We develop control laws for the pendular motion of a spinning tethered satellite system. The control laws are developed using a simplified system model that contains unknown disturbance terms that account for unmodeled dynamics in the simplified model. Principles of sliding mode and adaptive nonlinear control design are used to develop control laws that drive the pendular motion to a desired reference motion in spite of the effects of unmodeled dynamics. The effectiveness of the control laws is tested by applying them to a much more physically realistic system model than that used in their derivation. In all cases tested, the control laws are successful in controlling the pendular motion of the more physically realistic model about the desired reference motion. The main disadvantage of the control laws is that they require relatively large steady-state control inputs to account for the unmodeled dynamics and hold the pendular motion on the reference motion. Any problems associated with these large steady-state inputs could be avoided by simply turning the control off after the reference motion has been reached and accepting the small deviations from the reference motion due to unmodeled dynamics.

Because of this fact, we develop control laws using a simplified system model that isolates the pendular motion. The system is modeled as two point masses connected by a rigid tether, with the system mass center constrained to an unperturbed circular orbit. In this manner, the system can provide propellantless orbital transfer capabilities for low-Earth orbit satellites.

The primary goal of the control development is to derive control laws that can be applied to more physically realistic system models, or even an actual spinning TSS. The simplified system model used in this work neglects many aspects of spinning TSS dynamics, such as the elastic vibrations of the tether, the attitude motion of the end bodies, and variations in the orbit of the system. To account for the effects of these unmodeled dynamics, unknown time-varying disturbance terms are included in the equations of motion governing the pendular motion. Principles of sliding mode control and adaptive control are used to develop control laws that control the pendular motion about a desired reference motion, in spite of the effects of the unmodeled dynamics. The effectiveness of the control laws is demonstrated using numerical simulations of a much more physically realistic system model.

In this work, we are concerned with controlling the pendular motion of a two body spinning TSS.
is the in-plane attitude angle of the tether is related to \( \mathcal{F}_o \) and \( \mathcal{F}_e \) by

\[
\mathbf{R}^{eo} = \begin{pmatrix}
\cos \alpha & 0 & -\sin \alpha \\
-\sin \alpha \sin \beta & \cos \beta & -\cos \alpha \sin \beta \\
\sin \alpha \cos \beta & \sin \beta & \cos \alpha \cos \beta
\end{pmatrix}
\]

and the angular velocity of \( \mathcal{F}_e \) relative to \( \mathcal{F}_o \) is

\[
\mathbf{\omega}^{en} = -\dot{\beta} \hat{e}_1 + \dot{\alpha} \cos \beta \hat{e}_2 + \dot{\alpha} \sin \beta \hat{e}_3
\]

Combining Eqs. (1) and (4) and making use of Eq. (3), the angular velocity of \( \mathcal{F}_e \) relative to \( \mathcal{F}_n \) is

\[
\mathbf{\omega}^{en} = -\dot{\beta} \hat{e}_1 + (\dot{\alpha} + \Omega) \cos \beta \hat{e}_2 + (\dot{\alpha} + \Omega) \sin \beta \hat{e}_3
\]

Because the system is modeled as a rigid body, the equations governing the pendular motion are determined using Euler’s rotational equations

\[
\mathbf{\ddot{\omega}}^{en} + \mathbf{\omega}^{en} \times (\mathbf{\ddot{\omega}}^{en} + \mathbf{\omega}^{en} \times \mathbf{\omega}^{en}) = \mathbf{\bar{M}}
\]

where \( \mathbf{\bar{I}} \) is the second moment of inertia tensor of the system about \( \mathcal{C} \), and \( \mathbf{\bar{M}} \) is the total external torque vector acting on the system about \( \mathcal{C} \). All of the mass of the system is distributed along \( \hat{e}_3 \), so \( \mathbf{\bar{I}} \) can be written as

\[
\mathbf{\bar{I}} = I(1 - \hat{e}_3 \hat{e}_3)
\]

where \( I \) is the transverse moment of inertia of the system and \( \mathbf{1} \) is the identity tensor. We assume that the only external torques acting on the system are the gravity-gradient torque, \( \mathbf{\bar{M}}_g \), and a control torque, \( \mathbf{\bar{M}}_c \). In this study we are not concerned with the physical mechanism used to provide the control torque, although we do note that it could be provided by small thrusters on the end bodies of the system. Because the length of a typical TSS is small relative to the orbit radius of the system, we use a linear approximation for the gravity-gradient torque

\[
\mathbf{\bar{M}}_g = 3\Omega^2 \hat{e}_3 \times \mathbf{\bar{I}} \cdot \hat{e}_3
\]

Expressing the control torque as

\[
\mathbf{\bar{M}}_c = -M_{\beta} \hat{e}_1 + M_{\alpha} \hat{e}_2
\]

the total external torque acting on the system is

\[
\mathbf{\bar{M}} = 3\Omega^2 \hat{e}_3 \times \mathbf{\bar{I}} \cdot \hat{e}_3 - M_{\beta} \hat{e}_1 + M_{\alpha} \hat{e}_2
\]

Using Eqs. (5), (7), and (10) in Eq. (6), the equations governing the pendular motion are

\[
\ddot{\alpha} \cos \beta - 2(\dot{\alpha} + \Omega) \dot{\beta} \sin \beta + 3\Omega^2 \sin \alpha \cos \alpha \cos \beta = \frac{M_{\alpha}}{I}
\]

\[
\dot{\beta} + [(\dot{\alpha} + \Omega)^2 + 3\Omega^2 \cos^2 \alpha] \sin \beta \cos \beta = \frac{M_{\beta}}{I}
\]

To cast Eqs. (11) and (12) in a nondimensional form, define the nondimensional time \( \tau = \Omega t \) and denote
derivatives with respect to $\tau$ by $\dot{\hat{z}}$. Using this definition of $\tau$, Eqs. (11) and (12) are written in nondimensional form as

$$
\ddot{\hat{\alpha}} \cos \beta - 2(\hat{\alpha} + 1)\beta \cos \beta + 3 \sin \alpha \cos \alpha \cos \beta = u_\alpha + \delta_\alpha 
$$

$$
\ddot{\hat{\beta}} + (\hat{\alpha} + 1)^2 + 3 \cos^2 \alpha \sin \beta \cos \beta = u_\beta + \delta_\beta 
$$

where the nondimensional control inputs are defined as

$$
u = \frac{M}{I \Omega^2} 
$$

Equations (13) and (14) govern the pendular motion of the simplified TSS model illustrated in Fig. 1. As mentioned in the Introduction, the ultimate goal of the control developments presented in this work is to design control laws that can be applied to more physically realistic TSS models, and perhaps an actual TSS. More realistic system models contain dynamics not modeled by the simplified system model in Fig. 1, so control laws developed using Eqs. (13) and (14) will not be entirely successful in controlling the pendular motion of the more realistic system model. We therefore modify Eqs. (13) and (14) to include effects of unmodeled dynamics by including unknown, time-varying disturbance terms $\delta_\alpha$ and $\delta_\beta$.

$$
\ddot{\hat{\alpha}} \cos \beta - 2(\hat{\alpha} + 1)\beta \cos \beta + 3 \sin \alpha \cos \alpha \cos \beta = u_\alpha + \delta_\alpha 
$$

$$
\ddot{\hat{\beta}} + (\hat{\alpha} + 1)^2 + 3 \cos^2 \alpha \sin \beta \cos \beta = u_\beta + \delta_\beta 
$$

The disturbance terms $\delta_\alpha$ and $\delta_\beta$ take into account all aspects of the dynamics not included in the simplified system model. In the next section, we use principles of sliding mode and adaptive control to derive control laws for $u_\alpha$ and $u_\beta$ that control the pendular motion about a desired reference motion in spite of the unknown disturbance terms.

### III. Control of Pendular Motion

For a typical spinning TSS, the nominal pendular motion is a planar spin in the orbit plane. For uncontrolled planar motion ($u_\alpha = u_\beta = 0, \beta = \tilde{\beta} = 0$), Eq. (13) admits the energy integral

$$
h = \ddot{\hat{\alpha}}^2 + 3 \sin^2 \alpha 
$$

where the constant $h$ is a measure of the energy of the planar pendular motion. For general uncontrolled motion, Eqs. (13) and (14) admit the energy integral

$$
\mathcal{H} = (\ddot{\hat{\alpha}}^2 + 3 \sin^2 \alpha) \cos^2 \beta + \dot{\hat{\beta}}^2 + 4 \sin^2 \beta 
$$

The quantity $\mathcal{H}$ is the Hamiltonian of the simplified system model, and is a measure of the energy of the pendular motion. Note that $\mathcal{H}$ reduces to $h$ for planar pendular motion. In the control developments that follow, we therefore seek control laws that drive the out-of-plane pendular motion to 0, and the quantity $\mathcal{H}$ to some desired value $h^*$. The result will be a natural planar trajectory with an energy level of $h^*$. Moreover, we seek control laws that drive the pendular motion to the desired reference motion in spite of the unknown disturbance terms $\delta_\alpha$ and $\delta_\beta$ that account for the effects of unmodeled dynamics in the simplified system model.

#### Sliding Mode Control

A common nonlinear control design technique used for systems with unknown disturbance terms is sliding mode control. Two excellent resources that cover the general theory of sliding mode control design are the books by Khalil and Slotine and Li. The basic concepts of sliding mode control are discussed as they are encountered in the control development presented below.

We first consider controlling the out-of-plane pendular motion to zero. Define the quantity

$$
z_1 = \ddot{\hat{\beta}} + k_\beta \beta 
$$

where $k_\beta$ is a positive constant. From Eq. (20), if we could drive $z_1$ to zero in a finite time and hold it there for all time, then $\beta$ and $\dot{\beta}$ would exponentially approach 0 as $\tau$ approaches infinity. The surface $z_1 = 0$ is known as the sliding manifold or the sliding mode for the $\beta$ dynamics, and the control developments presented below are geared toward driving the motion to the sliding manifold in a finite time. Moreover, we want to drive the motion to the sliding manifold in spite of the unknown disturbance terms.

Differentiating Eq. (20) and using Eq. (17), the $z_1$ dynamics are governed by

$$
z_1 = -[(\ddot{\hat{\alpha}} + 1)^2 + 3 \cos^2 \alpha] \sin \beta \cos \beta + k_\beta \beta + u_\beta + \dot{\delta}_\beta 
$$

From Eq. (21), let the control input be

$$
u_\beta = [(\ddot{\hat{\alpha}} + 1)^2 + 3 \cos^2 \alpha] \sin \beta \cos \beta - k_\beta \beta + v_\beta 
$$

such that Eq. (21) becomes

$$
z_1 = v_\beta + \dot{\delta}_\beta 
$$

Ellis
Note that we have chosen $u_\beta$ to cancel the known, nonlinear components of Eq. (21) while leaving the unknown disturbance term. The term $v_\beta$ must now be chosen to account for $\delta_\beta$.

Define the candidate Lyapunov function for the $z_1$ dynamics as

$$V_1 = \frac{1}{2} z_1^2$$  \hspace{1cm} (24)

which is positive definite about $z_1 = 0$. Differentiating Eq. (24) and using Eq. (23), the rate-of-change of $V_1$ along the system trajectory is

$$\dot{V}_1 = z_1 \ddot{z}_1 = z_1 (v_\beta + \delta_\beta) \leq z_1 v_\beta + |z_1| |\delta_\beta|$$  \hspace{1cm} (25)

At this point, we make the assumption that the unknown disturbance $\delta_\beta$ is bounded, such that

$$|\delta_\beta| \leq \delta_\beta^*$$  \hspace{1cm} (26)

where $\delta_\beta^*$ is a positive constant. Using this definition in Eq. (25), the function $V_1$ satisfies

$$\dot{V}_1 \leq z_1 v_\beta + |z_1| \delta_\beta^*$$  \hspace{1cm} (27)

Now, let $v_\beta$ be given by

$$v_\beta = -(b_\beta + \delta_\beta^*) \text{sgn}(z_1)$$  \hspace{1cm} (28)

where $b_\beta > 0$ is a control gain and $\text{sgn}(x)$ is the signum function. Using Eq. (28) in Eq. (27), we have

$$\dot{V}_1 \leq -z_1 (b_\beta + \delta_\beta^*) \text{sgn}(z_1) + |z_1| \delta_\beta^*$$

$$= -b_\beta |z_1|$$  \hspace{1cm} (29)

$$\leq 0$$

So, $V_1$ is non-increasing, and is therefore a Lyapunov function for the $z_1$ dynamics. The control law given by the combination of Eqs. (22) and (28) therefore stabilizes the pendular motion about $z_1 = 0$.

To gain further insight about the controlled motion of the system, we must consider two cases. First, if $|z_1| = 0$, then from the definition of $V_1$ we must have $V_1 = 0$. But, we also know that $V_1$ is non-increasing, so if $V_1$ becomes 0, it must remain zero for all time, implying that $z_1$ remains zero for all time as well. Next, consider the case when $|z_1| \neq 0$. Noting that $\dot{V}_1$ can also be written as

$$\dot{V}_1 = |z_1| \frac{d}{dt}(|z_1|)$$  \hspace{1cm} (30)

from Eq. (29) we have

$$\frac{d}{dt}(|z_1|) \leq -b_\beta$$  \hspace{1cm} (31)

Integrating Eq. (31), we obtain

$$|z_1| \leq -b_\beta \tau + |z_1(0)|$$  \hspace{1cm} (32)

which implies that $|z_1|$ reaches 0 in a finite time, $T_1$, that satisfies

$$T_1 \leq \frac{|z_1(0)|}{b_\beta}$$  \hspace{1cm} (33)

Once $|z_1|$ reaches zero, we know that it remains 0 for all time. The control therefore drives the out-of-plane pendular motion to the sliding mode in finite time, after which $\beta$ and $\tilde{\beta}$ approach 0 according to $z_1 = 0$. This convergence is achieved in spite of the presence of the unknown disturbance term $\delta_\beta$. Combining Eqs. (22) and (28), the composite control law is

$$u_\beta = -\alpha(\alpha + 1) \cos \alpha \sin \beta \cos \beta - k_\beta \tilde{\beta}$$

$$- (b_\beta + \delta_\beta^*) \text{sgn}(z_1)$$  \hspace{1cm} (34)

To determine an expression for the control input $u_\alpha$, define the quantity

$$z_2 = \mathcal{H} - h^*$$  \hspace{1cm} (35)

The surface $z_2 = 0$ defines the sliding mode for the in-plane pendular motion. If we can drive $z_2$ to 0 and hold it there, $\mathcal{H}$ will remain at the desired value for all time. Because the control law given by Eq. (34) drives the out-of-plane pendular motion to 0, the pendular motion will therefore approach a natural planar trajectory with $h = h^*$.

Differentiating Eq. (35) and using Eqs. (16) and (17), the $z_2$ dynamics are governed by

$$\dot{z}_2 = 2[\ddot{\alpha} (u_\alpha + \delta_\alpha) \cos \beta + \tilde{\beta} (u_\beta + \delta_\beta)]$$  \hspace{1cm} (36)

From Eq. (36), let $u_\alpha$ be given by

$$u_\alpha = \frac{1}{\alpha \cos \beta} \left( \frac{v_\alpha}{2} - \tilde{\beta} u_\beta \right)$$  \hspace{1cm} (37)

such that the closed-loop dynamics for $z_2$ become

$$\dot{z}_2 = v_\alpha + 2(\delta_\alpha \ddot{\alpha} \cos \beta + \delta_\beta \tilde{\beta})$$  \hspace{1cm} (38)

As done in the development of $u_\beta$, we have chosen $u_\alpha$ to cancel certain nonlinear terms in the $z_2$ dynamics, while also containing a term, $v_\alpha$, that will be used to compensate for the uncertainties $\delta_\alpha$ and $\delta_\beta$. 
Define the candidate Lyapunov function for the $z_2$ dynamics as

$$V_2 = \frac{1}{2}z_2^2$$

(39)

which is positive definite about $z_2 = 0$. Differentiating Eq. (39) and using Eq. (38), the time rate-of-change of $V_2$ along the system trajectory is

$$\dot{V}_2 = z_2\ddot{z}_2$$

$$= z_2 v_\alpha + 2z_2(\delta_\alpha \dot{\alpha} \cos \beta + \delta_\beta \dot{\beta})$$

(40)

As done with $\delta_\beta$, we assume that $\delta_\alpha$ is bounded by

$$|\delta_\alpha| \leq \delta_\alpha^*$$

(41)

such that Eq. (40) becomes

$$\dot{V}_2 \leq z_2 v_\alpha + 2|z_2|(\delta_\alpha^* |\dot{\alpha}| + \delta_\beta^* |\dot{\beta}|)$$

(42)

From Eq. (42), let $v_\alpha$ be given by

$$v_\alpha = -[b_\alpha + 2(\delta_\alpha^* |\dot{\alpha}| + \delta_\beta^* |\dot{\beta}|)] \text{sgn}(z_2)$$

(43)

where $b_\alpha > 0$ is a constant control gain, such that

$$\dot{V}_2 \leq -z_2[b_\alpha + 2(\delta_\alpha^* |\dot{\alpha}| + \delta_\beta^* |\dot{\beta}|)] \text{sgn}(z_2)$$

$$+ 2|z_2|(\delta_\alpha^* |\dot{\alpha}| + \delta_\beta^* |\dot{\beta}|)$$

$$= -b_\alpha |z_2|$$

$$\leq 0$$

(44)

The function $V_2$ is therefore non-increasing, and is a Lyapunov function for the $z_2$ dynamics. The control law given by the combination of Eqs. (37) and (43) stabilizes $z_2$ about 0. Following similar logic as outlined previously for $z_1$, we can further state that the control law drives $z_2$ to 0 in a finite time, $T_2$, that satisfies

$$T_2 \leq \frac{|z_2(0)|}{b_\alpha}$$

(45)

and hold $z_2$ at zero for all time. The control law given by Eqs. (37) and (43) therefore drives $\mathcal{H}$ to the desired value and holds it at that value, in spite of the unknown disturbance terms. Combining Eqs. (37) and (43), the composite control law is

$$u_\alpha = -\beta \dot{u}_\beta - \frac{1}{2}[b_\alpha + 2(\delta_\alpha^* |\dot{\alpha}| + \delta_\beta^* |\dot{\beta}|)] \text{sgn}(z_2)$$

$$\frac{\dot{\alpha} \cos \beta}{\dot{\beta} + k_\beta \beta}$$

(46)

The control laws defined by Eqs. (34) and (46) drive the pendular motion to the desired reference motion in spite of the unknown disturbance terms. The only restrictions that we have to place of the system dynamics are that the disturbance terms are bounded, which is not a particularly restrictive assumption. However, we do note that the control laws are discontinuous because of their reliance on the signum function, which is discontinuous when its argument is 0. Such discontinuous control inputs can never be achieved by a real physical system, so we must modify the control laws to remove these discontinuities.

Following Refs. [5] and [6], let the control inputs have the modified form

$$u_\beta = ([\dot{\alpha} + 1]^2 + 3 \cos^2 \alpha] \sin \beta \cos \beta - k_\beta \dot{\beta}$$

$$- (b_\beta + \delta_\beta^*) \text{sat} \left( \frac{z_1}{\varepsilon_1} \right)$$

(47)

$$u_\alpha = \frac{-\beta u_\beta - \frac{1}{2}[b_\alpha + 2(\delta_\alpha^* |\dot{\alpha}| + \delta_\beta^* |\dot{\beta}|)] \text{sat} \left( \frac{z_2}{\varepsilon_2} \right)}{\dot{\alpha} \cos \beta}$$

(48)

where $\varepsilon_i$ are small positive constants and sat($x$) is the saturation function

$$\text{sat}(x) = \begin{cases} x, & |x| \leq 1 \\ \text{sgn}(x), & |x| > 1 \end{cases}$$

(49)

Because of the form of the saturation function, the control laws defined by Eqs. (47) and (48) are continuous; however, the use of the saturation function in place of the signum function means that the motion is not driven exactly to the sliding mode. Instead, the motion is driven to a small neighborhood of the sliding mode called the boundary layer, the size of which is determined by the small positive constants $\varepsilon_i$. As the $\varepsilon_i$ decrease, the boundary layer decreases in size, such that it vanishes when both $\varepsilon_i$ are 0. Once the control drives the system to the boundary layer, $z_1$ and $z_2$ remain in it for all time. In light of this fact, using Eqs. (47) and (48) means that we can never exactly reach the desired reference motion, but we can come arbitrarily close to it by appropriate choice of $\varepsilon_i$.

The control gains for the sliding mode controller can be determined as follows. First, $b_\beta$ and $b_\alpha$ can be determined from Eqs. (33) and (45) by specifying values for $T_1$ and $T_2$. The resulting values of $b_\alpha$ and $b_\beta$ will ensure that $z_1$ and $z_2$ reach the boundary layer in times that satisfy Eqs. (33) and (45); however, we should note that the boundary layer may be reached in much faster times than $T_1$ and $T_2$. To determine an appropriate value for $k_\beta$, we note that the out-of-plane motion is governed by

$$\dot{\beta} + k_\beta \beta = 0$$

(50)
on the sliding mode. The solution to Eq. (50) is

\[ \beta(\tau) = \beta(\tau^*) e^{-k_\beta(\tau-\tau^*)} \]  

(51)

where \( \tau^* \) is the time that the sliding manifold is reached. Defining the ratio \( r = \beta(\tau)/\beta(\tau^*) \), we can rewrite Eq. (51) as

\[ k_\beta = -\frac{\ln(r)}{\tau - \tau^*} \]  

(52)

So, we can use Eq. (52) to determine \( k_\beta \) such that \( \beta \) is approximately reduced by a factor of \( r \) in a time \( \tau - \tau^* \) after the boundary layer is reached. Note that the reduction in \( \beta \) is approximate because the sliding mode is never exactly reached.

Adaptive Sliding Mode Control

The sliding mode control laws derived previously in this section drive the pendular motion to the desired reference motion in spite of the unknown disturbance terms. However, the control laws require a priori knowledge of the disturbance terms in the form of the upper bounds \( \delta^* \) and \( \delta^*_B \). For systems such as a spinning TSS, this type of knowledge may not be available, or it may be impractical to determine the upper bounds of the disturbances for the entire range of system parameters. We would therefore like to determine control laws that can account for the unknown disturbances without any a priori knowledge of them. To determine such control laws, we once again use the principles of sliding mode control used previously in this section, but also make use of some basic principles of adaptive nonlinear control.6

Returning to Eq. (25), recall that the time-rate-of-change of the Lyapunov function for the \( z_1 \) dynamics satisfies

\[ \dot{V}_1 \leq z_1 v_\beta + |z_1| |\delta_\beta| \]  

(53)

In the sliding mode control law development, we assumed that we knew an upper bound on \( |\delta_\beta| \) and proceeded to derive an expression for \( v_\beta \) using that knowledge. We now relax the assumption that we know anything about \( |\delta_\beta| \) and determine an expression for \( v_\beta \). Let \( v_\beta \) be given by

\[ v_\beta = -b_\beta \text{sgn}(z_1) \]  

(54)

where \( b_\beta = b_\beta(\tau) \) is a positive, time-varying control gain. Using Eq. (54) in Eq. (53), we have

\[ \dot{V}_1 \leq -(b_\beta - |\delta_\beta|)|z_1| \]  

(55)

If \( b_\beta \geq |\delta_\beta| \), then \( \dot{V}_1 \leq 0 \) and we have a sliding mode controller similar to the one developed previously. If \( b_\beta < |\delta_\beta| \), however, then the Lyapunov rate will be positive and \( z_1 \) does not approach the sliding mode. In light of these facts, let \( b_\beta \) be governed by the equation

\[ \dot{b}_\beta = (d_{\beta 1} b_\beta + d_{\beta 2} \dot{V}_1) H(\text{sgn}(\dot{V}_1)) \]  

(56)

where \( d_{\beta 1} > 0 \) are constant control gains and \( H(x) \) is the Heaviside function

\[ H(x) = \begin{cases} 1, & x \geq 0 \\ 0, & x < 0 \end{cases} \]  

(57)

From Eq. (56), when \( \dot{V}_1 < 0 \) we have \( \dot{b}_\beta = 0 \), and \( b_\beta \) remains constant. When \( \dot{V}_1 \geq 0 \) we have \( \dot{b}_\beta > 0 \), and \( b_\beta \) increases; the positive constants \( d_{\beta 1} \) dictate the rate of growth of \( b_\beta \), which is proportional to the current value of \( b_\beta \) and the current value of the Lyapunov rate. In this manner, \( b_\beta \) increases whenever the Lyapunov rate is positive until it reaches a value that is larger than the upper bound of \( |\delta_\beta| \), after which it remains constant and the control laws acts like the sliding mode control law developed previously. The update expression given by Eq. (56) allows \( b_\beta \) to adapt according to the current behavior of the system. In a sense, Eq. (56) allows the control law to “learn” the value of \( b_\beta \) that results in a negative definite Lyapunov rate.

To ensure that the control law is continuous, let Eq. (54) be modified as

\[ v_\beta = -b_\beta \text{sat}(\frac{z_1}{\varepsilon_1}) \]  

(58)

When used with the adaptation mechanism of Eq. (56), this control law ensures that \( z_1 \) is driven to, and held within, the boundary layer. Once \( z_1 \) is held inside the boundary layer, there may be times at which the Lyapunov rate is positive, and \( b_\beta \) will be updated according to Eq. (56). However, because \( z_1 \) has already reached the boundary layer, we no longer need to update \( b_\beta \). We therefore set \( \dot{b}_\beta = 0 \) if \( |z_1| \leq \varepsilon_1 \). Combining Eqs. (34), the composite control law for the out-of-plane pendular motion is

\[ u_\beta = ([\dot{\alpha} + 1]^2 + 3 \cos^2 \alpha) \sin \beta \cos \beta - k_\beta \dot{\beta} \]  

\[ - b_\beta \text{sat}(\frac{z_1}{\varepsilon_1}) \]  

(59)

Returning to Eq. (40), recall that the Lyapunov rate for the \( z_2 \) dynamics satisfies

\[ \dot{V}_2 \leq z_2 v_\alpha + 2|z_2|(|\delta_\alpha||\dot{\alpha}| + |\delta_\beta||\dot{\beta}|) \]  

(60)
Similar to the procedure used to determine \( v_\beta \), let \( v_\alpha \) be given by

\[
v_\alpha = -b_\alpha \text{sgn}(z_2)
\]  

(61)

where \( b_\alpha = b_\alpha(\tau) \) is a positive, time-varying control gain. Using Eq. (61) in Eq. (60), we have

\[
\dot{V}_2 \leq -[b_\alpha - 2(||\delta_\alpha|| + ||\delta_\beta||)||z_2|]
\]  

(62)

In light of Eq. (62), let \( b_\alpha \) be governed by

\[
\dot{b}_\alpha = (d_{\alpha 1} b_\alpha + d_{\alpha 2} \dot{V}_2)H(\text{sgn}(\dot{V}_2))
\]  

(63)

where \( d_{\alpha i} > 0 \) are constant control gains. As discussed previously when considering \( b_\beta \), Eq. (63) eventually results in a sliding mode controller that drives \( z_2 \) to 0. To ensure that the control input is continuous, however, we modify Eq. (61) as

\[
v_\alpha = -b_\alpha \text{sat}\left(\frac{z_2}{\varepsilon_2}\right)
\]  

(64)

such that the composite control law is

\[
u_\alpha = -u_\beta \dot{\beta} - \frac{1}{2} b_\alpha \text{sat}\left(\frac{z_2}{\varepsilon_2}\right)
\]  

(65)

Equations (59) and (65), in combination with the adaptation mechanisms of Eqs. (56) and (63), drive the pendular motion to the desired reference motion. Moreover, they do so in spite of the unknown disturbance terms, and without any a priori knowledge of the disturbance terms.

### IV. Validation of Control Laws

The control laws developed in the previous section are validated by applying them to a more physically realistic system model. Detailed descriptions of the more physically realistic system model used in this work can be found in Refs. [2] and [7]. This model treats the system as two finite, rigid bodies connected by a flexible tether. The orbit of the system is also allowed to vary over time. The more realistic model therefore contains effects due to the end body attitude motion, elastic vibrations of the tether, and variation of the system orbit that are not accounted for by the simplified system model presented in Sec. II. The effects of these unmodeled dynamics can be grouped into the disturbance terms \( \delta_\alpha \) and \( \delta_\beta \) in Eqs. (16) and (17), such that the control laws derived in Sec. III can be applied to the more realistic model. In the remainder of this section, numerical solutions of the more physically realistic system model are presented to assess the effectiveness of the control laws developed in Sec. III.

**Sliding Mode Control**

An example of the controlled response of the pendular motion of the more realistic system model under the action of the sliding mode control laws is shown in Figs. 2 and 3. From Fig. 2 we can see that \( H \) is driven to the desired value of 25 and held near 25 for all time. Similarly, we can see from Fig. 3 that the out-of-plane pendular motion is driven to the sliding mode in finite time, after which the out-of-plane motion converges to zero. One point to note about the response shown in Figs. 2 and 3 is that the pendular motion is not driven exactly to the sliding mode, but, as mentioned previously, is instead driven to an arbitrarily small neighborhood of the sliding mode called the boundary layer. We have chosen the size of the boundary layer in Figs. 2 and 3 to be quite small to better illustrate the effectiveness of the control laws.

Figures 4 and 5 show the required control inputs corresponding to the system response shown in Figs. 2 and 3. Note that these control inputs are assumed to be thrust forces from small thrusters on the end bodies, and that \( F_{B1} \) and \( F_{B2} \) correspond to dimensional forms of \( u_\alpha \) and \( u_\beta \), respectively. As Figs. 4 and 5 show, both control inputs are on the order of 1 N in magnitude, meaning that they could easily be supplied by small thrusters on the end bodies. We do note, however, that the control laws require relatively large steady-state inputs to account for the effects of unmodeled dynamics and hold the pendular motion in the boundary layer. These steady-state inputs are one of the main drawbacks of the sliding mode control laws, because steady-state inputs could require large amounts of propellant if they are provided by thrusters on the end bodies. The simplest solution to this problem is to turn off the control once the reference motion has been reached and accept the small deviations from the reference motion that will arise.
Adaptive Sliding Mode Control

An example of the controlled response of the pendular motion of the more realistic system model under the action of the adaptive sliding mode control laws is shown in Figs. 6 and 7. As Fig. 6 shows, $H$ is driven to the boundary layer in finite time and held there for all time. Note that the boundary layer is once again chosen to be quite small to illustrate the effectiveness of the control laws. Figure 7 shows the controlled response of the out-of-plane pendular motion. The motion is driven to the boundary layer in finite time, but cannot be held there because the control gains have not sufficiently adapted. The motion undergoes several periods in which it is driven in and out of the boundary layer before the gains have become sufficiently large to hold the motion in the boundary layer. After this time, the out-of-plane motion approaches the desired reference motion. Time histories of the control gains are shown in Figs. 8 and 9. Note that both gains start at 0, then grow before leveling-off at steady-state values.

Figures 10 and 11 show the required control inputs corresponding to the system response shown in Figs. 6 and 7. As with the regular sliding mode control laws, the adaptive laws require control forces on the order of 1 N. The adaptive laws also require the same sort of large steady-state inputs as required by the regular sliding mode control laws. As mentioned previously, the simplest solution to the problems that arise from these large steady-state inputs is to simply turn off the control once the reference motion is reached and accept any resulting small deviations from the desired reference motion.
Figure 6: Controlled response of $H$ under the action of the adaptive sliding mode control laws

Figure 7: Controlled response of out-of-plane pendular motion under the action of the adaptive sliding mode control laws

Figure 8: Time history of the adaptive control gain $b_\alpha$

Figure 9: Time history of the adaptive control gain $b_\beta$

Figure 10: Control force $F_{B1}$ required for adaptive sliding mode control

Figure 11: Control force $F_{B2}$ required for adaptive sliding mode control
V. Conclusions

Both the sliding mode control laws and the adaptive sliding mode control laws are successful in controlling the pendular motion of the more physically realistic system model about the desired reference motion. This control is robust, in that it is achieved in spite of the fact that the control laws are based upon a simplified system model that does not account for several aspects of the system dynamics, such as end body attitude motion, variation of the system orbit, and the elastic vibrations of the tether. The main drawback of both sets of control laws is that they require relatively large steady-state inputs in order to account for the effects of the unmodeled dynamics. If the inputs are provided by small thrusters on the end bodies, these steady-state inputs could result in excessive propellant requirements. The simplest solution to this problem is to turn off the control once the desired reference motion is reached and accept the small deviations from the reference motion that will inevitably arise.

References


