# Derivation of a linear Boltzmann equation for a lattice gas

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**Abstract**

We consider a Lorentz gas in the plane where the scatterers have random positions on a square lattice. The scatterers are identical disks of diameter $\epsilon$, which is also the size of the side of a cell and the probability, for a given cell, to be occupied by a scatterer. A point particle moves freely between the scatterers, interacting with them through elastic collisions. We show that, when $\varepsilon \to 0$, the probability density of such a light particle converges to the solution of the linear Boltzmann equation with the hard-sphere cross section.

# 1 Introduction

The Lorentz model consists in a fixed configuration of identical scatterers and a light particle interacting with them. It has been introduced for the study of transport phenomena in the solid state physics.

It is well known that, under a suitable scaling, usually called the Boltzmann-Grad limit, the probability density of the light particle is expected to approach the solution of a linear Boltzmann equation, with a cross section given by the potential of a single scatterer. More precisely, the Boltzmann-Grad limit consists in letting $R \to 0$, $\mu \to \infty$, $R^{d-1}\mu \to$const, where $R$ is the interaction range of each scatterer, $\mu$ is the density of the configuration and $d$ is the dimension of the physical space. It is clear that this is a low density limit in which the light particle performs a finite number of collisions for a finite time.

The first rigorous result was obtained by Gallavotti [5] for a random (Poisson) distribution of hard disks. Later on Spohn [7] and Boldrighini, Bunimovich and Sinai [1] investigated the same problem from different points of view. More recently Desvillettes and Pulvirenti [4] dealt with the same problem for long-range potentials.

All such results were obtained by assuming a random distribution of the obstacles. On the other hand, if the scatterer distribution is periodic, the probability density of the light particle cannot converge to the solution of a transport equation, under the same scaling limit. This was shown, among other things, by Bourgain, Golse and Wennberg [2]. Therefore some stochasticity in the scatterer distribution has to be postulated if one wants that the motion of the light particle is asymptotically equivalent to a random flight process, as it is necessary to derive a linear transport equation.

In this paper we consider a light test particle moving in a lattice gas with occupation number $n_x = 0, 1$; elements of the lattice gas are disks of diameter $\epsilon$ with centers stochastically distributed on a square lattice of cell size $\epsilon$. The light particle moves freely between the disks, interacting with them through elastic collisions. For this system we prove that, when the radius of the disks goes to zero, properly scaling the probability for a point of the lattice to be the center of a scatterer, the distribution density of the test particle converges to the solution of the linear Boltzmann equation in an uniform background, with the hard-disk cross section.

This result is obtained by proving the same convergence for a Markovian model, by mean of a separation between the quick and slow variables and the application of the ergodic theorem. Since this Markovian model is equivalent to the original one in the Boltzmann-Grad limit, the convergence result can be extended to the lattice gas model.

## 2 The lattice gas model and the result

Let $\mathbb{Z}^2_\varepsilon$ be a two-dimensional lattice whose cells have size $\varepsilon \in [0,1]$:

$$\mathbb{Z}^2_\varepsilon = \{(j_1\varepsilon, j_2\varepsilon) | j_i \in \mathbb{Z}, \quad i = 1,2\}.$$

We consider also the lattice formed by the centers:

$$\mathcal{C} = \{((j_1 + 1/2)\varepsilon, (j_2 + 1/2)\varepsilon) | j_i \in \mathbb{Z}, \quad i = 1,2\}.$$

A lattice gas is a sequence

$$\{n_c\}_{c \in \mathcal{C}}$$

where $n_c$, the occupation number, is a random variable taking the value $0$ or $1$ with probability $1 - \varepsilon$ and $\varepsilon$ respectively, indipendently for all $c \in \mathcal{C}$.

For all $c \in \mathcal{C}$ for which $n_c = 1$, we put an open disk of radius $\varepsilon$ which will be called *obstacle* or *scatterer* in the sequel. We denote by $\mathbf{c}$ a scatterer configuration and by $\Lambda_{\mathbf{c}}$ the region occupied by the obstacles.

Consider now a test point particle, initially localized in $x \in \mathbb{R}^2 \setminus \partial\Lambda_{\mathbf{c}}$. We denote by $v \in S^1$ its initial velocity. The dynamics of the test particle is given by elastic reflection by the scatterer configuration. Namely, the particle moves freely up to the first instant $\tau$ when it meets an obstacle with precollisional velocity:

$$\tau = \min\{t \in \mathbb{R}^+ | x + vt \in \partial\Lambda_{\mathbf{c}}\}, \quad v \cdot n \leq 0\}.$$

Then the particle performs an elastic collision. The postcollisional velocity is given by:

$$v' = v - 2(v \cdot n) \quad n. \tag{2.1}$$

where $n$ is the outward normal to $\partial\Lambda_{\mathbf{c}}$.

We denote by $\tilde{z}_\epsilon(t) = \tilde{T}^t_\varepsilon(x,v)$ the flow constructed in this way. A typical path is illustrated in Fig. 1.

*Remark 0*

Note that $\tilde{T}^t_\varepsilon(x,v)$ is defined even though $x \in \Lambda_{\mathbf{c}} \setminus \partial\Lambda_{\mathbf{c}}$. This is done for technical reasons only and the same results which are valid for the flow $\tilde{T}^t_\varepsilon(x,v)$ are

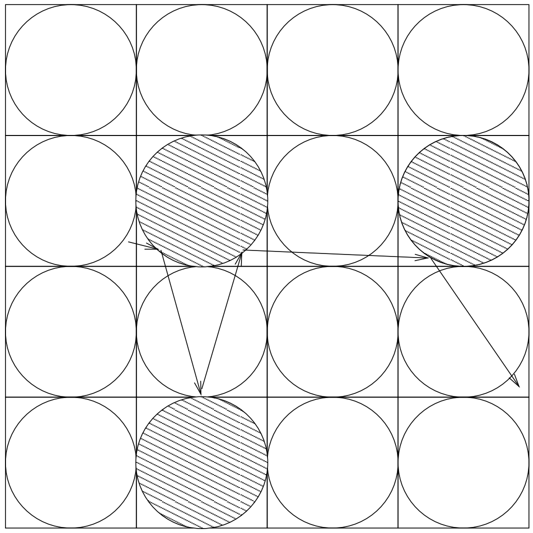

Figure 1: Typical path for the lattice gas model

also valid for the more natural flow which is not defined if $x \in \Lambda_{\mathbf{c}}$, having the two flows the same asymptotic behaviour.

If $f_0 = f_0(x, v)$ is the initial distribution density for the particle, its distribution at time $t > 0$, denoted by $\tilde{f}_\varepsilon = \tilde{f}_\varepsilon(x, v, t)$, is given by the following formula:

$$\int dxdv \tilde{f}_\varepsilon(x, v, t) g(x, v) = \int dxdv f_0(x, v) \tilde{\mathbb{E}}(g(\tilde{T}_\varepsilon^t(x, v))), \tag{2.2}$$

where $g$ is any continuous function and $\tilde{\mathbb{E}}$ denotes the expectation with respect to the lattice gas distribution.

We shall prove:

**Theorem 1** *Let $f_0 : \mathbb{R}^2 \times S^1 \to \mathbb{R}^+$ be the initial probability density (so that $f_0$ is assumed positive and in $L_1(\mathbb{R}, S^1)$ with integral one). Then, for any $t > 0$:*

$$\lim_{\varepsilon \to 0} \tilde{f}_\varepsilon(\cdot, t) = f(\cdot, t) \tag{2.3}$$

*in $\mathcal{D}'$. $f(\cdot, t)$ is the unique solution of the transport equation:*

$$(\partial_t + v \cdot \nabla_x) f(x, v, t) = \frac{1}{2} \int_{S^-} d\omega \{f(x, v', t) - f(x, v, t)\} |v \cdot \omega| \tag{2.4}$$

*where $S^- = \{\omega \in S^1 | v \cdot \omega < 0\}$, $v'$ is the outgoing velocity after a collision with outward normal $\omega$ and ingoing velocity $v$ (see formula ( 2.1)) and $f(x, v, 0^+) = f_0(x, v)$.*

# 3 The proof

As a first step, Theorem 1 will be proved for a simpler, Markovian process $z_\epsilon$ which we are going to define; this process is then shown to be equivalent to $\tilde{z}_\varepsilon$ in the Boltzmann-Grad limit, and, consequently, we prove Theorem 1.

## 3.1 The Markovian model

Consider the special lattice gas configuration in which $n_c = 1$, $c \in \mathcal{C}$, and the corrisponding (close-packing) scatterer configuration $\mathbf{c}$. Set $\mathcal{S} = \partial\Lambda_{\mathbf{c}}$. We define the stochastic process $z_\epsilon(t) = T^t_\epsilon(x, v)$ in the following way. A test particle initially localized in $x \in \mathbb{R}^2 \setminus \mathcal{S}$, with velocity $v \in S^1$, moves freely up to the first time $\tau$ when it meets an obstacle with incoming velocity:

$$\tau = \min\{t \in \mathbb{R}^+ | x + vt \in \mathcal{S}, \quad v \cdot n \leq 0\}.$$

Then the particle performs an elastic collision with probability $\epsilon$ or goes ahead with probability $1 - \epsilon$. After the first collision the procedure is iterated. A typical path is illustrated in Fig. 2

The distribution density for the particle at time $t > 0$, $f_\varepsilon = f_\varepsilon(x, v, t)$, is given by:

$$\int dx dv f_\varepsilon(x, v, t) g(x, v) = \int dx dv f_0(x, v) \mathbb{E}(g(T^t_\varepsilon(x, v))), \tag{3.1}$$

where $\mathbb{E}$ denotes the expectation with respect to the process $z_\epsilon$.

Let $P^\varepsilon_t(x, v|y, w)$ be the transition probability density associated to the process. For any sample $\gamma^t_\varepsilon(x, v)$ of the process, let $p(\gamma^t_\varepsilon(x, v))$ be the probability of its realization, namely $p(\gamma^t_\varepsilon(x, v)) = \varepsilon^k (1 - \varepsilon)^h$, being $k$ and $h$ the number of obstacles on which the test particle represented by the path $\gamma^t_\varepsilon(x, v)$ collides and the number of obstacles crossed by it respectively. Then

$$P^\varepsilon_t(x, v|y, w) = \sum_{\gamma^t_\varepsilon(x,v)} p(\gamma^t_\varepsilon(x, v)) \delta(\gamma^t_\varepsilon(x, v) - (y, w)).$$

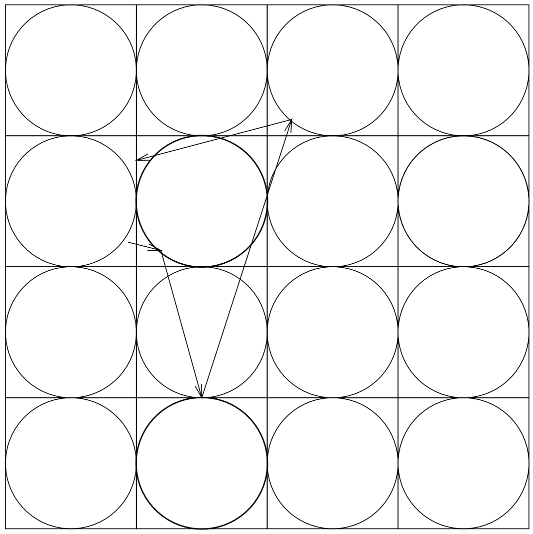

Figure 2: Typical path for the Markovian model

It is immediate to verify the following remarkable property inherited by the Hamiltonian nature of the elastic collision mechanism:

$$P_t^{\varepsilon}(x, v|y, w) = P_t^{\varepsilon}(y, -w|x, -v). \tag{3.2}$$

As a consequence, by ( 3.2) and ( 3.1) we can write:

(3.3)
$$f_{\varepsilon}(x, v, t) = \mathbb{E}[(Rf_0)(T_{\varepsilon}^t(x, -v))] = \sum_{\gamma_{\varepsilon}^t(x,-v)} p(\gamma_{\varepsilon}^t(x, -v))(Rf_0)(\gamma_{\varepsilon}^t(x, -v))$$

where $(Rf)(x, v) = f(x, -v)$.

The advantage in considering $z_{\varepsilon}$ with respect to $\tilde{z}_{\varepsilon}$ is that the first is described by a semigroup $V_{\varepsilon}^t$: $V_{\varepsilon}^t g(x, v) = \mathbb{E}(g(T_{\varepsilon}^t(x, v)))$ for which, by virtue of 3.2, we have:

$$\|V_{\varepsilon}^t g\|_{L_1} \leq \|g\|_{L_1}. \tag{3.4}$$

For the evolution associated to this model it is possible to prove the following theorem:

**Theorem 2** *Let $f_0 : \mathbb{R}^2 \times S^1 \to \mathbb{R}^+$ be the initial probability density (so that $f_0$ is assumed positive and in $L_1(\mathbb{R}, S^1)$ with integral one). Then, for any $t > 0$:*

(3.5) $$\lim_{\varepsilon\to 0} f_\varepsilon(\cdot,t) = f(\cdot,t)$$

*in $\mathcal{D}'$. $f(\cdot,t)$ is the unique solution of the transport equation:*

(3.6) $$(\partial_t + v\cdot\nabla_x) f(x,v,t) = \frac{1}{2}\int_{S^-} d\omega \{f(x,v',t) - f(x,v,t)\}|v\cdot\omega|$$

*where $S^- = \{\omega\in S^1 | v\cdot\omega < 0\}$, $v'$ is the outgoing velocity after a collision with outward normal $\omega$ and ingoing velocity $v$ (see formula ( 2.1)) and $f(x,v,0^+) = f_0(x,v)$.*

*Remark 1*

In the proof of Thm. 2 it will be clear that, using ( 3.3), one can improve the convergence ( 3.5). Namely $f_\varepsilon \to f$ in $L_1$, with an explicit rate of convergence, provided that $f_0$ is sufficiently smooth.

Let $g \in \mathcal{D}(\mathbb{R}^2\times S^1)$ be a test function. Then, if $f(\cdot,t)\in L_1(x,v)$ solves weakly Eq. ( 2.4), with initial datum $f_0\in L_1(x,v)$, we can define $g = g(x,v,t)$ by:

$$\int dx dv f_0(x,v) g(x,v,t) = \int dx dv f(x,v,t) g(x,v).$$

It follows that $g(x,v,t)$ can be expressed by the following series expansion:

(3.7) $$g(x,v,t) := V^t g(x,v) = \sum_{n\geq 0} (V^t g)_n (x,v,t) = \sum_{n\geq 0} g_n(x,v,t)$$

where $V^t$ is a linear semigroup, $g_0(x,v,t) = e^{-t} f_0(x+vt,v)$ and, for $n>1$,

$$g_n(x,v,t) = e^{-t}2^{-n}\int_0^t dt_1 \int_{t_1}^t dt_2 \cdots \int_{t_{n-1}}^t dt_n \int_{S^-} d\omega_1 \cdots \int_{S^-} d\omega_n \prod_{k=1}^n |\omega_k\cdot v_{k-1}|$$

$$g(x+vt_1+v_1(t_2-t_1)+\ldots v_{n-1}(t_n-t_{n-1})+v_n(t-t_{n-1}),v_n),$$

where:

$$v_0 = v$$

$$v_k = v'_{k-1} = v_{k-1} - 2(\omega_k\cdot v_{k-1})\omega_k.$$

To prove Thm. 2 we shall show that $g(t)$ is the $L_1$-limit for $\varepsilon \to 0$ of the sequence:

$$g_\varepsilon(x,v,t) = \mathbb{E}\Big[g(T_\varepsilon^t(x,v))\Big] = V_\varepsilon^t g(x,v)$$

for all $t \in \mathbb{R}$. In order to do this we expand $g_\varepsilon$ in a similar way. For a fixed initial condition $(x,v)$ and $t>0$, let us define the following sets:

$$S_1(t) = \{t_1 \in (0,t) | x + v^{(1)} t_1 \in \mathcal{S}, v^{(1)} \cdot n \le 0\}$$

$$S_2(t,t_1) = \{t_2 \in (t_1,t) | t_1 \in S_1(t), x + v^{(1)} t_1 + v^{(2)}(t_2 - t_1) \in \mathcal{S}, v^{(2)} \cdot n \le 0\}$$

$$\dots$$

$$S_n(t,t_1 \dots t_{n-1}) = \{t_n \in (t_{n-1},t) | t_i \in S_i, i = 1 \dots n-1,$$
$$x + v^{(1)} t_1 \dots + v^{(n)}(t_n - t_{n-1}) \in \mathcal{S}, v^{(n)} \cdot n \le 0\}$$
$$S_{n+1}(t,t_1 \dots t_n) = \{t_{n+1} \in (t_n,t) | t_i \in S_i, i = 1 \dots n,$$
$$x + v^{(1)} t_1 \dots + v^{(n+1)}(t_{n+1} - t_n) \in \mathcal{S}, v^{(n+1)} \cdot n \le 0\}$$

$S_n(\dots)$ denotes all possible collision instants, up to the time $t$, of the $n-$th collision, given the previous collision instants $t_1 \dots t_{n-1}$. The velocity $v^{(i)}$ is the precollisional velocity of the $i-$th collision (we are consistently setting $v^{(1)} = v$) that is also the postcollisional velocity of the $(i-1)$-th collision.

We have:

$$g_\varepsilon(x,v,t) = \sum_{n\ge 0} (V_\epsilon^t g)_n(x,v,t) = \sum_{n \ge 0} g_{\varepsilon,n}(x,v,t) \tag{3.8}$$

where $g_{\varepsilon,0}(x,v,t) = (1-\varepsilon)^{k_1} f_0(x+vt,v)$ and, for $n>1$,

$$g_{\varepsilon,n}(x,v,t) = \sum_{t_1 \in S_1(t)} \dots \sum_{t_n \in S_n(t,t_1 \dots t_{n-1})} \varepsilon^n (1-\varepsilon)^{k^{(n+1)}}$$

$$g(x + vt_1 + v_1(t_2 - t_1) + \dots v_{n-1}(t_n - t_{n-1}) + v_n(t - t_{n-1}), v_n).$$

Here $v_0 = v$ and $v_i = v'_{i-1}$ is the postcollisional velocity with incoming velocity $v_{i-1}$. Finally $k^{(n+1)}$ denotes the number of crossing of the scatterers (with precollisional velocity) that is:

$$k^{(n+1)} = \sum_{i=1}^{n+1} k_i; \qquad k_i = \sum_{\tau \in S_i(\dots); \tau < t_i} 1 \qquad t_{n+1} = t.$$

We now compare the two evolutions in terms of the semigroup operators $V^t$ and $V^t_\varepsilon$. We fix an arbitrary time $t > 0$, and $\sigma = \varepsilon^\beta$, with $\beta \in (0,1)$ to be fixed later and such that $t = k\sigma$. Therefore, using the semigroup property:

(3.9)
$$(V^t - V^t_\varepsilon)g = (V^{k\sigma} - V^{k\sigma}_\varepsilon)g = V^\sigma_\varepsilon(V^{(k-1)\sigma} - V^{(k-1)\sigma}_\varepsilon)g + (V^\sigma - V^\sigma_\varepsilon)V^{(k-1)\sigma}g.$$

We estimate the $L_1(x,v)$ norm of ( 3.9) using ( 3.4)

(3.10)
$$\|(V^{k\sigma} - V^{k\sigma}_\varepsilon)g\|_{L_1} \leq \|(V^{(k-1)\sigma} - V^{(k-1)\sigma}_\varepsilon)g\|_{L_1} + \|(V^\sigma - V^\sigma_\varepsilon)V^{(k-1)\sigma}g\|_{L_1}.$$

We evaluate the second term in the right hand side of ( 3.10) expanding both $V^\sigma$ and $V^\sigma_\varepsilon$, according to ( 3.7) and ( 3.8), with respect to the number of collisions. The idea is that, since $\sigma$ is small, only the terms with $n = 0$ and $n = 1$ are relevant. Indeed for $F \in L_1 \cap L_\infty(x,v)$ and compactly supported:

$$\|(V^\sigma F)_n\|_{L_\infty} \leq \frac{(C\sigma)^n}{n!}\|F\|_{L_\infty}$$

and therefore:

$$\|V^\sigma F - (V^\sigma F)_0 - (V^\sigma F)_1\|_{L_1} \leq C\varepsilon^{2\beta}. \tag{3.11}$$

The same inequality holds for $V^\sigma_\varepsilon$:

$$\|V^\sigma_\varepsilon F - (V^\sigma_\varepsilon F)_0 - (V^\sigma_\varepsilon F)_1\|_{L_1} \leq C\varepsilon^{2\beta}. \tag{3.12}$$

In order to control the difference:

$$\|(V^\sigma_\varepsilon F)_0 + (V^\sigma_\varepsilon F)_1 - (V^\sigma F)_0 - (V^\sigma F)_1\|_{L_1} \tag{3.13}$$

we shall make use of the following Proposition, which is the key ingredient in the proof.

**Proposition 1** *Let $\{F_\varepsilon\}$, $\varepsilon \in (0,1)$, be a family of non-negative functions, $F_\varepsilon \in C^\infty_0(\mathbb{R}^2 \times S^1)$ with the following properties.*

*i) Each $F_\varepsilon$ is compactly supported in $x$, uniformly in $\varepsilon$.*

*ii)For all $k \leq 3$, $\sup_{x,v} |D^k F_\varepsilon(x,v)| \leq C$, where $D^k$ denotes any $k$-th partial derivative with respect to $x, v$ and $C$ is independent of $\varepsilon$.*

*Under these hypotheses, for $\sigma = \varepsilon^\beta$ and $\beta < \frac{1}{33}$:*

$$(1-\varepsilon)^{k_1} F_\varepsilon(x+v\sigma, v) = e^{-\sigma} F_\varepsilon(x+v\sigma, v) + \varphi_0(x,v,\sigma) \tag{3.14}$$

*and*

$$\sum_{t_1 \in S_1(\sigma)} \varepsilon(1-\varepsilon)^{k_1+k_2} F_\varepsilon(x+vt_1-v_1(\sigma-t_1), v_1) = \tag{3.15}$$

$$e^{-\sigma}\int_0^\sigma dt_1 \int_{S^-} d\omega \frac{|\omega\cdot v|}{2} F_\varepsilon(x+vt_1-v_1(\sigma-t_1), v_1) + \varphi_1(x,v,\sigma)$$

*where*

$$\|\varphi_0(\sigma)\|_{L_1} + \|\varphi_1(\sigma)\|_{L_1} \leq C\varepsilon^{2\beta}.$$

Before proving Proposition 1, we shall use it to prove Thm. 2. Putting $F_\varepsilon = V^{(k-1)\sigma} g$, we easily verify that hypotheses i) and ii) of Proposition 1 are satisfied. Therefore the quantity ( 3.13) can be estimated by $C\varepsilon^{2\beta}$. As a consequence, from ( 3.10) using ( 3.11) and ( 3.12), we arrive to:

$$\|(V^{k\sigma} - V_\varepsilon^{k\sigma})g\|_{L_1} \leq \|(V^{(k-1)\sigma} - V_\varepsilon^{(k-1)\sigma})g\|_{L_1} + C\varepsilon^{2\beta}. \tag{3.16}$$

Moreover we can iterate ( 3.16) so that:

$$\|(V^{k\sigma} - V_\varepsilon^{k\sigma})g\|_{L_1} \leq Ck\varepsilon^{2\beta} = Ct\varepsilon^{\beta}. \tag{3.17}$$

This concludes the proof.

Proof of Proposition 1

We start by proving ( 3.15).

Let us introduce the set of the crossing times of the vertical lines of the lattice, specifically:

$$\Sigma_1 = \{\tau_1 \in (0,\sigma) | |x_1 + \cos\alpha\tau_1| = \varepsilon j; j = 1,2\dots\}$$

where $v = (\cos\alpha, \sin\alpha)$. Moreover, we also assume $\alpha \in (0, \pi/4)$, the general case being recovered by a similar argument. We observe that, as illustrated by Fig. 3, the trajectory between two successive crossing of the vertical lines meets a disk once or twice according if it crosses the sets $(0, \varepsilon y_1) \cup (\varepsilon y_2, \varepsilon]$ or $[\varepsilon y_1, \varepsilon y_2]$ respectively. Here and in the sequel, we shall denote by $y, y_i \dots$ the quotes of the vertical left side of a reference box of side one.

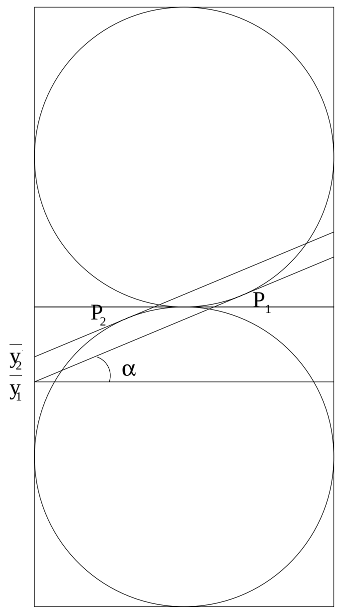


Figure 3: crossing of vertical lines

The points $y_1$ and $y_2$ are determined by the tangence points $P_1$ and $P_2$. As a consequence of these geometrical considerations, we cannot recover the impact times $t_1$ from the crossing times $\tau_1$ if the trajectory crosses $(\varepsilon y_1, \varepsilon y_2)$ because, in this case, we have two possibilities for the collision. Therefore we write the left hand side of ( 3.15) as

$$T_\varepsilon = \sum_{t_1 \in S_1(\sigma)} \varepsilon(1-\varepsilon)^{k_1+k_2} F_\varepsilon(x + vt_1 + v_1(t_1)(\sigma - t_1), v_1(t_1)) =$$

$$\sum_{\tau_1 \in \Sigma_1} \varepsilon(1-\varepsilon)^{k_1+k_2} F_\varepsilon(x + vt_1 + v_1(t_1)(\sigma - t_1), v_1(t_1)) +$$

$$\sum_{\tau_1 \in \Sigma_1} \varepsilon(1-\varepsilon)^{k_1+k_2} F_\varepsilon(x + v\bar{t}_1 + v_1(\bar{t}_1)(\sigma - \bar{t}_1), v_1(\bar{t}_1))\chi(y(\tau_1) \in (y_1, y_2)) \tag{3.18}$$

where $t_1 = t_1(\tau_1)$ and $\bar{t}_1 = \bar{t}_1(\tau_1)$ are the first and the second impact time with the disk configuration after $\tau_1$, $y(\tau_1)$ is the vertical quote of the particle at the entrance time $\tau_1$, seen in the unit reference box, $v_1(t_1)$ is the velocity after collision at time $t_1$, $\chi($something$)$ is the characteristic function of ”something”.

We denote by $T_\varepsilon^1$ and $T_\varepsilon^2$ the two terms in the right hand side of Eq. ( 3.18) respectively. We analyze in detail $T_\varepsilon^2$ being the behavior of $T_\varepsilon^1$ completely analogous.

We observe preliminarily that the outgoing velocity $v_1 = v'$ is quickly varying with respect to $t_1$, while $F_\varepsilon$ is slowly varying with respect to the space variable. With this in mind we decompose $\Sigma_1$ in blocks, large enough to extract averages for $v_1$, but sufficiently small to ensure that $F_\varepsilon$ is practically constant as regards the other dependence. Denoting by $|A|$ the cardinality of a set $A$, we set

$$\Sigma_1 = \bigcup_{q=1}^{Q} I_q$$

where $Q = [|\Sigma_1|^{1/2}]$, and the sets $I_q$ are choosen disjoint and approximatively of the same cardinality $\approx Q$. Note that $|\Sigma_1| \approx \frac{\sigma \cos\alpha}{\varepsilon}$.

By the smoothness of $F_\varepsilon$ we can write:

$$(3.19)\quad T_\varepsilon^2 = \sum_{q=1}^{Q} \sqrt{\varepsilon} \sum_{\tau_1 \in I_q} \sqrt{\varepsilon}(1-\varepsilon)^{k_1+k_2} F_\varepsilon(x + v\tau_1^q + v_1(\sigma - \tau_1^q), v_1)$$
$$\chi(y(\tau_1) \in (y_1, y_2)) + \circ(\sqrt{\varepsilon})$$

where $\tau_1^q$ denotes the entrance time in the first box of the block $I_q$. Eq. ( 3.19) follows by the fact that $|\tau_1^q - t_1| \leq C\sqrt{\varepsilon}$. Moreover in Eq. ( 3.19) the variables $k_1$, $k_2$ and $v_1$ are thought expressed in terms of $\tau_1$ by means of $\bar{t}_1$ (which fix in which box the scattering takes place).

We also note that we can replace $k_i$ by $\bar{k}_i^q$ (with $i = 1, 2$), which are the numbers of crossing the disks before and after $\tau_1^q$ for $i = 1$ or $i = 2$ respectively. Indeed we have:

$$k_i = \bar{k}_i^q + O(\frac{1}{\sqrt{\varepsilon}})$$

so that this change produces an error $\circ(\sqrt{\varepsilon})$. Note also that now the only dependence on $\tau_1$ is confined on $v_1$ and $y(\tau_1)$, the other quantities depending on the block $I_Q$ only.

We now fix a time $t_1 \in (0, \sigma)$ and let $q = q(t_1, \varepsilon)$ be such that $\tau_1^q \leq t_1 < \tau_1^{q+1}$. We want to analyze the factor:

$$(3.20)\quad \frac{1}{|I_q|} \sum_{\tau_1 \in I_q} F_\varepsilon(x + v\tau^{q(t_1,\varepsilon)} + v_1(\sigma - \tau^{q(t_1,\varepsilon)}), v_1)\chi(y(\tau_1) \in (y_1, y_2)).$$

Since $v_1 = v_1(y)$ (the velocity after the collision) depends on the vertical quote $y$ of the particle at the entrance time $\tau_1$, seen in the reference unit box, we can rewrite ( 3.20) as:

$$\frac{1}{N}\sum_{n=0}^{N-1} H_\varepsilon(\phi^n(y_\varepsilon)) \tag{3.21}$$

where $N = |I_q|$, the function $H_\varepsilon$ is defined, for fixed $x, v, t_1$, by:

$$H_\varepsilon : y \in [0,1) \to F_\varepsilon(x + v\tau^{q(t_1,\varepsilon)} + v_1(\sigma - \tau^{q(t_1,\varepsilon)}), v_1)\chi(y(\tau_1) \in (y_1, y_2))$$

and

$$\phi(y) = (y + \tan\alpha) \bmod 1,$$

is the rotation in $S^1$. The initial point $y_\varepsilon$ in the ergodic sum ( 3.21) is given by the relation:

$$x + v\tau_1^q = \Big(\Big[\frac{x + v\tau_1^q}{\varepsilon}\Big] + (0, y_\varepsilon)\Big)\varepsilon.$$

Here $[x] = ([x_1],[x_2])$ denotes the element of $\mathbb{R}^2$ which has as components the integer parts of the components of $x \in \mathbb{R}^2$. To make $H_\varepsilon$ smooth we replace $\chi(y(\tau_1) \in (y_1, y_2))$ by $\chi_\varepsilon(y)$, that is a $C^\infty$ function such that $\chi(y(\tau_1) \in (y_1, y_2)) \geq \chi_\varepsilon(y)$,

$$\|\chi_\varepsilon - \chi(y \in (y_1, y_2))\|_{L_1(S^1)} \leq \varepsilon^\gamma$$

and

$$|\chi_\varepsilon'''(y)| \leq \varepsilon^{-3\gamma}.$$

$\gamma \in (0,1)$ will be fixed later on.

Defining

$$\tilde H_\varepsilon(y) = F_\varepsilon(x + v\tau^{q(t_1,\varepsilon)} + v_1(y)(\sigma - \tau^{q(t_1,\varepsilon)}), v_1(y))\chi_\varepsilon(y),$$

by the invariance of $dy$ with respect to $\phi$, we suddenly get that:

$$\|\frac{1}{N}\sum_{n=0}^{N-1}(\tilde H_\varepsilon(\phi^n(\cdot)) - H_\varepsilon(\phi^n(\cdot))\|_{L_1(y)} \leq \|\tilde H_\varepsilon - H_\varepsilon\|_{L_1(y)} \leq C\varepsilon^\gamma. \tag{3.22}$$

In dealing with

$$\frac{1}{N}\sum_{n=0}^{N-1} \tilde H_\varepsilon(\phi^n y_\varepsilon) \tag{3.23}$$

we shall use the fact that

$$|\tilde H_\varepsilon'''(y)| \leq C\varepsilon^{-3\gamma}.$$

Writing ( 3.23) in terms of the Fourier transform of $\tilde{H}_\varepsilon$, we find:

$$(3.24)\qquad ( 3.23) = \int_0^1 dy \tilde{H}_\varepsilon(y) + \frac{1}{N}\sum_{k\neq 0} e^{i2\pi k y_\varepsilon} \frac{e^{i2\pi k \tan\alpha N} - 1}{e^{i2\pi k\tan\alpha} - 1}\hat{\tilde{H}}_\varepsilon(k).$$

Setting:

$$(3.25)\qquad B_N = \{\xi \in [0,1) | \sup_{k\neq 0} \frac{1}{k^2|e^{i2\pi k\xi} - 1|} > \sqrt{N}\}$$

and

$$B_N(k) = \{\xi \in [0,1) | \frac{1}{|e^{i2\pi k\xi} - 1|} \geq k^2\sqrt{N}\},$$

by the Tchebychev inequality:

$$\text{meas}(B_N(k)) \leq |k|^{-4/3} N^{-1/3} \int_0^1 d\xi \frac{1}{|e^{i2\pi k\xi} - 1|^{\frac{2}{3}}}$$

for which

$$(3.26)\qquad \text{meas}(B_N) \leq \sum_{k\neq 0} \text{meas}(B_N(k)) \leq C\frac{1}{N^{1/3}}.$$

If $\tan\alpha \notin B_N$, the sum in the right hand side of ( 3.24) can be estimated by:

$$\frac{2}{N^{1/2}}\sum_{k\neq 0} k^2 |\hat{\tilde{H}}_\varepsilon|(k) \leq \frac{C}{N^{1/2}}\Big(\int_0^1 |\tilde{H}'''_\varepsilon(y)|^2 dy\Big)^{1/2} = C\varepsilon^{\frac{1-\beta}{4}}\varepsilon^{-3\gamma},$$

where we have used that $N \geq C\varepsilon^{-(1-\beta)/2}$ and ( 3.25).
Therefore if $\tan\alpha \notin B_N$, we have:

$$(3.27)\qquad |\frac{1}{N}\sum_{n=0}^{N-1} \tilde{H}_\varepsilon(\phi^n(y_\varepsilon)) - \int_0^1 H_\varepsilon(y)| \leq C\varepsilon^{2\beta}$$

after having choosen $\gamma = 2\beta$, $\beta < 1/33$.
As a consequence we can write:

$$T_\varepsilon^2 = \int_0^\sigma dt_1 (1-\varepsilon)^{\bar{k}_1^{q(t_1,\varepsilon)} + \bar{k}_2^{q(t_1,\varepsilon)}} \cos\alpha \cdot$$

$$(3.28)\qquad \int_0^1 dy F_\varepsilon(x + v\tau^{q(t_1,\varepsilon)} + v_1(\sigma - \tau^{q(t_1,\varepsilon)}), v_1)\chi(y \in (y_1, y_2)) + \varphi_\varepsilon(x, v)$$

where the error $\varphi_\varepsilon$ can be estimated in the $L_1$ norm. Indeed the error we made is a sum of various terms which we now summarize. If we integrate in $dv$ over the region $\tan\alpha \in B_N$, using ( 3.26) we get an error bounded by $\varepsilon^{\frac{1-\beta}{6}} < \varepsilon^{2\beta}$. On the other hand, if $\tan\alpha \notin B_N$ the errors we made are bounded by $C\sqrt{\varepsilon}$ (continuity arguments), or $C\varepsilon^{2\beta}$ (see ( 3.27)) in $L_\infty$, which, because of the compactness of the support of $F_\varepsilon$, can be transformed in $L_1$ estimates. Then the regularization of the characteristic function $\chi(y \in (y_1, y_2))$ produce an error $C\varepsilon^{2\beta}$ in $L_1$ as follows by ( 3.22). In facts the integration over $dy$ corresponds to an integration over the ordinate of $x$. The last error is to replace the sum $\sum_{q=1}^{Q} \sqrt{\varepsilon}\sqrt{\frac{t}{\cos\alpha}}$ by the integral $\int_0^t dt_1$, which, due to the smoothness of the integrand, is $\circ(\sqrt{\varepsilon})$.

In conclusion

$$\|\varphi_\varepsilon\|_{L_1} \le C\varepsilon^{2\beta}.$$

The same estimate holds for $T^1_\varepsilon$:

$$T^1_\varepsilon = \int_0^\sigma dt_1 (1-\varepsilon)^{\bar{k}_1^q + \bar{k}_2^q} \cdot$$

$$\int_0^1 dy F_\varepsilon(x + v\tau^q + v_1(\sigma - \tau^q), v_1) + \varphi_\varepsilon(x, v)$$

(here we neglect the dependence of $q$ by $t_1$ and $\varepsilon$ for notational simplicity) with the convention that here $v_1$ is computed according to the collision with the first disk.

We now consider the mapping $\rho \to \xi(\rho)$ as in Fig. 4 (where $\rho$ is the impact parameter).

$\xi \in [\bar{y}_1, y_2]$ and this mapping is one-to-one and $\frac{d\rho}{d\xi} = \cos\alpha$. The collisions related to the interval $(\bar{y}_1, \bar{y}_2)$ with the second disk correspond to the set $\rho \in [-\frac{1}{2}, -\frac{1}{2} + (\bar{y}_2 - \bar{y}_1)\cos\alpha]$. The collisions relative to the set $\rho \in [-\frac{1}{2} + (\bar{y}_2 - \bar{y}_1)\cos\alpha, \frac{1}{2}]$ are those relative to the interval $[\bar{y}_2, y_2]$ with the first disk, described by the term $T^1_\varepsilon$. Therefore we can express the integration over $dy$ by means of a single integral on $d\rho$. It is more convenient, however, to introduce the hard-sphere cross section namely: $1/2|\omega \cdot v| = \frac{d\rho}{d\theta}$ where $\theta$ is the scattering angle. As a consequence:

$$T_\varepsilon = \int_0^\sigma dt_1 (1-\varepsilon)^{\bar{k}_1^q + \bar{k}_2^q} \cdot$$

$$1/2 \int d\omega |\omega \cdot v| F_\varepsilon(x + v\tau^q + v_1(\sigma - \tau^q), v_1) + \varphi_\varepsilon(x, v)$$

where $v_1$ is the outgoing velocity relative to $\omega$ and incoming velocity $v$.

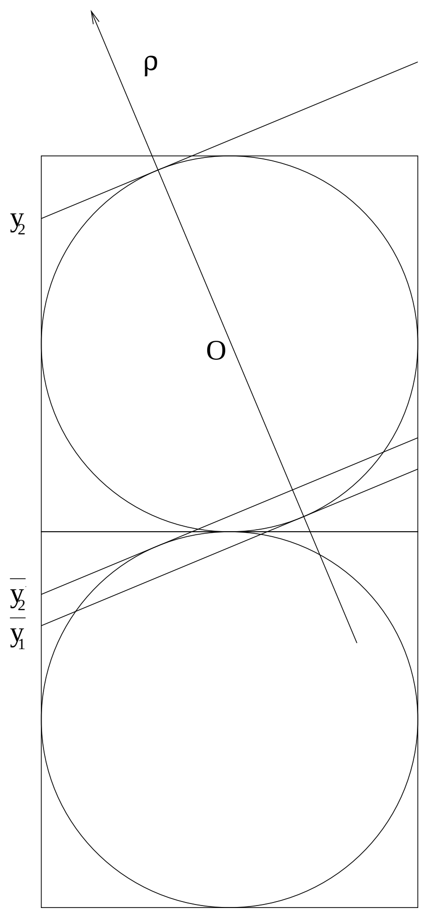


Figure 4: the mapping $\rho \to \xi(\rho)$

Setting now $\psi : [0,1) \to \mathbb{R}$ defined as:

$$\psi(y) = 1 \quad \text{if} \quad y \in [0, y_1) \cup (y_2, 1]$$

$$\psi(y) = 2 \quad \text{if} \quad y \in [y_1, y_2],$$

we have

$$\bar{k}_1^q = \sum_{r=1}^{q-1} \sum_{h=1}^{|I_r|} \psi(\phi^h(y_\varepsilon)). \tag{3.29}$$

for some initial point $y_\varepsilon$.

We now apply again the ergodic theorem, after having smoothed $\psi$ as before, so that we can replace $(1-\varepsilon)^{\bar{k}_1^q}$ by

$$\exp\{-t_1 \cos\alpha \int_0^1 \psi(y)dy\} = e^{-t_1}.$$

Finally we have:

$$\bar{k}_2^q = \sum_{r \geq q+1} \sum_{h=1}^{|I_r|} \psi(\phi_1^h(y_\varepsilon)). \tag{3.30}$$

Here the rotation map $\phi_1$ is associated to the scattering angle $\theta$. We perform the same construction, depending whether $\tan\theta$ belong to $B_N$ or not. In the first case we have a negligible set of $\omega$'s. In the second case, we can apply the ergodic estimate for which we can replace $(1-\varepsilon)^{\bar{k}_1^q}$ by $e^{-(\sigma - t_1)}$.

This concludes the proof of ( 3.15).

We observe that ( 3.14) has been also proven in the treatment of the expressions ( 3.29) and ( 3.30) .

*Remark 2*

The same procedure can be applied to prove an analogous result for other periodic configurations of obstacles. For instance, consider the triangular lattice (see Fig. 5)

$$\mathcal{T} = \{((n_1 + \frac{n_2}{2})a, \frac{\sqrt{3}}{2} n_2 a) | n_i \in \mathbb{Z} \quad i = 1, 2\}.$$

The radius $r$ of the scatterers must satisfies the condition $r \geq \frac{\sqrt{3}}{4} a$ so to have finite horizon.

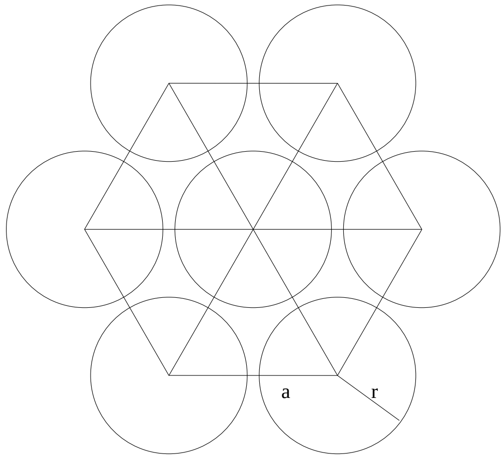


Figure 5: triangular lattice

## 3.2 The lattice gas: asymptotic equivalence to the Markovian model

Let $z(t) = T^t(x, v)$ be the Markov process associated to the evolution equation ( 2.4).

The three processes $z$,$z_\epsilon$,$\tilde{z}_\varepsilon$ have the same initial distribution density $f_0 = f_0(x, v)$ and can be represented with probability $1$ on the Skorohod space

$$\begin{aligned} D_{[0,T]}(\mathbb{R}^2 \times S^1) = \{z : [0, T] \longrightarrow \mathbb{R}^2 \times S^1 | \forall t \;\; &\in \;\; [0, T) \; z(t) = \lim_{s \to t^+} z(s), \\ z(T) = \lim_{s \to T^-} z(s), \forall t \;\; &\in \;\; (0, T] \; \exists \lim_{s \to t^-} z(s)\} \end{aligned}$$

equipped with the distance

$$\begin{aligned} d_S(x, y) &= \inf_{\lambda \in \Lambda} \{ \sup_{0 \leq t \leq T} \|x(t) - y(\lambda(t))\|_{\mathbb{R}^2 \times S^1} + \sup_{0 \leq t \leq T} |t - \lambda(t)| \} \\ \Lambda &= \{\lambda \in C([0, T]) : \lambda(t) > \lambda(s) \;\; t > s, \lambda(0) = 0, \;\; \lambda(T) = T\}. \end{aligned}$$

Indeed the three processes $z$,$z_\epsilon$,$\tilde{z}_\varepsilon$ perform a finite number of jumps with probability one. The statement is obvious for $z$ and $z_\epsilon$. By a standard argument based on Poincaré lemma the same property holds also for $\tilde{z}_\varepsilon$ (see for instance [3], p. 142).

Given a trajectory $z \in D_{[0,T]}(\mathbb{R}^2 \times S^1)$, we call *jumping time* for $z$ each time $t^*$ such that $\lim_{t \to t^*+} z(t) \neq \lim_{t \to t^*-} z(t)$.

Let $\mu$,$\mu_\epsilon$,$\tilde{\mu}_\varepsilon$ be the measures induced by the processes $z$,$z_\epsilon$,$\tilde{z}_\varepsilon$ resp. (we denote in the following by $\mu^*$, $z^*$, $P^*$, etc. one of the three possible choices for the measures, processes, transition probability densities, etc.). Each $\mu^*$ is defined on cylindrical functions by the integrals:

$$\int \mu^*(dz)F(z) = \int f_0(z)P^*_{t_1}(z|z_1)P^*_{t_2-t_1}(z_1|z_2)\dots P^*_{t_n-t_{n-1}}(z_{n-1}|z_n) \\ F_n(z_1,\dots,z_n)dzdz_1\dots dz_n$$

where $F(z) = F_n(z(t_1),\dots,z(t_n))$, $F_n \in C(\mathbb{R}^2 \times S^1)$.

As a consequence of the convergence obtained in Theorem 2, it is possible to show that $\mu_\varepsilon \to \mu$ weakly.

**Lemma 1** *For each cylindrical function $F$*

$$\lim_{\varepsilon\to 0}\int \mu_\varepsilon(dz)F(z) = \int \mu(dz)F(z)$$

*(The marginal distributions of $\mu^\varepsilon$ converge to the marginal distributions of $\mu$)*

The convergence is evident if we consider the cylindrical functions of the form

$$F(z) = \{\prod_{i=1}^n F_i(z(t_i))\}, \qquad F_i \in C_0^\infty(\mathbb{R}^2 \times S^1)$$

On these particular functions we have:

$$\int \mu^*(dz)F(z) = \\ \int f_0(z)P^*_{t_1}(z|z_1)F_1(z_1)P^*_{t_2-t_1}(z_1|z_2)F_2(z_2)\dots P^*_{t_n-t_{n-1}}(z_{n-1}|z_n)F_n(z_n)dzdz_1\dots dz_n = \\ \int dz[V_*^{t_n-t_{n-1}}(F_nV_*^{t_{n-1}-t_{n-2}}F_{n-1}\dots F_2V_*^{t_1}F_1](z)f_0(z)$$

so that, if we define

$$\begin{aligned} G^n_\varepsilon &= V_\varepsilon^{t_n-t_{n-1}}(F_nV_\varepsilon^{t_{n-1}-t_{n-2}}F_{n-1}\dots F_2V_\varepsilon^{t_1}F_1) \\ G^n &= V^{t_n-t_{n-1}}(F_nV^{t_{n-1}-t_{n-2}}F_{n-1}\dots F_2V^{t_1}f_1) \end{aligned}$$

and consider ( 3.17), we obtain the following bound:

$$|\int \mu_\varepsilon(dz)F(z) - \int \mu(dz)F(z)| \leq \\ M\|G^n_\varepsilon - G^n\|_{L_1(\mathbb{R}^2\times S^1)} \leq \tilde{C}t\varepsilon^\beta$$

The convergence on the set of cylindrical functions is then obtained by linearity and density.

To extend the result to the set of continues bounded functions we use the following theorem ( [6], p. 431):

**Theorem 3** *Let $\xi_\varepsilon(t)$ be a sequence of stochastic processes such that:*

*a) $\xi_\varepsilon(t)$ belongs with probability 1 to $D_{[0,T]}(\mathcal{H})$, where $\mathcal{H}$ is a complete metric space with metric $d_\mathcal{H}$;*

*b) the marginal distributions of $\xi_\varepsilon(t)$ converge to the marginal distributions of $\xi_0(t)$ as $\varepsilon \to 0$;*

*c) there exist positive $\alpha$, $\beta$ and $H$ such that*

$$\mathbb{E}[d_\mathcal{H}(\xi_\varepsilon(t_1)\xi_\varepsilon(t_2))d_\mathcal{H}(\xi_\varepsilon(t_2)\xi_\varepsilon(t_3))]^\beta \leq H(t_3 - t_1)^{1+\alpha}$$

*for $\varepsilon \geq 0$, $t_1 < t_2 < t_3$.*

*Then for each continuous functional $\phi : D_{[0,T]}(\mathcal{H}) \longrightarrow \mathbb{R}$ the distribution of $\phi(\xi_\varepsilon)$ converges to the distribution of $\phi(\xi)$.*

If $z_\varepsilon$ verifies the hypothesis of the theorem 3, the convergence of the distribution of $\phi(z_\varepsilon)$ to the distribution of $\phi(z)$ for each continuos functional implies the convergence

$$\lim_{\varepsilon\to 0} \int \mu_\epsilon(dz)f(z) = \int \mu(dz)f(z) \tag{3.31}$$

for each bounded continuos function $f$.

Let $\mathcal{H} = \mathbb{R}^2 \times S^1$ and $d_\mathcal{H}(x, y) = d(x, y) = \|x - y\|_{\mathbb{R}^2\times S^1} \wedge 1$. Then

$$\mathbb{E}[d(z_\varepsilon(t_1), z_\varepsilon(t_2))d(z_\varepsilon(t_2), z_\varepsilon(t_3))] \leq 9(t_3 - t_1)^2.$$

In fact

$$\begin{aligned}
&d(z_\varepsilon(t_1), z_\varepsilon(t_2))d(z_\varepsilon(t_2), z_\varepsilon(t_3)) \leq \|z_\varepsilon(t_1) - z_\varepsilon(t_2)\|_{\mathbb{R}\times S^1}\|z_\varepsilon(t_2) - z_\varepsilon(t_3)\|_{\mathbb{R}\times S^1} \\
&\quad \leq \|x_\varepsilon(t_1) - x_\varepsilon(t_2)\|_{\mathbb{R}^2}\|x_\varepsilon(t_2) - x_\varepsilon(t_3)\|_{\mathbb{R}^2} + \|x_\varepsilon(t_1) - x_\varepsilon(t_2)\|_{\mathbb{R}^2}\|v_\varepsilon(t_2) - v_\varepsilon(t_3)\|_{S^1} \\
&\quad +\|v_\varepsilon(t_1) - v_\varepsilon(t_2)\|_{S^1}\|x_\varepsilon(t_2) - x_\varepsilon(t_3)\|_{\mathbb{R}^2} + \|v_\varepsilon(t_1) - v_\varepsilon(t_2)\|_{S^1}\|v_\varepsilon(t_2) - v_\varepsilon(t_3)\|_{S^1}.
\end{aligned}$$

Observing that $\|v_\varepsilon(t_1) - v_\varepsilon(t_2)\|_{S^1} = O(1)$ if it exists at least one jumping time $\bar{t} \in (t_1, t_2)$ and $\|v_\varepsilon(t_1) - v_\varepsilon(t_2)\|_{S^1}\|v_\varepsilon(t_2) - v_\varepsilon(t_3)\|_{S^1} = O(1)$ if there exist at least two jumping times $\bar{t}_1 \in (t_1, t_2)$ and $\bar{t}_2 \in (t_2, t_3)$, if we denote by $\chi_1(t_1, t_2)$ and $\chi_2(t_1, t_2, t_3)$ the characteristic functions resp. of the sets

$$\begin{aligned}
A_1(t_1, t_2) &= \{z \in D_{[0,T]}(\mathbb{R}^2 \times S^1) : \exists t_s \in (t_1, t_2) \;\; s.t. \;\; v(t_s^-) \neq v(t_s^+))\} \\
A_2(t_1, t_2, t_3) &= \{z \in D_{[0,T]}(\mathbb{R}^2 \times S^1) : \exists t_{s_1} \in (t_1, t_2), t_{s_2} \in (t_2, t_3) \;\; s.t. \;\; v(t_{s_i}^-) \neq v(t_{s_i}^+))\}
\end{aligned}$$

and if we note that

$$\begin{aligned}\|x_\varepsilon(t_i)-x_\varepsilon(t_{i+1})\|_{\mathbb{R}^2} &\le |t_i-t_{i+1}|\\ \|v_\varepsilon(t_i)-v_\varepsilon(t_{i+1})\|_{S^1} &\le 2\\ \chi_2(t_1,t_2,t_3) &= \chi_1(t_1,t_2)\chi_1(t_2,t_3),\end{aligned}$$

we have

$$\begin{aligned}&\mathbb{E}[d(z_\varepsilon(t_1),z_\varepsilon(t_2))d(z_\varepsilon(t_2),z_\varepsilon(t_3))] \le |t_2-t_1||t_3-t_2| + 4\mathbb{E}(\chi_2(t_1,t_2,t_3))\\ &\quad +2[|t_2-t_1|\mathbb{E}(\chi_1(t_2,t_3)) + |t_3-t_2|\mathbb{E}(\chi_1(t_1,t_2))]\\ &\quad \le |t_3-t_1|^2 + 4|t_3-t_2||t_2-t_1| + 4|t_3-t_2||t_2-t_1| \le 9|t_3-t_1|^2\end{aligned}$$

Since, for each $\varepsilon \ge 0$, $z_\varepsilon$ verifies the hypothesis of Theorem 3 with $\beta = 1$, $\alpha = 1$, $H = 9$, the convergence ( 3.31) is proved.

Let $D^f_{[0,T]}(\mathbb{R}^2\times S^1) \subset D_{[0,T]}(\mathbb{R}^2\times S^1)$ be the set of all trajectories with a finite number of jumps and define

$$A^-_\eta = \{z\in D^f_{[0,T]}(\mathbb{R}^2\times S^1) | \exists s : \min_{k\le s-1}\inf_{t\in(t_{k-1},t_k)} \|x(t)-x(t_s)\|_{\mathbb{R}^2} \le \eta \quad t_j \text{ jumping time}\}$$

$$A^+_\eta = \{z\in D^f_{[0,T]}(\mathbb{R}^2\times S^1) | \exists s : \min_{k\ge s+1}\inf_{t\in(t_k,t_{k+1})} \|x(t)-x(t_s)\|_{\mathbb{R}^2} \le \eta \quad t_j \text{ jumping time}\}$$

and

$$A_\eta = A^+_\eta \bigcup A^-_\eta .$$

Note that when $\eta \ge \varepsilon$, $A_\eta$ contains all the trajectories of the process $z_\varepsilon$ exhibiting recollisions (i.e. the trajectory hits the same obstacle twice) or interferences in the future or in the past (there exists an obstacle which is crossed and hited by the same trajectory) (see e. g. Fig. 2).

Moreover, $A_\eta$ is closed in the Skorohod topology, $\mu(A_\eta)$ is vanishing when $\eta$ goes to $0$ and $\mu(\partial A_\eta) = 0$ ($A_\eta$ is a continuity set for $\mu$). As a consequence of ( 3.31)

$$\lim_{\varepsilon\to 0}\mu_\epsilon(A_\eta) = \mu(A_\eta) \quad \forall\eta. \tag{3.32}$$

For any continuous bounded function $f : D_{[0,T]}(\mathbb{R}^2\times S^1) \longrightarrow \mathbb{R}$, we have:

$$\begin{aligned}&|\int \tilde\mu_\varepsilon(dz)f(z) - \int \mu_\varepsilon(dz)f(z)| \le \|f\|_\infty(\tilde\mu_\varepsilon(A_\eta)+\mu_\varepsilon(A_\eta)) +\\ &\qquad |\int_{A^c_\eta}\tilde\mu_\varepsilon(dz)f(z) - \int_{A^c_\eta}\mu_\varepsilon(dz)f(z)|\end{aligned} \tag{3.33}$$

Note that $\tilde{\mu}_\varepsilon|_{A^c_\eta} \geq \mu_\varepsilon|_{A^c_\eta}$. Indeed, for a given $z$, set

$$n_\varepsilon(z) = n^1_\varepsilon(z) - n^2_\varepsilon(z)$$

where $n^1_\varepsilon(z)$ is the number of crossing the obstacles by the trajectory and $n^2_\varepsilon(z)$ is the number of crossed obstacles. Therefore

$$\int_{A^c_\eta} \tilde{\mu}_\varepsilon(dz) f(z) = \int_{A^c_\eta} \mu_\varepsilon(dz) f(z) (\frac{1}{1-\varepsilon})^{n_\varepsilon(z)} \tag{3.34}$$

As a consequence $\tilde{\mu}_\varepsilon(A_\eta) \leq \mu_\varepsilon(A_\eta)$, which, by virtue of ( 3.32), is vanishing as $\eta \to 0$.

By virtue of ( 3.34), it follows that the left hand side of ( 3.33) is bounded by

$$F_1(\eta) + \int_{A^c_\eta} \mu_\varepsilon(dz) f(z) [(\frac{1}{1-\varepsilon})^{n_\varepsilon(z)} - 1] \tag{3.35}$$

where $F_i(\eta) \to 0$ as $\eta \to 0$, $i = 1, 2, \dots$.

We preliminary observe that by an elementary geometrical argument:

$$n_\varepsilon(z) \leq \sum_{i=1}^{k(z)} \sum_{j=i+1}^{k(z)} \frac{C}{\tan \alpha_{ij}} \tag{3.36}$$

where $2\alpha_{ij}$ is the angle formed by the lines which contain the i-th and j-th branch of the trajectory and $k(z)$ is the number of jumps.

In addition we have the obvious estimate

$$(\frac{1}{1-\varepsilon})^{n_\varepsilon(z)} \leq e^{cT}.$$

The integral in 3.35 is bounded by:

$$\|f\|_{L_\infty} \{ \int_{A^c_\eta} \mu_\varepsilon(dz) \prod_{i<j} \chi(\tan \alpha_{ij} > \xi) \chi(k(z) < N) [(\frac{1}{1-\varepsilon})^{\frac{ck(z)^2}{\xi}} - 1] +$$
$$\int_{A^c_\eta} \mu_\varepsilon(dz) \chi(k(z) \geq N) e^{cT} + \int_{A^c_\eta} \mu_\varepsilon(dz) \sum_{i<j} \chi(\tan \alpha_{ij} < \xi) e^{cT} \} \chi(k(z) < N).$$

The first integral is bounded by

$$[(\frac{1}{1-\varepsilon})^{\frac{cN^2}{\xi}} - 1] \overset{\varepsilon \to 0}{\to} 0. \tag{3.37}$$

The second integral is bounded by

$$e^{cT}\frac{T^N}{N!} + F_2(\varepsilon).$$

The third integral is bounded by

$$CN^2(\xi + F_3(\varepsilon)).$$

Finally, using 3.35 and choosing suitable scalings for $N$ and $\xi$, we conclude that

$$|\int f(z)\mu_\varepsilon(dz) - \int f(z)\tilde{\mu}_\varepsilon(dz)| \leq F_1(\eta) + F_4(\varepsilon).$$

Therefore, due to the arbitrariness of $\eta$, we conclude:

$$\lim_{\varepsilon\to 0}\tilde{\mu}_\varepsilon = \lim_{\varepsilon\to 0}\mu_\varepsilon = \mu$$

weakly.

This concludes the proof of Theorem 2.

*Remark 3*

Of course, Theorem 2 can be proved for other periodic configurations of obstacles, such as that on a triangular lattice (see Remark 2).

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