty$, the matrix elements (58) are given by,

$$\cdot \text{ If } k_i \neq k_0 \text{ or } k_f \neq k_0 :$$
$$\langle k_f; m_f = 0; -+ |\hat{U}_{\text{eff}}(t_f, t_i)|k_i; m_i = 0; -+\rangle = 0;$$

(69)

$$\cdot \text{ If } k_i = k_{0,i} \text{ and } k_f = k_{0,f} :$$
$$\langle k_f; m_f = 0; -+ |\hat{U}_{\text{eff}}(t_f, t_i)|k_i; m_i = 0; -+\rangle = [-\beta\Delta t\langle -+ | +-\rangle]\; \langle k_{0,f}|e^{-\frac{i}{\hbar}\Delta t\hat{H}}|k_{0,i}\rangle.$$

Hence indeed, up to a β-dependent normalisation, these matrix elements reproduce those in (10) representing within Dirac's quantisation the propagation of physical states only. Given the representation in (11), one thus concludes that the choice $F(\lambda) = 0$ defines an admissible gauge fixing.

Let us now turn to the general case of an arbitrary function $F(\lambda)$. Given the result (67), it is clear that whenever $\phi_{k_i} = 0$ and $\phi_{k_f} = 0$, the matrix element (58) reduces again to the same value as in (68) and (69). However, it is the decoupling of the unphysical states which may not be realised[6], implying specific restrictions on the choice for $F(\lambda)$. In order to apply the limit $L \to \infty$ to these matrix elements, it is best to introduce a rescaled variable $\lambda = L\tilde{\lambda}$ with $-1 \leq \tilde{\lambda} < 1$. Given the general expression (67), it should be clear that in order to reproduce the same results as in the admissible case $F(\lambda) = 0$, the following limit

$$\lim_{L \to \infty} \frac{1}{L}F(L\tilde{\lambda}) = \tilde{F}(\tilde{\lambda})$$

(70)

is to define a finite function $\tilde{F}(\tilde{\lambda})$ of $\tilde{\lambda}$ for all values of $\tilde{\lambda}$. Whenever this criterion is met, the choice of gauge fixing function in (17) defines an admissible gauge fixing of the system, for which the BRST invariant matrix elements (58) are given as in (69), and do indeed reproduce, up to some normalisation factor which is also function of the parameter β, the correct time evolution of Dirac's physical states only. Note, however, that the resulting matrix elements in (69) are nonetheless functions of the parameter β appearing in such choices of admissible functions Ψ. Furthermore, when the criterion (70) is not met, the associated choice of gauge fixing is not admissible, since the BRST invariant matrix elements (58) then do not coincide with (69), and thus cannot be expressed through a single integral covering of Teichmüller space as in (11). In other words, the BFV-PI, which provides the

phase space path integral representation for the BRST invariant matrix elements (58), cannot be entirely independent of the choice of gauge fixing "fermion" function Ψ, in contradiction with the FV theorem as usually stated.

The conclusion reached in (70) is also consistent with the explicit examples available in the literature and recalled in (43) and (44). Note that all these examples do indeed agree with the general criterion for admissibility established in (70).

4.2   Deconstructing the Fradkin-Vilkovisky theorem in discretized form

Let us now address, within the discretized Lagrange multiplier sector, the general argument claiming to confirm independence of the BFV-PI on the choice of gauge fixing fermion, based on the expressions (45) and (46).

First consider again the states of ghost number (-1), spanned by $|k; m; --\rangle$ for all values of k and m. One readily finds

$$\langle k_1; m_1; -- |\{\hat{\Psi}, \hat{Q}_B\}|k_2; m_2; --\rangle = 0, \tag{71}$$

as it must since $\{\hat{\Psi}, \hat{Q}_B\}$ is of zero ghost number while the (-1) ghost number sector is spanned only by $|--\rangle$ which is such that $\langle --|--\rangle = 0$. Note that this result also agrees with that established in (57), which applies for the same reasons. Thus the conclusion in (45) is valid on the BRST cohomology class at ghost number (-1) on account of these simple and general facts, totally independently of the choice for Ψ, and, for that matter, of the argument in (45) itself.

Let us now consider the BRST invariant states $|k; m = 0; -+\rangle$ used in the evaluation of the BFV-PI, and more generally the matrix elements of the operator in (46), at a specific eigenvalue ϕ_{k_i} of the constraint $\hat{\phi}$, for the states $|m; -+\rangle$,

$$\mathcal{N}_L(m_f, m_i; \phi_{k_i}) = \langle m_f; -+|e^{-\frac{i}{\hbar}\Delta t \frac{i}{\hbar}\{\hat{\Psi}, \hat{Q}_B\}}|m_i; -+\rangle, \tag{72}$$

it being understood that the action of the operators on these external states is evaluated along the same lines as in Sect.3.4. Hence, this evaluation shall also be done for the specific choice $F(\lambda) = 0$ known to be admissible and to lead to the results in (68) and (69).

The explicit expansion of the above matrix elements then reduces to the following series of expressions, in perfect analogy with the calculation in (49),

$$\begin{aligned}
\mathcal{N}_L(m_f, m_i; \phi_{k_i}) = & \ i\hbar\beta\frac{\pi\hbar}{L}(m_i - m_f)\langle -+|+-\rangle \sum_{k=1}^{\infty} \frac{1}{k!}\left(\frac{\Delta t}{\hbar^2}\right)^k (-i\hbar\beta\phi_{k_i})^{k-1} \langle m_f|\hat{\lambda}^k|m_i\rangle \\[2mm]
= & -\frac{1}{\phi_{k_i}}\frac{\pi\hbar}{L}(m_i - m_f)\langle -+|+-\rangle \langle m_f|\left[e^{-\frac{i}{\hbar}\Delta t\beta\phi_{k_i}\hat{\lambda}} - 1\right]|m_i\rangle \\[2mm]
= & -\frac{1}{\phi_{k_i}}\frac{\pi\hbar}{L}(m_i - m_f)\langle -+|+-\rangle \left\{\langle m_f|e^{-\frac{i}{\hbar}\Delta t\beta\phi_{k_i}\hat{\lambda}}|m_i\rangle - \delta_{m_f,m_i}\right\} \\[2mm]
= & -\beta\Delta t\langle -+|+-\rangle \frac{1}{\beta\Delta t\phi_{k_i}}\frac{\pi\hbar}{L}(m_i - m_f)\int_{-L}^{L}\frac{d\lambda}{2L} e^{i\frac{\pi}{L}(m_i-m_f)\lambda} e^{-\frac{i}{\hbar}\beta\Delta t\phi_{k_i}\lambda}.
\end{aligned} \tag{73}$$

Note that in this form, setting from the outset the values $m_f = 0 = m_i$ leads to a vanishing expression, as it did in the analysis of Sect.3.4. Furthermore, if from the outset we take the physical values $\phi_{k_i} = 0 = \phi_{k_f}$, only the term with $k = 1$ in the above sum survives, leading to the following values,

$$\begin{aligned}
&\cdot \text{If } m_f = m_i \ : \ \mathcal{N}_L(m_f, m_i; \phi_{k_i}) = 0; \\[2mm]
&\cdot \text{If } m_f \neq m_i \ : \ \mathcal{N}_L(m_f, m_i; \phi_{k_i}) = \beta\Delta t\langle -+|+-\rangle \cos\pi(m_i - m_f).
\end{aligned} \tag{74}$$

None of these results thus reproduce the correct ones in (68). However, in the plane wave representations of Sect.3.4 these quantities being distribution-valued, a final integration by parts had to be applied before recovering the correct result. Likewise in the present discretized representation, the final evaluation of the above expression finally leads to

$$\mathcal{N}_L(m_f, m_i; \phi_{k_i}) =$$
$$= -\beta \Delta t \langle -+|+-\rangle \left\{ \int_{-L}^{L} \frac{d\lambda}{2L} e^{i\frac{\pi}{L}(m_i - m_f)\lambda} e^{-\frac{i}{\hbar}\beta \Delta t \phi_{k_i} \lambda} - e^{i\pi(m_i - m_f)} \frac{\sin(\beta \Delta t L \phi_{k_i}/\hbar)}{(\beta \Delta t L \phi_{k_i}/\hbar)} \right\}. \tag{75}$$

Setting now $m_f = 0 = m_i$, still the value for $\mathcal{N}_L(m_f = 0, m_i = 0; \phi_{k_i})$ vanishes identically, irrespective of whether the constraint eigenvalue ϕ_{k_i} is physical or not. Nevertheless, by having compactified the degree of freedom $\lambda(t)$ onto a circle thus leading only to a discrete spectrum of quantum states in the Lagrange multiplier sector $(\hat{\lambda}, \hat{\pi})$, we have avoided any use of distribution-valued matrix elements. Why, then, does the argument based on (45) and (46) still not lead to the correct result?

The fact of the matter is that the adjoint action from the right onto the external states $\langle m_f; -+|$ of the operators $(\hat{Q}_B \hat{\Psi} \hat{Q}_B \hat{\Psi} \cdots)$ in (46) is not necessarily warranted when the operators $\hat{\lambda}$ and $\hat{\pi}$ appear in combination for the compactified regularisation. For example, consider the matrix elements

$$\langle m_f | \hat{\lambda} \hat{\pi} | m_i \rangle = \frac{\pi\hbar}{L} m_i \langle m_f | \hat{\lambda} | m_i \rangle \quad , \quad \langle m_f | \hat{\pi} \hat{\lambda} | m_i \rangle = \frac{\pi\hbar}{L} m_f \langle m_f | \hat{\lambda} | m_i \rangle, \tag{76}$$

where in the second expression the adjoint action of the operator $\hat{\pi}$ onto the state $\langle m_f |$ is used. However, one must then conclude that

$$\langle m_f | \left[\hat{\lambda}, \hat{\pi} \right] | m_i \rangle = \frac{\pi\hbar}{L}(m_i - m_f) \langle m_f | \hat{\lambda} | m_i \rangle , \tag{77}$$

in obvious contradiction with the Heisenberg algebra,

$$\langle m_f | \left[\hat{\lambda}, \hat{\pi} \right] | m_i \rangle = i\hbar \langle m_f | m_i \rangle = i\hbar \delta_{m_f, m_i}. \tag{78}$$

In presence of the operator $\hat{\lambda}$, the adjoint action of $\hat{\pi}$ on bra-states should be avoided. Rather, one should evaluate the action of all operators from the left onto ket-states and only at the very end project the result onto the relevant bra-states. For instance,

$$\hat{\lambda} \hat{\pi} | m_i \rangle = \frac{\pi\hbar}{L} m_i \hat{\lambda} | m_i \rangle \quad , \quad \hat{\pi} \hat{\lambda} | m_i \rangle = -i\hbar | m_i \rangle + \frac{\pi\hbar}{L} m_i \hat{\lambda} | m_i \rangle, \tag{79}$$

so that

$$\langle m_f | \hat{\lambda} \hat{\pi} | m_i \rangle = \frac{\pi\hbar}{L} m_i \langle m_f | \hat{\lambda} | m_i \rangle \quad , \quad \langle m_f | \hat{\pi} \hat{\lambda} | m_i \rangle = -i\hbar \delta_{m_f, m_i} + \frac{\pi\hbar}{L} m_i \langle m_i | \hat{\lambda} | m_i \rangle, \tag{80}$$

in obvious agreement with the Heisenberg algebra $\left[\hat{\lambda}, \hat{\pi} \right] = i\hbar$. The same conclusions may be reached by considering the explicit wave function representations of the Heisenberg algebra given in (52) and (53) for the circle topology. In fact, the operator $\hat{\lambda}$ being represented through multiplication by λ of single-valued wave functions $\langle \lambda | \psi \rangle$ on the circle for which the operator $\hat{\pi} = -i\hbar \partial/\partial \lambda$ is self-adjoint, leads to wave functions that are no longer single-valued on the circle. In particular, the required integration by parts corresponding to the adjoint action of the derivative operator $\hat{\pi} = -i\hbar \partial/\partial \lambda$ induces a nonvanishing surface term because of the lack of single-valuedness of the wave function

$\lambda\langle\lambda|\psi\rangle$, in direct correspondence with the second relation in (80). In other words, even though both operators are well defined on the space of normalisable wave functions on the circle, the operator $\hat{\lambda}$ maps outside the domain of states for which the operator $\hat{\pi}$ is self-adjoint.

This is the thus the core reason why the evaluation of the matrix element (58) according to the argument in (46) in which the strings of operators $(\cdots\hat{\Psi}\hat{Q}_B\hat{\Psi}\hat{Q}_B)$ and $(\hat{Q}_B\hat{\Psi}\hat{Q}_B\hat{\Psi}\cdots)$ act separately from the left onto the ket-states $|m_i;-+\rangle$ and from the right onto the bra-states $\langle m_f;-+|$, respectively, is unwarranted. Indeed, even when $F(\lambda)=0$, precisely the combination $\hat{\pi}\hat{\lambda}$ appears in the product $\hat{Q}_B\hat{\Psi}$ for which, as detailed above, the adjoint action of $\hat{\pi}$ from the right onto the bra-states is not justified unless the proper surface term contributions are accounted for as well (whereas for the product $\hat{\Psi}\hat{Q}_B$ the relevant combination is $\hat{\lambda}\hat{\pi}$ which unambiguously acts from the left onto the ket-states).

Nevertheless, such ambiguities do not arise for the actual anticommutator $\left\{\hat{\Psi},\hat{Q}_B\right\}$ when it is explicitly evaluated, without keeping the two classes of terms separate as done in the argument based on (45) and (46). For example when $F(\lambda)=0$, the potentially troublesome term that is then left over is simply

$$\left\{-i\hbar\beta\hat{b}_2\hat{\lambda},\hat{c}^1\hat{\pi}\right\} = -i\hbar\beta\left[\hat{\lambda},\hat{\pi}\right]\hat{b}_2\hat{c}^1 = \hbar^2\beta\hat{b}_2\hat{c}^1, \tag{81}$$

and is thus responsible for the transformation of the ghost ket-state $|-+\rangle$ into the state $|+-\rangle$ possessing a nonvanishing overlap with the ghost bra-state $\langle -+|$.

Applying this prescription for the evaluation of the matrix elements in (72), in fact one is brought back to the approach used in Sect.4.1, thereby reproducing the general results established in that context. For example when $F(\lambda)=0$, a direct calculation along the lines of (73) readily finds

$$\mathcal{N}_L(m_f,m_i;\phi_{k_i}) = -\beta\Delta t\langle -+|+-\rangle\,\langle m_f|e^{-\frac{i}{\hbar}\beta\Delta t\phi_{k_i}\hat{\lambda}}|m_i\rangle, \tag{82}$$

hence finally

$$\cdot\;\text{If }\phi_{k_i}=0:\;\;\mathcal{N}_L(m_f,m_i;\phi_{k_i}) = -\beta\Delta t\,\langle -+|+-\rangle\,\delta_{m_f,m_i};$$

$$\cdot\;\text{If }\phi_{k_i}\neq 0:\;\;\mathcal{N}_L(m_f,m_i;\phi_{k_i}) = -\beta\Delta t\,\langle -+|+-\rangle\,\int_{-L}^{L}\frac{d\lambda}{2L}\,e^{-\frac{i}{\hbar}\beta\Delta t\phi_{k_i}\lambda}\,e^{i\frac{\pi}{L}(m_i-m_f)\lambda}, \tag{83}$$

a result to be compared to (75) in light of the remarks in (79) and (80). In particular, setting then $m_f=0=m_i$, exactly the same results as in (68) and (69) in the $L\to\infty$ limit are thus recovered.

In conclusion, even though the compactification regularisation of the Lagrange multiplier sector was introduced to circumvent the subtle issues explaining why the argument, based on (45) and (46) and plane wave representations of the Heisenberg algebra and claiming to confirm that the BFV-PI indeed ought to be totally independent of the choice of gauge fixing fermion Ψ, is unwarranted, new subtleties arise for a finite value of L implying again that this argument does not stand up to closer scrutiny. When properly analysed, the argument rather confirms once again the results obtained by direct evaluation of the relevant matrix elements. In particular, these matrix element corresponding to the BFV-PI, even though gauge invariant, are not independent of the choice of gauge fixing procedure. The general criterion for the admissibility of the class of gauge fixing fermions defined in (17) is provided in (70).

5 Conclusions

Rather than gauge fixing the system through its Lagrange multiplier sector, as is achieved through the choice made in (17), it is also possible to contemplate gauge fixing in phase space through some

condition of the form $\chi(q^n, p_n) = 0$, which, within the BFV-BRST formalism, is related to the following choice of gauge fixing "fermion",

$$\Psi = \rho \mathcal{P}_1 \chi(q^n, p_n) + \beta \mathcal{P}_2 \lambda, \tag{84}$$

ρ being an arbitrary real parameter. In the same manner as described in this note for the class of gauge choices (17), it would be of interest to identify a criterion that the function $\chi(q^n, p_n)$ should meet in order that the associated gauge fixing be admissible. However, this issue turns out to be quite involved, and we have not been able to develop a general solution. In fact, in contradistinction to the class of gauge fixings analysed in this note, the answer to this problem in the case of the choices in (84) would also depend on more detailed properties of the first-class Hamiltonian H, the structure of the original configuration space q^n, and how the local gauge transformations generated by the first-class constraint ϕ act on that space. In Ref.[6], two specific models are considered for which the criterion of admissibility in terms of the function $\chi(q^n, p_n)$ is indeed different for each model.

Another simple model which was considered is defined by the seemingly trivial action

$$S[q] = \int dt N \dot{q}, \tag{85}$$

where the single degree of freedom $q(t)$ takes its values in a circle of radius R, while N is some normalisation factor. The associated first-class constraint $\phi = p - N$ generates arbitrary redefinitions of the coordinate $q(t)$, while in this case the first-class Hamiltonian H vanishes, $H = 0$. An admissible phase space gauge fixing condition is $\chi(q, p) = q - q_i$, q_i being some initial value for $q(t)$. At the quantum level, and when taking due account of a possible nontrivial U(1) holonomy[17] for the representation of the Heisenberg algebra $[\hat{q}, \hat{p}] = i\hbar$, it turns out that the factor N is quantised and that the physical spectrum is reduced to a single $\hat{p}$-eigenstate. When computing the BRST invariant matrix elements (58) of interest for the choice of gauge fixing (84) with $\chi = q - q_i$, the admissibility of this gauge fixing is confirmed, once again up to a normalisation factor stemming from the ghost and Lagrange multiplier sectors which is explicitly dependent on the parameters ρ and β appearing in Ψ. In particular, and as is also the case with the result established in (69), the BRST invariant matrix elements (58) vanish identically in the limit $\beta \to 0$, including those for $k_i = k_{0,i}$ and $k_f = k_{0,f}$ which should correspond to the nonvanishing physical ones in (10).

Hence, contrary to the usual statement of the Fradkin-Vilkovisky theorem, the BRST/gauge invariant BFV path integral is not totally independent of the choice of gauge fixing "fermion" Ψ. This note revisited once again this issue, with two main conclusions. First, for a general class of gauge fixing "fermions", it identified a general criterion for admissibility within a simple general class of constrained systems with a single first-class constraint which commutes with the first-class Hamiltonian. This criterion is in perfect agreement with the conclusions of explicit counter-examples to the usual statement of the FV theorem, and it may be seen as a continuation of the work in Ref.[6]. Second, the basic reasons why the general argument claiming to establish complete independence of the BFV-PI on the gauge fixing "fermion" is unwarranted in the case of the associated BRST invariant matrix elements, have been addressed in simple terms. It has been shown that the lack of total independence from Ψ of the BFV-PI arises because, whereas the action of the anticommutator $\{\hat{\Psi}, \hat{Q}_B\}$ on BRST invariant states is unambiguous, that of the operators $\hat{\Psi}\hat{Q}_B$ and $\hat{Q}_B\hat{\Psi}$ separately is not.

These conclusions were reached by two separate routes, namely by working either with the plane wave representation on the real line for the Lagrange multiplier sector Heisenberg algebra, or else by compactifying that sector onto a circle in order to avoid having to deal with non-normalisable states and

a continuous spectrum of eigenstates. In the first approach, it was shown that due to the distribution-valued character of the relevant matrix elements, usual arguments claiming to establish complete independence from Ψ have to be considered with greater care, thereby confirming the lack of total independence, even though manifest gauge invariance is preserved throughout. In the compactified approach, it was shown that the usual argument is beset by another ambiguity, namely the fact that the Lagrange multiplier operator $\hat{\lambda}$ maps outside of the domain of normalisable states for which the conjugate operator $\hat{\pi}$ is self-adjoint, inducing further crucial surface terms which are ignored by the usual argument. Incidentally, were it not for such subtle points, the usual statement of the FV theorem would be correct, so that the BFV-PI would always be vanishing, irrespective of the choice of gauge fixing, clearly an undesirable situation since the correct quantum evolution operator could then not be reproduced. This is explicitly illustrated by the fact that using the compactification regularisation and in the limit $\beta \rightarrow 0$, the BFV-PI vanishes for the choices (17) and (84), and this independently of the functions $F(\lambda)$ or the parameter ρ. Indeed for these two choices, it is precisely the parameter β which controls any contribution from the gauge fixing "fermion" to the BFV-PI.

The actual and precise content of the FV theorem is already described in the Introduction. As mentioned there, its relevance is really within a nonperturbative context, while for ordinary perturbation theory, there is no reason to doubt that the BFV-PI integral should be independent of the gauge fixing fermion[10]. However, the subtle and difficult issues raised by the correct statement of the Fradkin-Vilkovsky theorem are certainly to play an important role in the understanding of nonperturbative and topological features of strongly interacting nonlinear dynamics, such as that of Yang-Mills theories.

Faced with this situation, it thus appears that the admissibility of any given gauge fixing procedure must be addressed on a case by case basis, once a dynamical system is considered. In particular, this requires the knowledge of the modular space of gauge orbits of the system, in general a difficult problem in itself. However, it should be recalled that any quantisation procedure of a constrained system not involving any gauge fixing procedure, such as that based on the physical projector[13] which is set within precisely Dirac's quantisation approach only, avoids having to address these difficult problems of identifying modular space and assessing admissibility. Indeed, through the physical projector approach, an admissible covering of modular space is always achieved implicitly[14], as illustrated for example in (11).

References

[1] E.S. Fradkin and G.A. Vilkovisky, *Phys. Lett.* **B55** (1975) 224;
I.A. Batalin and G.A. Vilkovisky, *Phys. Lett.* **B69** (1977) 309;
E.S. Fradkin and T.E. Fradkina, *Phys. Lett.* **B72** (1978) 343;
I.A. Batalin and E.S. Fradkin, *Rivista del Nuovo Cimento* **9** (1986) 1.

[2] M. Henneaux, *Physics Reports* **126** (1985) 1;
M. Henneaux and C. Teitelboim, *Quantization of Gauge Systems* (Princeton University Press, Princeton, 1992).

[3] J. Govaerts, *Int. J. Mod. Phys.* **A4** (1989) 173; *ibid* 4487.

[4] J. Govaerts and W. Troost, *Class. Quantum Grav.* **8** (1991) 1723.

[5] J. Govaerts, *Hamiltonian Quantisation and Constrained Dynamics* (Leuven University Press, Leuven, 1991).

[6] F.G. Scholtz and S.V. Shabanov, *Ann. Phys.* **263** (1998) 119.

[7] A. Rogers, *Class. Quantum Grav.* **17** (2000) 389.

[8] V.M. Villanueva, J. Govaerts and J.-L. Lucio-Martinez, *J. Phys.* **A33** (2000) 4183.

[9] V.N. Gribov, *Nucl. Phys.* **B139** (1978) 1.

[10] B.L. Voronov, P.M. Lavrov and I.V. Tyutin, *Sov. J. Nucl. Phys.* **36** (1982) 292.

[11] I.M. Singer, *Comm. Math. Phys.* **60** (1978) 7.

[12] J. Govaerts and J.R. Klauder, *Ann. Phys.* **274** (1999) 251.

[13] J.R. Klauder, *Ann. Phys.* **254**(1997) 419 ;
J.R. Klauder, *Nucl. Phys.* **B547** (1999) 397;
J.R. Klauder, *Quantization of Constrained Systems*, *Lect. Notes Phys.* **572** (2001) 143, `hep-th/0003297`.

[14] J. Govaerts, *J. Phys.* **A30** (1997) 603.

[15] C. Teitelboim, *Phys. Rev.* **D25** (1982) 3159.

[16] M. Henneaux, private communication.

[17] J. Govaerts and V.M. Villanueva, *Int. J. Mod. Phys.* **A15** (2000) 4903.