On the number of maximal independent sets in a graph

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Miller and Muller (1960) and independently Moon and Moser (1965) determined the maximum number of maximal independent sets in an \( n \)-vertex graph. We give a new and simple proof of this result.

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Let \( G \) be a (simple, undirected, finite) graph. A set \( S \subseteq V(G) \) is independent if no edge of \( G \) has both its endpoints in \( S \). An independent set \( S \) is maximal if no independent set of \( G \) properly contains \( S \). Let \( \text{MIS}(G) \) be the set of all maximal independent sets in \( G \). Miller and Muller (1960) and Moon and Moser (1965) independently proved that the maximum, taken over all \( n \)-vertex graphs \( G \), of \( |\text{MIS}(G)| \) equals

\[
g(n) := \begin{cases} 
3^{n/3} & \text{if } n \equiv 0 \pmod{3} \\
4 \cdot 3^{(n-4)/3} & \text{if } n \equiv 1 \pmod{3} \\
2 \cdot 3^{(n-2)/3} & \text{if } n \equiv 2 \pmod{3} 
\end{cases}
\]

This result is important for various reasons. For example, \( g(n) \) bounds the time complexity of various algorithms that output all maximal independent sets (Bron and Kerbosch, 1973; Lawler et al., 1980; Tsukiyama et al., 1977; Tomita et al., 2006; Johnson et al., 1988; Eppstein, 2003; Eppstein et al., 2010). Here we give a new and simple proof of this upper bound on \( |\text{MIS}(G)| \).

**Theorem 1** (Miller and Muller [1960], Moon and Moser [1965]) For every \( n \)-vertex graph \( G \),

\[
|\text{MIS}(G)| \leq g(n)
\]

**Proof:** We proceed by induction on \( n \). The base case with \( n \leq 2 \) is easily verified. Now assume that \( n \geq 3 \). Let \( G \) be a graph with \( n \) vertices. Let \( d \) be the minimum degree of \( G \). Let \( v \) be a vertex of degree

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Let $N[v]$ be the closed neighbourhood of $v$. If $I \in \text{MIS}(G)$ then $I \cap N[v] \neq \emptyset$, otherwise $I \cup \{v\}$ would be an independent set. Moreover, if $w \in I \cap N[v]$ then $I \setminus \{w\} \in \text{MIS}(G - N[w])$. Thus
\[
|\text{MIS}(G)| \leq \sum_{w \in N[v]} |\text{MIS}(G - N_G[w])|,
\]
Since $\deg(w) \geq d$ and $g$ is non-decreasing, by induction,
\[
|\text{MIS}(G)| \leq (d + 1) \cdot g(n - d - 1).
\]
Note that
\[
4 \cdot 3^{(n-4)/3} \leq g(n) \leq 3^{n/3}.
\]
If $d \geq 3$ then
\[
|\text{MIS}(G)| \leq (d + 1) \cdot 3^{(n-d-1)/3} \leq 4 \cdot 3^{(n-4)/3} \leq g(n).
\]
If $d = 2$ then
\[
|\text{MIS}(G)| \leq 3 \cdot g(n - 3) = g(n).
\]
If $d = 1$ and $n \equiv 1 \pmod{3}$ then since $n - 2 \equiv 2 \pmod{3}$,
\[
\text{MIS}(G) \leq 2 \cdot g(n - 2) \leq 2 \cdot 2 \cdot 3^{(n-2-2)/3} = 4 \cdot 3^{(n-4)/3} = g(n).
\]
If $d = 1$ and $n \equiv 0 \pmod{3}$ then since $n - 2 \equiv 1 \pmod{3}$,
\[
\text{MIS}(G) \leq 2 \cdot g(n - 2) \leq 2 \cdot 4 \cdot 3^{(n-2-4)/3} \leq 3^{n/3} = g(n).
\]
If $d = 1$ and $n \equiv 2 \pmod{3}$ then since $n - 2 \equiv 0 \pmod{3}$,
\[
\text{MIS}(G) \leq 2 \cdot g(n - 2) \leq 2 \cdot 3^{(n-2)/3} = g(n).
\]
This proves that $|\text{MIS}(G)| \leq g(n)$, as desired.

For completeness we describe the example by Miller and Muller (1960) and Moon and Moser (1965) that proves that Theorem 1 is best possible. If $n \equiv 0 \pmod{3}$ then let $M_n$ be the disjoint union of $n/3$ copies of $K_3$. If $n \equiv 1 \pmod{3}$ then let $M_n$ be the disjoint union of $K_4$ and $2n/3$ copies of $K_3$. If $n \equiv 2 \pmod{3}$ then let $M_n$ be the disjoint union of $K_2$ and $2n/3$ copies of $K_3$. Observe that $|\text{MIS}(M_n)| = g(n)$.

Note that Vatter (2011) gave another proof of Theorem 1, and also described a connection between this result and the question, “What is the largest integer that is the product of positive integers with sum $n$?” Also note that Dieter Kratsch proved that $|\text{MIS}(G)| \leq 3^{n/3}$ using a similar proof to that presented here; see Gaspers (2010, page 177). Thanks to the authors of Vatter (2011) Gaspers (2010) for pointing out these references.
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References


