Soliton perturbation theory for the fifth order KdV-type equations with power law nonlinearity

Anjan Biswas\textsuperscript{a,\,*}, Swapan Konar\textsuperscript{b}

\textsuperscript{a} Department of Applied Mathematics and Theoretical Physics, Center for Research and Education in Optical Sciences and Applications, Delaware State University, Dover, DE 19901-2277, USA
\textsuperscript{b} Department of Applied Physics, Birla Institute of Technology, Mesra, Ranchi-835215, India

Received 20 July 2006; accepted 1 February 2007

Abstract

The adiabatic parameter dynamics of solitons, due to fifth order KdV-type equations with power law nonlinearity, is obtained with the aid of soliton perturbation theory. In addition, the small change in the velocity of the soliton, in the presence of perturbation terms, is also determined in this work.

© 2007 Elsevier Ltd. All rights reserved.

Keywords: Soliton perturbation; Adiabatic parameter dynamics; Integrals of motion; Integrability; Nonlinear evolution equation

1. Introduction

The dimensionless form of the fifth order KdV (fKdV) equation with power law nonlinearity is given by

\[ q_t + a q_{xxx} + b q_{xxxxx} - c \left( q^{n+1} \right)_x = 0 \]  \hspace{1cm} (1)

where \( a, b \) and \( c \) are constant parameters and \( n > 0 \). This equation has applications in fluid dynamics and plasma physics particularly for \( n = 1 \) and \( n = 2 \). According to Karpman [3], the nonlinear term in (1) dictates the stability of the soliton and it is necessary to have \( n \leq 6 \) for the soliton to remain stable. In this work, the focus is going to be on solitons that are due to this fKdV equation given by (1) although there are other types of solutions to (1) including the fifth type kovatons which are glued kink–antikink pairs which can be of arbitrary width.

It needs to be noted that (1) will fail the Painleve test of integrability for arbitrary \( n \). Thus the classical methods of studying this fKdV equation, namely the Inverse Scattering Transform, the Backlund Transform or Hirota’s bilinear method, will not work. Incidentally, the tanh method that was introduced in 1992 [5,8], and was improved in 2003 [2], serves as a powerful technique for integrating the nonlinear evolution equations, even if the Painleve test of...
integrability fails. Using this method, the one-soliton solution of (1) is given by
\[ q(x, t) = \frac{A}{\cosh^\frac{4}{n} B (x - \bar{x})} \]  
where
\[ A = \left[ \frac{a^2(n + 2)(n + 4)(3n + 4)}{2bc(n^2 + 4n + 8)^2} \right]^\frac{1}{n} \]  
\[ B = \frac{n}{2b} \sqrt{\frac{ab}{n^2 + 4n + 8}}. \]  
From (4), it needs to be noted that, for (1), solitons exist for \( ab < 0 \). Here, \( A \) represents the amplitude of the soliton that is given by (3) while \( B \) is the width of the soliton that is given by (4), while \( \bar{x} \) represents the center position of the soliton and therefore the velocity of the soliton is given by
\[ v = \frac{d\bar{x}}{dt}. \]  

2. Mathematical properties

Eq. (1) has at least two integrals of motion that are known as the linear momentum \( (M) \) and energy \( (E) \) [1,4,10]. These are respectively given by
\[ M = \int_{-\infty}^{\infty} q dx = \frac{A}{B} \frac{\Gamma\left(\frac{1}{2}\right) \Gamma\left(\frac{\frac{2}{n}}{n}\right)}{\Gamma\left(\frac{1}{2} + \frac{\frac{2}{n}}{n}\right)} \]  
and
\[ E = \int_{-\infty}^{\infty} q^2 dx = \frac{A^2}{B} \frac{\Gamma\left(\frac{1}{2}\right) \Gamma\left(\frac{\frac{4}{n}}{n}\right)}{\Gamma\left(\frac{1}{2} + \frac{\frac{4}{n}}{n}\right)}. \]  
These conserved quantities are calculated by using the one-soliton solution given by (2). Also, in (6) and (7), \( \Gamma(x) \) is the usual gamma function that is defined as
\[ \Gamma(x) = \int_{0}^{\infty} e^{-t} t^{-x} dt. \]  
The center of the soliton \( \bar{x} \) is given by the definition
\[ \bar{x} = \frac{\int_{-\infty}^{\infty} x q dx}{\int_{-\infty}^{\infty} q dx} = \frac{\int_{-\infty}^{\infty} x q dx}{M} \]  
where \( M \) is defined in (6). Thus, the velocity of the soliton is given by
\[ v = \frac{d\bar{x}}{dt} = \frac{\int_{-\infty}^{\infty} x q_t dx}{\int_{-\infty}^{\infty} q dx} = \frac{\int_{-\infty}^{\infty} x q_t dx}{M}. \]  
On using (1), (2) and (6), the velocity of the soliton reduces to
\[ v = -\frac{2a A^2(n + 2)^2}{b (n^2 + 4n + 8)^2}. \]
3. Perturbation terms

The perturbed fKdV equation that is going to be studied in this work is given by

$$q_t + a q_{xxx} + b q_{xxxxx} - c \left( q^{n+1} \right)_x = \epsilon R$$  \hspace{1cm} (12)

where, in (12), \( \epsilon \) is the perturbation parameter and \( 0 < \epsilon \ll 1 \), while \( R \) gives the perturbation terms. In the presence of perturbation terms, the momentum and the energy of the soliton do not stay conserved. Instead, they undergo adiabatic changes that lead to the adiabatic deformation of the soliton amplitude, width and a slow change in the velocity \([3]\). Using (7), the law of adiabatic deformation of the soliton energy is given by \([1]\)

$$\frac{dE}{dt} = 2\epsilon \int_{-\infty}^{\infty} q R dx$$  \hspace{1cm} (13)

while the adiabatic law of change of the velocity of the soliton, from (10), is given by

$$v = -\frac{2aA^2(n+2)^2}{b(n^2+4n+8)^2} + \frac{\epsilon}{M} \int_{-\infty}^{\infty} x R dx.$$  \hspace{1cm} (14)

In this work, the perturbation terms that are going to be considered are

$$R = \alpha q + \beta q_{xx} + \gamma q_x q_{xx} + \delta q^m q_x + \lambda q_{xxx} + \nu q_x q_{xx} + \sigma q^2 q_x \quad \text{in} \quad \mathbb{R}$$

$$+ \xi q_x q_{xxx} + \eta q_x q_{xxxx} + \rho q_{xxx} + \psi q_{xxxx} + \kappa q q_{xxx}. \quad (15)$$

In \( R \), dissipation gives rise to the first two terms and so \( \alpha \) and \( \beta \) are small dissipative coefficients \([4]\). Also, \( \delta \) or \( \psi \) represent the coefficient of the higher order nonlinear dispersive term \([4]\) and \( m \) is a positive integer with \( 1 \leq m \leq 4 \) \([1, 7]\). The coefficient of \( \rho \) provides a higher order stabilizing term and must therefore be taken into account \([4]\). The perturbation term given by the coefficient of \( \eta \) was recently considered \([6]\) while the remaining perturbation terms arise in the context of the extended version of integrable equations \([7]\).

3.1. Applications

In the presence of these perturbation terms, the adiabatic variation of the energy of the soliton is given by

$$\frac{dE}{dt} = 2\epsilon A^2 Bn^2 \Gamma \left( \frac{1}{2} \right) \left[ \frac{256\kappa (n^2 + 4n - 72) A B^4}{2n(n+4)(n+12)} \frac{\Gamma \left( \frac{3}{n} \right)}{\Gamma \left( \frac{1}{2} + \frac{6}{n} \right)} \right]$$

$$\times \frac{1}{n(n+8)(3n+8)} \left\{ \alpha n^2(n+8)(3n+8) - 16\beta B^2 n(n+8) + 256\rho(n+3)B^2 \right\} \frac{\Gamma \left( \frac{4}{n} \right)}{\Gamma \left( \frac{1}{2} + \frac{4}{n} \right)} \right] . \quad (16)$$

The law of the change of velocity for the given perturbation terms in (15) is given by

$$v = -\frac{2aA^2(n+2)^2}{b(n^2+4n+8)^2} + \epsilon \frac{\Gamma \left( \frac{2}{n} + \frac{1}{2} \right)}{\Gamma \left( \frac{2}{n} \right)} \left[ \frac{\delta A^m}{m+1} \frac{\Gamma \left( \frac{2m+2}{n} \right)}{\Gamma \left( \frac{1}{2} + \frac{2m+2}{n} \right)} \right]$$

$$+ \frac{8A B^2}{n^2(n+8)(3n+8)} \left\{ n(3n+8)(\gamma - 2\lambda) - 16B^2(n+3)(3\xi - \eta) \right\} \frac{\Gamma \left( \frac{4}{n} \right)}{\Gamma \left( \frac{1}{2} + \frac{4}{n} \right)}$$

$$+ \frac{8A^2B^2}{3n(n+6)(n+12)} \left\{ vn - 2\sigma(n+18) \right\} \frac{\Gamma \left( \frac{6}{n} \right)}{\Gamma \left( \frac{1}{2} + \frac{6}{n} \right)} \right] . \quad (17)$$
4. Conclusions

In this work, soliton perturbation theory is used to study the perturbed fifth order KdV equation. This theory is used to establish the adiabatic parameter dynamics of the soliton energy. Also, it is shown that the velocity undergoes a slow change due to these perturbation terms. In future, it will be possible to extend these perturbation terms to include other perturbation terms that include the non-local ones too. The quasi-stationary aspects of the perturbed soliton in the presence of such perturbation terms will be studied and reported in future publications.

Acknowledgement

The research of the first author (AB) was fully supported by NSF Grant No: HRD-0630388 and the support is very gratefully appreciated.

References