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Spectral Solutions for Some Hyperbolic Partial Differential Equations by the Ultraspherical Tau Method

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Abstract: Ultraspherical spectral methods with Chebyshev and Legendre polynomials that allow efficient approximation solutions for the hyperbolic partial differential equations in a square have been used to solve one and two-dimensional wave equations. These methods reduce the problem into a system of ordinary differential equations in the time-dependent expansion coefficients. The resulting system may be solved numerically by using several newly improved methods some of which are totally new. These methods are the eigenvalue method, the variational iteration method, Picard's method, Runge-Kutta method, and the analytical approximation method. Comparisons of results due to various methods with the exact solutions have been made.

Keywords: Wave equation; Chebyshev; Legendre; Systems of ordinary differential equations; Eigenvalues; Variational iterations; Picard; Runge-kutta; and Analytical approximation solution

1 Introduction

There has been a great interest in approximating the solution of an initial boundary value problem by using spectral methods as a truncated series of globally defined smooth functions in each space variable. This can be considered as a development of the known weighted residual method. Choice of the trial functions distinguishes these methods from finite difference and finite element methods in such a way that it is responsible for the superior approximation properties of the spectral methods. For problems with smooth solutions the method is highly accurate and in many other equations, especially the nonlinear problems, it gives an exceedingly approximation solution with substantially fewer degrees of freedom.

The trial functions considered here for the expansion solution are to be taken as the well known Chebyshev and Legendre polynomials. Explicit expressions for the expansion coefficients of the solution have been obtained by Karageorghis [1] when the basis functions of expansion are shifted Chebyshev polynomials $T_n^*(x)$, $x \in [0, 1]$. A corresponding formula for Legendre polynomials $P_n(x)$, $x \in [-1, 1]$ has been derived in [2]. Doha [3] has obtained a more general formula when the basis functions are the ultraspherical polynomials $C_n^{(\alpha)}(x)$, $x \in [-1, 1]$ and $\alpha \in (-\frac{1}{2}, \infty)$. Formulae for first and second kinds Chebyshev polynomials $T_n(x)$, $U_n(x)$ and Legendre polynomials $P_n(x)$ are given as special cases of $C_n^{(\alpha)}(x)$.

Spectral methods based on double Chebyshev polynomials for solving numerically partial differential equations in two space variables have been used by many authors, among them, Dew and Scraton [4], Doha [5, 6], Haidvogel and Zang [7] and Horner [8]. The existence and use of Fast Fourier Transform for Chebyshev polynomials to compute efficiently the matrix-vector product has made them more widely used than other sets of orthogonal polynomials, e.g. Legendre, or ultraspherical polynomials, Doha and Abd-Elhameed [9]. In that work they presented a double ultraspherical spectral methods for solving some parabolic and elliptic partial differential equations in a square subject to the most general inhomogeneous mixed boundary conditions e.g. the one and two-dimensional heat equation as well as Poisson's and Helmholtz equations. It is worthy to mention that ultraspherical tau methods with Chebyshev, or Legendre polynomials are extremely sensitive to the proper formulation of boundary conditions. When proper boundary conditions are imposed

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so that the problem is well-posed, the methods yield very accurate results. When improper boundary conditions are applied, the methods are likely to be explosively unstable. An example is given by Gottlieb and Orszag [10]. Some algorithms for solving one and two-dimensional problems have been proposed by Elliott [11], Knibb and Scraton [12] and Dew and Scraton [13]. Other useful references are Doha [14], Golub and Loan [15], Noor and Din [16], He [17–20], Inokuti et al. [21], Noor [22], Graham [23], and Loan [24].
In the present work we are going to use ultraspherical spectral methods based on Chebyshev and Legendre polynomials for solving some hyperbolic partial differential equations, namely, the wave equation in one and two space variables in addition to the time variable. The resulting systems of ordinary differential equations have been solved for the time-dependent expansion coefficients by using several numerical methods some of which are totally new.

2 Formula for expansion coefficients of the derivatives [9, 14]

For the coefficients of a differentiated function of ultraspherical polynomials, let $u(x)$ be an infinitely differentiable function defined on the interval $[-1,\,1]$, then we can write

$$u(x)=\sum_{n=0}^{\infty} a_n\, C_n^{(\alpha)}(x),\quad (\alpha>-\frac{1}{2}), \tag{1}$$

where $C_n^{(\alpha)}(x)$ are the ultraspherical polynomials for which $C_n^{(\alpha)}(1)=1, n=0,\,1,\,2,\,\cdots$ and has the desirable properties that $C_n^{(0)}(x)$ are identical with the Chebyshev polynomials of the first kind $T_n(x)$, $C_n^{(1)}(x)$ are equal to $[1/(n+1)]U_n(x)$ where $U_n(x)$ are the Chebyshev polynomials of the second kind, and $C_n^{(\frac{1}{2})}(x)$ are the Legendre polynomials $P_n(x)$. For the q^{th} derivative of $u(x)$

$$u^{(q)}(x)=\sum_{n=0}^{\infty} a_n^{(q)}\, C_n^{(\alpha)}(x),\qquad a_n^{(0)}=a_n,\quad q\geq 1, \tag{2}$$

then

Theorem 1: [3]

$$a_n^{(q)}=\frac{2^q\,(n+\alpha)\,\Gamma(n+2\alpha)}{(q-1)!\,n!}\sum_{j=1}^{\infty}\frac{(j+q-2)!\,(n+2j+q-2)!\,\Gamma(n+j+q+\alpha-1)}{(j-1)!\,\Gamma(n+j+\alpha)\,\Gamma(n+2j+q+2\alpha-2)} \times\; a_{n+2j+q-2},\quad n\geq 0,\;\; q\geq 1. \tag{3}$$

3 Coefficients of differentiated expansions of double ultraspherical polynomials [14]

Define the double ultraspherical polynomials as:

$$C_{mn}^{(\alpha)}(x,\,y)=C_m^{(\alpha)}(x)\,C_n^{(\alpha)}(y), \tag{4}$$

where $C_m^{(\alpha)}(x)$ and $C_n^{(\alpha)}(y)$ are ultraspherical polynomials of degrees m and n in the space variables x and y, respectively. These polynomials are satisfying the biorthogonality relation

$$\int_{-1}^{1}\int_{-1}^{1}\left[(1-x^2)\,(1-y^2)\right]^{\alpha-\frac{1}{2}}\ C_{ij}^{(\alpha)}(x,\,y)\,C_{kl}^{(\alpha)}(x,\,y)\,dx\,dy=$$

$$\begin{cases} \frac{\pi\,i!\,j!}{(i+\alpha)\,(j+\alpha)\,\Gamma(i+2\alpha)\,\Gamma(j+2\alpha)}\left(\frac{\Gamma(2\alpha)\,\Gamma(\alpha+\frac{1}{2})}{\Gamma(\alpha)}\right)^2; & i=k,\ j=l,\ \ \alpha\neq 0,\\ \pi^2; & i=j=k=l=0,\ \ \alpha=0,\\ \frac{\pi^2}{4}; & i=k\neq 0,\ j=l\neq 0,\ \ \alpha=0,\\ \frac{\pi^2}{2}; & i=k\neq 0,\ j=l=0,\ or\\ & i=k=0,\ j=l\neq 0,\ \alpha=0,\\ 0; & otherwise. \end{cases} \tag{5}$$

It is worthy to note here that

$$T_{mn}(x,\,y)=C_{mn}^{(0)}(x,\,y)=T_m(x)\,T_n(y), \tag{6}$$

$$U_{mn}(x,\,y)=C_{mn}^{(1)}(x,\,y)=\frac{1}{(m+1)(n+1)}\,U_m(x)\,U_n(y), \tag{7}$$

$$P_{mn}(x,\,y)=C_{mn}^{(\frac{1}{2})}(x,\,y)=P_m(x)\,P_n(y). \tag{8}$$

Let $u(x,\,y)$ be a continuous function defined on the square $-1\leq x,\,y\leq 1$ and has continuous and bounded partial derivatives of any order with respect to its variables x and y. Then it is possible to express

$$u(x,\,y)=\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}a_{mn}\,C_m^{(\alpha)}(x)\,C_n^{(\alpha)}(y), \tag{9}$$

$$u^{(p,\,q)}(x,\,y)=\frac{\partial^{p+q}u(x,\,y)}{\partial x^p\,\partial y^q}=\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}a_{mn}^{(p,\,q)}\,C_m^{(\alpha)}(x)\,C_n^{(\alpha)}(y), \tag{10}$$

where $a_{mn}^{(p,\,q)}$ denote the ultraspherical expansion coefficients of $u^{(p,\,q)}(x,\,y)$ and $a_{mn}^{(0,\,0)}=a_{mn}$. The relations among the coefficients $a_{mn}^{(p,\,q)}$, $a_{mn}^{(0,\,q)}$, $a_{mn}^{(p,\,0)}$ and a_{mn} are given by the following theorem:

Theorem 2: [14]

$$a_{mn}^{(p,\,q)}=\frac{2^p\,(m+\alpha)\,\Gamma(m+2\alpha)}{(p-1)!\,m!}\sum_{i=1}^{\infty}\frac{(i+p-2)!\,\Gamma(m+i+p+\alpha-1)\,(m+2i+p-2)!}{(i-1)!\,\Gamma(m+i+\alpha)\,\Gamma(m+2i+p+2\alpha-2)}\times a_{m+2i+p-2,\,n}^{(0,\,q)},\qquad p\geq 1, \tag{11}$$

$$a_{mn}^{(p,\,q)}=\frac{2^q\,(n+\alpha)\,\Gamma(n+2\alpha)}{(q-1)!\,n!}\sum_{j=1}^{\infty}\frac{(j+q-2)!\,\Gamma(n+j+q+\alpha-1)\,(n+2j+q-2)!}{(j-1)!\,\Gamma(n+j+\alpha)\,\Gamma(n+2j+q+2\alpha-2)}\times a_{m,\,n+2j+q-2}^{(p,\,0)},\quad q\geq 1, \tag{12}$$

$$a_{mn}^{(p,\,q)}=\frac{2^{p+q}\,(m+\alpha)\,(n+\alpha)\,\Gamma(m+2\alpha)\,\Gamma(n+2\alpha)}{(p-1)!\,(q-1)!\,m!\,n!}\sum_{j=1}^{\infty}\sum_{i=1}^{\infty}\frac{(i+p-2)!\,(j+q-2)!}{(i-1)!\,(j-1)!}\times\frac{\Gamma(m+i+p+\alpha-1)\,\Gamma(n+j+q+\alpha-1)\,(m+2i+p-2)!\,(n+2j+q-2)!}{\Gamma(m+i+\alpha)\,\Gamma(n+j+\alpha)\,\Gamma(m+2i+p+2\alpha-2)\,\Gamma(n+2j+q+2\alpha-2)}\,a_{m+2i+p-2,\,n+2j+q-2},\quad p,\,q\geq 1. \tag{13}$$

In particular, the special cases for the "bivariate" Chebyshev polynomials of the first and second kinds may be obtained directly by taking $\alpha = 0$ and 1, respectively, and for the "bivariate" Legendre polynomials by taking $\alpha = \frac{1}{2}$ in the previous theorem.

Problem statements and methods of solution

The purpose of this section is to implement the ultraspherical Tau spectral methods using Chebyshev and Legendre polynomials for solving numerically the wave equations in one and two space variables in addition to the time variable.

4 One-dimensional wave equation

Consider the following problem

$$\frac{\partial^2 u(x,\, t)}{\partial t^2} = K^2\, \frac{\partial^2 u(x,\, t)}{\partial x^2}, \qquad (-1 \le x \le 1),\ \ t > 0, \tag{14}$$

subject to the boundary conditions

$$u(-1,\, t) = c_1, \qquad u(1,\, t) = c_2, \qquad t > 0, \tag{15}$$

and the initial conditions

$$u(x,\, 0) = f(x), \qquad \frac{\partial u(x,\, 0)}{\partial t} = g(x), \qquad -1 \le x \le 1, \tag{16}$$

where c_1, c_2 and K are constants and $f(x)$ and $g(x)$ are known functions of x. It is assumed that the solution of the above problem can be expressed in a uniformly convergent ultraspherical series expansion of the form

$$u(x,\, t) = \sum_{n=0}^{\infty} a_n(t)\, C_n^{(\alpha)}(x), \tag{17}$$

We assume that the functions $f(x)$ and $g(x)$ are satisfying the boundary conditions Eq.11 in order to be sure that the solution of equation Eq.10 is free of discontinuities. Assume also that $f(x)$ and $g(x)$ have series expansions of the form

$$f(x) = \sum_{n=0}^{\infty} f_n\, C_n^{(\alpha)}(x), \ \text{ and } \ g(x) = \sum_{n=0}^{\infty} g_n\, C_n^{(\alpha)}(x), \tag{18}$$

which are uniformly convergent in $-1 \le x \le 1$. Then the solution of equation Eq.10 has a series expansion solution that is free of discontinuities. Assuming that $a_n(t)$ and $\ddot{a}_n(t)$ are negligible for $n \ge N+1$ and differentiating the $N+1$ term approximation equation Eq.13 and substituting in equation Eq.10 we get:

$$\sum_{n=0}^{N} \ddot{a}_n(t)\, C_n^{(\alpha)}(x) = K^2 \sum_{n=0}^{N} a_n(t)\, C''^{(\alpha)}_n(x), \tag{19}$$

where $\dot{}$ and $'$ denote differentiation with respect to time and space, respectively. Notice here that $C''^{(\alpha)}_n(x)$ is of degree $n-2$ and we aim to express the right-hand side of equation Eq.15 in terms of $C_n^{(\alpha)}(x)$. Thus equation Eq.2 gives

$$\sum_{n=0}^{N} \ddot{a}_n(t)\, C_n^{(\alpha)}(x) = K^2 \sum_{n=0}^{N} a_n^{(2)}(t)\, C_n^{(\alpha)}(x), \tag{20}$$

where $a_n^{(2)}(t)$ is given by equation Eq.3 with $q = 2$. After a straight forward calculations with the help of initial and boundary conditions and use of the orthogonality property of ultraspherical polynomials we arrive at the following 2^{nd} order system of ordinary differential equations:

$$\underline{\ddot{a}}(t) + A^{(\alpha)}\, \underline{a}(t) = 0, \tag{21}$$

where $A^{(\alpha)}$ is a constant nonsingular square matrix and

$$\underline{a}(t) = [\,a_0(t)\; a_1(t) \;\; \cdots \;\; a_{N-2}(t)\,]^T \,. \tag{22}$$

Solving system Eq.17 for the unknown coefficients $a_n(t),\; n = 0,\, 1,\, \cdots,\, N-2$ under the initial conditions $a_n(0) = f_n$, and $\dot{a}_n(0) = g_n$, the solution equation Eq.13 is therefore completely known.

Solution of the matrix differential equation:

In this section we are going to solve the resulting 2^{nd} order linear system of ordinary differential equations of the form Eq.17 by using each of the following five newly improved methods. Thus it is the time now to take $K = 4$, $c_1 = c_2 = 0$, $f(x) = \sin(\frac{\pi\,(x+1)}{2})$, $g(x) = 0$, and $N = 8$. Then the matrices $A = A^{(\alpha)}$ of constant coefficients corresponding to $\alpha = 0$ and $\alpha = \frac{1}{2}$, respectively, are as follows:

$$A^{(0)} = \begin{pmatrix} 4096 & 0 & 8064 & 0 & 7168 & 0 & 4736 \\ 0 & 5376 & 0 & 4992 & 0 & 3456 & 0 \\ 3840 & 0 & 7680 & 0 & 6912 & 0 & 4608 \\ 0 & 4480 & 0 & 4480 & 0 & 3200 & 0 \\ 3072 & 0 & 6144 & 0 & 6144 & 0 & 4224 \\ 0 & 2688 & 0 & 2688 & 0 & 2688 & 0 \\ 1792 & 0 & 3584 & 0 & 3584 & 0 & 3584 \end{pmatrix},$$

and

$$A^{(\frac{1}{2})} = \begin{pmatrix} 576 & 0 & 528 & 0 & 416 & 0 & 240 \\ 0 & 1296 & 0 & 1056 & 0 & 624 & 0 \\ 2640 & 0 & 2640 & 0 & 2080 & 0 & 1200 \\ 0 & 2464 & 0 & 2464 & 0 & 1456 & 0 \\ 3744 & 0 & 3744 & 0 & 3744 & 0 & 2160 \\ 0 & 2288 & 0 & 2288 & 0 & 2288 & 0 \\ 3120 & 0 & 3120 & 0 & 3120 & 0 & 3120 \end{pmatrix},$$

and the boundary conditions $a_n^{(\alpha)}(0) = f_n$ and $\dot{a}_n^{(\alpha)}(0) = g_n$ corresponding to $\alpha = 0$ and $\alpha = \frac{1}{2}$, respectively, are thus:

$$\begin{aligned} &\underline{a}^{(0)}(0) = [\,0.944002,\, 0,\, -0.499403,\, 0,\, 0.0279921,\, 0,\, -0.000596695\,]^T \,, \\ &\underline{\dot{a}}^{(0)}(0) = [\,0,\, 0,\, 0,\, 0,\, 0,\, 0,\, 0\,]^T \,, \end{aligned} \tag{23}$$

and

$$\begin{aligned} &\underline{a}^{(\frac{1}{2})}(0) = [\,0.63662,\, 0,\, -0.687085,\, 0,\, 0.051779,\, 0,\, -0.00133046\,]^T \,, \\ &\underline{\dot{a}}^{(\frac{1}{2})}(0) = [\,0,\, 0,\, 0,\, 0,\, 0,\, 0,\, 0\,]^T \,. \end{aligned} \tag{24}$$

It is to be noted here that the exact solution of this problem is given by

$$u(x,\, t) = \sin(\frac{\pi\,(x+1)}{2})\;\cos(2\,\pi\, t). \tag{25}$$

4.1 The eigenvalue method (denoted by EVM)

Starting with the assumption that

$$\underline{a}(t) = V_n\;\sin(\lambda_n\, t + b_n), \quad \text{with } b_n \text{ as constants}, \tag{26}$$

is a nonzero solution of the matrix equation Eq.17 leads to the following eigenvalue problem

$$A\,V_n = \lambda_n^2\, V_n, \quad n = 0,\, 1,\, \cdots,\, N-2. \tag{27}$$

This means that λ_n^2 is an eigenvalue of A with corresponding eigenvector $V_n = [v_{0n}\; v_{1n}\; \cdots\; v_{(N-2)n}]^T$. Hence

$$\underline{a}(t) = \sum_{n=0}^{N-2} V_n\;\left(\pm\,\xi_n\;\sin(\sqrt{\lambda_n'}\, t) + \eta_n\;\cos(\sqrt{\lambda_n'}\, t)\right), \tag{28}$$

with $\lambda'_n = eignv(A)$ is also a solution and the constants are to be determined from the following initial conditions:

$$V\,\underline{\eta} = \underline{a}(0), \quad \text{and} \quad V\,\underline{\xi}' = \underline{\dot{a}}(0), \tag{29}$$

where

$$V = [V_0\; V_1\; \cdots\; V_{N-2}], \qquad \underline{\eta} = [\eta_0\; \eta_1\; \cdots\; \eta_{N-2}]^T, \qquad \text{and}$$
$$\underline{\xi}' = [\xi_0\sqrt{\lambda'_0}\quad \xi_1\sqrt{\lambda'_1}\quad \cdots\quad \xi_{N-2}\sqrt{\lambda'_{N-2}}\,]^T. \tag{30}$$

Now after getting $\underline{a}(t)$ completely determined the values $a_{N-1}(t)$ and $a_N(t)$ will come from the boundary conditions

$$a_{N-1}(t) = -a_1(t) - a_3(t) - \cdots - a_{N-3}(t), \tag{31}$$

$$a_N(t) = -\frac{1}{2}\,a_0(t) - a_2(t) - \cdots - a_{N-2}(t), \tag{32}$$

when $\alpha = 0$ and thus the solution $u(x,\,t)$ becomes explicitly known. Applying this method with $N = 8$, the nonzero coefficients solution values $a_n(t)$ of the system Eq.17 at $t = 0.1$, 0.15 and 0.2 have been computed and reported in Table 1 for both $\alpha = 0$ and $\alpha = \frac{1}{2}$, respectively. All the coming methods for solving system Eq.17 depend basically

Table 1: Nonzero coefficients $a_n(t)$ of one-dimensional wave equation by using the EVM

	$\alpha = 0$			$\alpha = \frac{1}{2}$		
$a_n(t)$	$t = 0.1$	$t = 0.15$	$t = 0.2$	$t = 0.1$	$t = 0.15$	$t = 0.2$
$a_0(t)$	0.763714	0.554871	0.291713	0.515036	0.374196	0.196726
$a_2(t)$	−0.404026	−0.293542	−0.154324	−0.555864	−0.403859	−0.212321
$a_4(t)$	0.0226461	0.0164533	0.00865001	0.0418901	0.0304349	0.0160005
$a_6(t)$	−0.000482573	−0.000350694	−0.000184406	−0.00107606	−0.000781958	−0.00041123
$a_8(t)$	$5.2914E-6$	$3.83691E-6$	$1.9948E-6$	0.0000134527	$9.93432E-6$	$5.35903E-6$

on transforming the second order system into a first order system of ordinary differential equations. For this purpose the following transformations will be used:

$$y_{2i}(t) = a_{2i}(t), \;\; and \;\; y_{2i+1}(t) = \dot{a}_{2i}(t), \quad i = 0,\,1,\,\cdots,\,\frac{N-2}{2}, \tag{33}$$

this leads to the following first order system:

$$\left.\begin{array}{l} \dot{y}_{2i}(t) = y_{2i+1}(t), \qquad \text{with} \qquad y_{2i}(0) = a_{2i}(0), \\ \dot{y}_{2i+1}(t) = \sum_{k=0,\,k-even}^{N-2} A_{(2i)k}\; y_k, \qquad \text{with} \\ \qquad\qquad y_{2i+1}(0) = \dot{a}_{2i}(0), \end{array}\right\}, i = 0,\,1,\,\cdots,\,\frac{N-2}{2}. \tag{34}$$

This simplified system can now be solved by using any of the following efficient methods:

4.2 The variational iteration method (denoted by 'VIM')

One of the most efficient methods for solving system Eq.27 is the variational iteration method. To illustrate the basic concept of the technique, Noor et al. [16], consider the following differential equation:

$$L\,y(t) + N\,y(t) = g(t), \tag{35}$$

where L is a linear operator, N is the nonlinear operator and $g(x)$ is a forcing term. According to variational iteration method [17–22] we can construct a correct functional as follows:

$$y_{n+1}(t) = y_n(t) + \int_0^t \lambda\,[L\,y_n(s) + N\,\widetilde{y}_n(s) - g(s)]\,ds, \tag{36}$$

where λ is a Lagrange multiplier which can be identified optimally by variational iteration method. The subscript n denote the n^{th} approximation, $\widetilde{y}_n$ is considered as a restricted variation i.e. $\delta\,\widetilde{y}_n = 0$. Equation Eq.29 is known as a correct functional. The solution of the linear problems can be solved in a single iteration step due to the exact identification of the Lagrange multiplier. For the sake of simplicity and to convey the idea of the technique we consider the following system of ordinary differential equations

$$y_i'(t) = f_i(y_i,\, t), \qquad i = 1,\, 2,\, \cdots,\, N, \tag{37}$$

subject to the initial conditions

$$y_i(0) = c_i, \qquad i = 1,\, 2,\, \cdots,\, N. \tag{38}$$

To solve this system by means of the variational iteration method it should be rewritten in the following form

$$y_i'(t) = f_i(y_i) + g_i(t), \qquad i = 1,\, 2,\, \cdots,\, N, \tag{39}$$

subject to the initial conditions Eq.31 and g_i are as defined in equation Eq.28. The correct functional for the system Eq.32 can be approximated as

$$y_i^{(k+1)}(t) = y_i^{(k)}(t) + \int_0^t \lambda_i \left[y_i'^{(k)}(s) - f_i\left(\widetilde{y}_1^{(k)}(s),\, \widetilde{y}_2^{(k)}(s), \cdots, \widetilde{y}_N^{(k)(s)}\right) - g_i(s) \right] ds,$$
$$i = 1,\, 2,\, \cdots,\, N, \tag{40}$$

where $\lambda_i = \pm 1,\ i = 1,\, 2, \cdots,\, N$ are Lagrange multipliers and $\widetilde{y}_1,\ \widetilde{y}_2,\ \cdots,\ \widetilde{y}_N$ denote the restricted variations. For $\lambda_i = -1,\ i = 1,\, 2, \cdots,\, N$ we have the following iterative scheme

$$y_i^{(k+1)}(t) = y_i^{(k)}(t) - \int_0^t \left[y_i'^{(k)}(s) - f_i\left(y_1^{(k)}(s),\, y_2^{(k)}(s), \cdots, y_N^{(k)}(s)\right) - g_i(s) \right] ds,$$
$$i = 1,\, 2,\, \cdots,\, N. \tag{41}$$

If we start with the initial approximation equation Eq.31 then successive approximations can be completely determined. Finally we approximate the solution $y_i(t) = \lim_{k\to\infty} y_i^{(k)}(t)$ by the N^{th} term $y_i^{(N)}(t)$ for $i = 1,\, 2,\, \cdots,\, N$. Application of this method to the system Eq.27 leads to the following system of integral equations with Lagrange multipliers $\lambda_i = -1$, $i = 0,\, 1,\, \cdots,\, N-2$

$$\left.\begin{array}{l} y_{2i}^{(k+1)}(t) = y_{2i}^{(k)}(t) - \int_0^t \left[\dot{y}_{2i}^{(k)}(s) - y_{2i+1}^{(k)}(s) \right] ds, \\ y_{2i+1}^{(k+1)}(t) = y_{2i+1}^{(k)}(t) - \\ \quad \int_0^t \left[\dot{y}_{2i+1}^{(k)}(s) - \sum_{j=0,\, j-even}^{N-2} A_{(2i)j}\, y_j^{(k)}(s) \right] ds, \end{array}\right\},\ i = 0,\, 1,\, \cdots,\, \frac{N-2}{2}. \tag{42}$$

By using $y_{2i}^{(0)}(t) = a_{2i}(0)$ and $y_{2i+1}^{(0)}(t) = \dot{a}_{2i}(0),\ i = 0,\, 1, \cdots,\, \frac{N-2}{2}$ we obtain the nonzero coefficients solutions $y_{2i}(t) = a_{2i}(t),\ \ 2i = 0,\, 2,\, 4,\, 6,$ and 8 at $t = 0.1, 0.15$ and 0.2 for both $\alpha = 0$ and $\alpha = \frac{1}{2}$, respectively, as reported in Table 2. Consequently, the expansion solutions $u(x,\, t)$, equation Eq.13, with the help of equations Eq.24 and Eq.25 at

Table 2: Nonzero coefficients $a_n(t)$ for one-dimensional wave equation by using the VIM

	$\alpha = 0$			$\alpha = \frac{1}{2}$		
$a_n(t)$	$t = 0.1$	$t = 0.15$	$t = 0.2$	$t = 0.1$	$t = 0.15$	$t = 0.2$
$a_0(t)$	0.763714	0.554871	0.291713	0.515036	0.374196	0.196726
$a_2(t)$	-0.404026	-0.293542	-0.154324	-0.555864	-0.403859	-0.212321
$a_4(t)$	0.0226461	0.0164533	0.00865012	0.0418901	0.0304349	0.0160005
$a_6(t)$	-0.000482573	-0.000350694	-0.000184338	-0.00107606	-0.000781958	-0.00041123
$a_8(t)$	$5.2914E-6$	$3.83653E-6$	$1.61779E-6$	0.0000134527	$9.93432E-6$	$5.35901E-6$

$t = 0.2$ and $N = 8$ for $\alpha = 0$ will be as follows:

$$u(x, 0.2) = 0.309017 - 0.381231\, x^2 + 0.0783711\, x^4 - 0.0064123\, x^6 + 0.000255653\, x^8. \tag{43}$$

Similar approximation solutions for $\alpha = \frac{1}{2}$ will be as follows:

$$u(x, 0.2) = 0.309017 - 0.381235\, x^2 + 0.0783886\, x^4 - 0.00644008\, x^6 + 0.000269436\, x^8. \tag{44}$$

In the limiting case as $N \to \infty$ those become exactly the same as the expansion of the exact solution at $t = 0.2$ whose expansion up to the eighth order is given by:

$$u_{exact}(x, 0.2) = 0.309017 - 0.381234\, x^2 + 0.0783882\, x^4 - 0.00644717\, x^6 + 0.000284067\, x^8. \tag{45}$$

4.3 Picard's method (denoted by PM)

In this section we are going to use Picard's method to solve iteratively the first order system of ordinary differential equations Eq.27 in the form

$$\left.\begin{aligned} & y_{2i}^{(k+1)}(t) = y_{2i}(0) - \int_0^t y_{2i+1}^{(k)}(s)\,ds, \text{ with } y_{2i}(0) = a_{2i}(0), \\ & y_{2i+1}^{(k+1)}(t) = y_{2i+1}(0) - \sum_{j=0,\,j-even}^{N-2} A_{(2i)j} \int_0^t y_j^{(k)}(s)\,ds, \\ & \qquad \text{with } y_{2i+1}(0) = \dot{a}_{2i}(0). \end{aligned}\right\}, i = 0, 1, \cdots, \frac{N-2}{2}. \tag{46}$$

This will converge uniformly to the exact solution as $k \to \infty$. Nonzero coefficients have been computed by this method starting with $y_{2i}(0) = a_{2i}(0)$, $2i = 0, 2, 4, 6,$ and 8 at $t = 0.1, 0.15$ and 0.2 for both $\alpha = 0$ and $\alpha = \frac{1}{2}$, respectively, and reported in Table 3.

Table 3: Nonzero coefficients $a_n(t)$ for one-dimensional wave equation by using PM

	$\alpha = 0$			$\alpha = \frac{1}{2}$		
$a_n(t)$	$t = 0.1$	$t = 0.15$	$t = 0.2$	$t = 0.1$	$t = 0.15$	$t = 0.2$
$a_0(t)$	0.763714	0.554871	0.291713	0.515036	0.374196	0.196726
$a_2(t)$	-0.404026	-0.293542	-0.154324	-0.555864	-0.403859	-0.212321
$a_4(t)$	0.0226461	0.0164533	0.00864979	0.0418901	0.0304349	0.0160005
$a_6(t)$	-0.000482573	-0.000350695	-0.000184545	-0.00107606	-0.000781958	-0.00041123
$a_8(t)$	$5.2914E-6$	$3.83758E-6$	$2.76596E-6$	0.0000134527	$9.93432E-6$	$5.35909E-6$

4.4 Runge-Kutta method (denoted by RKM)

The famous fourth order Runge-Kutta method has been applied to solve the system Eq.27. Partitioning the interval $[0,\, t]$ into n-subintervals through the points $t_i = i\,h$, $h = \frac{t}{n}$ and starting with

$$y_{2j,0} = a_{2j}(0), \quad \text{and} \quad y_{2j+1,0} = \dot{a}_{2j}(0), \quad j = 0,\, 1,\, \cdots,\, \frac{N-2}{2}. \tag{47}$$

then if the values $y_{0,i},\, y_{1,i},\, \cdots,\, y_{N-2,i}$ have been computed we can get $y_{0,i+1},\, y_{1,i+1},$ $\cdots,\, y_{N-2,i+1}$ by using the following algorithm:

$$\left.\begin{aligned} & K_{1,j} = f_j\left(t_i,\, y_{0,i},\, y_{1,i},\, \cdots,\, y_{N-1,i}\right), \qquad j = 0,\, 1,\, \cdots,\, N-1, \\ & K_{2,j} = f_j\left(t_i + \tfrac{h}{2},\, y_{0,i} + \tfrac{K_{1,0}}{2},\, y_{1,i} + \tfrac{K_{1,1}}{2},\, \cdots,\, y_{N-1,i} + \tfrac{K_{1,N-1}}{2}\right), \quad j = 0,\, 1,\, \cdots,\, N-1, \\ & K_{3,j} = f_j\left(t_i + \tfrac{h}{2},\, y_{0,i} + \tfrac{K_{1,0}}{2},\, y_{1,i} + \tfrac{K_{2,1}}{2},\, \cdots,\, y_{N-1,i} + \tfrac{K_{2,N-1}}{2}\right), \quad j = 0,\, 1,\, \cdots,\, N-1, \\ & K_{4,j} = f_j\left(t_i + h,\, y_{0,i} + K_{3,0},\, y_{1,i} + K_{3,1},\, \cdots,\, y_{N-1,i} + K_{3,N-1}\right), \quad j = 0,\, 1,\, \cdots,\, N-1, \end{aligned}\right\}, i = 0,\, 1,\, 2,\, \cdots, \tag{48}$$

and hence

$$y_{j,i+1} = y_{j,i} + \frac{h}{6}\left(K_{1,j} + 2\,K_{2,j} + 2\,K_{3,j} + K_{4,j}\right), \quad j = 0,\, 1,\, \cdots,\, N-1. \tag{49}$$

Here $i = 0,\, 1,\, 2,\, \cdots$. Note also that $K_{1,0},\; K_{1,1},\; \cdots,\; K_{1,N-1}$ must all be calculated before computing $K_{2,j}$. Applying this method with $h = 0.01$ we get results step by step and the values are reported in Table 4 for $\alpha = 0$ and $\alpha = \frac{1}{2}$, respectively.

Table 4: Nonzero coefficients $a_n(t)$ for one-dimensional wave equation by using RKM

	$\alpha = 0$			$\alpha = \frac{1}{2}$		
$a_n(t)$	$t = 0.1$	$t = 0.15$	$t = 0.2$	$t = 0.1$	$t = 0.15$	$t = 0.2$
$a_0(t)$	0.763714	0.554871	0.291713	0.515036	0.374196	0.196726
$a_2(t)$	−0.404026	−0.293542	−0.154324	−0.555864	−0.403859	−0.212321
$a_4(t)$	0.0226461	0.0164533	0.00865001	0.0418901	0.0304349	0.0160006
$a_6(t)$	−0.000482574	−0.000350699	−0.000184412	−0.00107606	−0.00078196	−0.000411226
$a_8(t)$	$5.29756E-6$	$3.86206E-6$	$2.02997E-6$	0.0000134583	$9.9395E-6$	$5.34673E-6$

4.5 The analytical approximation method (denoted by AAM)

System Eq.17, where $A = A^{(\alpha)}$ is a nonsingular square matrix, has the unique analytical approximation solution given by:

$$\underline{a}(t) = \cos(\sqrt{A}\, t)\ \underline{a}(0) + \sin(\sqrt{A}\, t)\ (\sqrt{A})^{-1}\ \underline{\dot{a}(0)}, \tag{50}$$

with $\underline{a}(0)$ and $\underline{\dot{a}}(0)$ are known from the initial conditions. This result can be easily proved by using the following theorem

Theorem 3:[15]
If $f(A)$ is defined and $A = Q\ Diag\,[\,b_1,\ \cdots,\ b_n\,]\ Q^{-1}$ then

$$f(A) = Q\ Diag\,[\,f(b_1),\ \cdots,\ f(b_n)\,]\ Q^{-1}. \tag{51}$$

Values of nonzero coefficients due to this analytical approximation solution are reported in Table 5. This serves as the exact solutions of system Eq.17 in its present form resulting by considering only nine terms in the ultraspherical series expansion solution Eq.13 with $N = 8$. Exact values of nonzero coefficients have been computed and reported in Table 6.

Table 5: Nonzero coefficients $a_n(t)$ for one-dimensional wave equation from the AAM

	$\alpha = 0$			$\alpha = \frac{1}{2}$		
$a_n(t)$	$t = 0.1$	$t = 0.15$	$t = 0.2$	$t = 0.1$	$t = 0.15$	$t = 0.2$
$a_0(t)$	0.763714	0.554871	0.291713	0.515036	0.374196	0.196726
$a_2(t)$	−0.404026	−0.293542	−0.154324	−0.555864	−0.403859	−0.212321
$a_4(t)$	0.0226461	0.0164533	0.00865001	0.0418901	0.0304349	0.0160005
$a_6(t)$	−0.000482573	−0.000350694	−0.000184406	−0.00107606	−0.000781958	−0.00041123
$a_8(t)$	$5.2914E-6$	$3.83691E-6$	$1.9948E-6$	0.0000134527	$9.93432E-6$	$5.35903E-6$

Comparison of the previous results with the exact values shows that the newly improved five methods discussed previously

Table 6: Exact values of nonzero coefficients $a_n(t)$ for one-dimensional wave equation

	$\alpha = 0$			$\alpha = \frac{1}{2}$		
$a_n(t)$	$t = 0.1$	$t = 0.15$	$t = 0.2$	$t = 0.1$	$t = 0.15$	$t = 0.2$
$a_0(t)$	0.763714	0.554871	0.291713	0.515036	0.374196	0.196726
$a_2(t)$	−0.404026	−0.293542	−0.154324	−0.555864	−0.403859	−0.212321
$a_4(t)$	0.0226461	0.0164533	0.00865003	0.0418901	0.0304349	0.0160006
$a_6(t)$	−0.000482737	−0.000350729	−0.000184389	−0.00107636	−0.000782023	−0.000411134
$a_8(t)$	$5.3865E-6$	$3.91352E-6$	$2.05746E-6$	0.0000137537	$9.99264E-6$	$5.25344E-6$

give a very good approximation solutions to the system of ordinary differential equations resulting due to the application of the ultraspherical tau method for the one-dimensional wave equation.
We end this section by reporting the solution values $u(x,\,t)$ of the problem using various methods for $x = \pm 0.8, \pm 0.4$ and 0.0 at $t = 0.2$. Tables 7 and 8 depict results corresponding to $N = 8$ for $\alpha = 0$ and $\alpha = \frac{1}{2}$, respectively. Results from the exact solution have also been reported for comparisons.
It is worthy to recall here that $\alpha = 0$ corresponds to using Chebyshev polynomials of the first kind and $\alpha = \frac{1}{2}$ corresponds to using Legendre polynomials in the spectral ultraspherical expansion solution.

Table 7: Solution values $u(x,\,t)$ for one-dimensional wave equation when $\alpha = 0$ and $N = 8$

method	$u(\pm 0.8,\,0.2)$	$u(\pm 0.4,\,0.2)$	$u(0.0,\,0.2)$
Exact	0.0954915	0.25	0.309017
EVM	0.0954916	0.25	0.309017
VIM	0.0954914	0.25	0.309016
PM	0.0954912	0.250001	0.309016
RKM	0.0954916	0.25	0.309017
AAM	0.0954916	0.25	0.309017

Table 8: Solution values $u(x,\,t)$ for one-dimensional wave equation when $\alpha = \frac{1}{2}$ and $N = 8$

method	$u(\pm 0.8,\,0.2)$	$u(\pm 0.4,\,0.2)$	$u(0.0,\,0.2)$
Exact	0.0954915	0.25	0.309017
EVM	0.0954916	0.25	0.309017
VIM	0.0954916	0.25	0.309017
PM	0.0954916	0.25	0.309017
RKM	0.0954916	0.25	0.309017
AAM	0.0954916	0.25	0.309017

5 Two-dimensional wave equation

Another illustrative example is the wave equation in two space variables in addition to the time variable. Consider the following hyperbolic partial differential equation

$$\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}, \qquad (-1 \le x,\ y \le 1), \quad t \ge 0. \tag{52}$$

under the boundary conditions

$$u(\pm 1,\ y,\, t) = c_1 \ \text{ and } \ u(x,\, \pm 1,\, t) = c_2, \qquad t \ge 0, \tag{53}$$

and the initial conditions

$$u(x,\, y,\, 0) = f(x,\, y) \ \text{ and } \ u_t(x,\, y,\, 0) = g(x,\, y), \quad (-1 \le\ x,\ y \le\ 1). \tag{54}$$

Taking $c_1 = c_2 = 0$ reduces the problem to the vibration problem of a membrane tightly stretched over a square frame bounded by $-1 \le x,\ y \le 1$ in the xy-plane. The problem is symmetric about both the axes. Utilizing the above spectral methods the solution of the problem is assumed to be expressed in the following double ultraspherical series expansion form

$$u(x,\, y,\, t) = \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} a_{mn}(t)\, C_m^{(\alpha)}(x)\, C_n^{(\alpha)}(y). \tag{55}$$

If the functions $f(x,\, y)$ and $g(x,\, y)$ are chosen so that the boundary conditions Eq.40 are satisfied and that they have a series expansion of the form

$$\begin{aligned} f(x,\, y) &= \textstyle\sum_{n=0}^{\infty} \sum_{m=0}^{\infty} f_{mn}\, C_m^{(\alpha)}(x)\, C_n^{(\alpha)}(y), \quad \text{and} \\ g(x,\, y) &= \textstyle\sum_{n=0}^{\infty} \sum_{m=0}^{\infty} g_{mn}\, C_m^{(\alpha)}(x)\, C_n^{(\alpha)}(y). \end{aligned} \tag{56}$$

which is uniformly convergent in $-1 \le x,\ y \le 1$, then it follows that the solution of Eq.39 in the form Eq.42 is uniformly convergent and is free of discontinuities. Substitution of equation Eq.42 into the differential equation Eq.39 gives

$$\ddot{a}_{mn}(t) = a_{mn}^{(2,\,0)}(t) + a_{mn}^{(0,\,2)}(t), \qquad m,\, n \ge 0, \tag{57}$$

where $\ddot{a}(t)$ denote the 2^{nd} derivative of $a_{mn}(t)$ with respect to t, $a_{mn}^{(2,0)}$ and $a_{mn}^{(0,2)}$ stand for the 2^{nd} derivatives of $a_{mn}(t)$ with respect to the space variables x and y, respectively. Direct substitution from equations Eq.7 and Eq.8 into equation Eq.44 leads to

$$\ddot{a}_{mn}(t) = \sum_{k=2}^{\infty} \xi_{mk}\, a_{kn}(t) + \sum_{l=2}^{\infty} a_{ml}(t)\, \eta_{ln}, \qquad m,\, n \geq 0, \tag{58}$$

with

$$\xi_{mk} = \begin{cases} k(k^2 - m^2), & k \geq m+2 \text{ and } (k-m) - even, \\ \\ 0, & otherwise, \end{cases} \tag{59}$$

and $\eta_{ij} = \xi_{ji}$. Following the same procedures as in [9], we assume that $a_{mn}(t)$ and $\ddot{a}_{mn}(t)$ are negligible for $m > M$ and $n > N$, respectively. Use of the boundary conditions Eq.40, with $c_1 = c_2 = 0$, enables us to express the coefficients $a_{(M-1)n}(t)$, $a_{Mn}(t)$, $a_{m(N-1)}(t)$ and $a_{mN}(t)$ as follows

$$\left.\begin{array}{ll} a_{(M-1)n}(t) = \sum_{m=0}^{M-2} \mu_m\, a_{mn}(t), & \mu_m = \frac{((-1)^m - 1)}{2}, \\ \\ a_{Mn}(t) = \sum_{m=0}^{M-2} \nu_m\, a_{mn}(t), & \nu_m = \frac{-(1+(-1)^m)}{2}, \end{array}\right\}, n = 0,\, 1,\, \cdots,\, N, \tag{60}$$

and

$$\left.\begin{array}{ll} a_{m(N-1)}(t) = \sum_{n=0}^{N-2} \mu_n\, a_{mn}(t), & \mu_n = \frac{((-1)^n - 1)}{2}, \\ \\ a_{mN}(t) = \sum_{n=0}^{N-2} \nu_n\, a_{mn}(t), & \nu_n = \frac{-(1+(-1)^n)}{2}. \end{array}\right\}, m = 0, 1,\, \cdots,\, M. \tag{61}$$

Making use of equations Eq.46 and Eq.47 to eliminate $a_{(M-1)n}(t)$, $a_{Mn}(t)$, $a_{m(N-1)}(t)$ and $a_{mN}(t)$ from the finite system Eq.45 yields

$$\ddot{A}(t) = H\, A(t) \;+\; A(t)\, T, \tag{62}$$

where

$$\begin{aligned} H &= [\xi_{mk} + \mu_k\, \xi_{m(N-1)} + \nu_k\, \xi_{mN}; \quad m,\, k = 0,\, 1,\, \cdots,\, M-2], \\ T &= [\eta_{ln} + \mu_l\, \eta_{(M-1)n} + \nu_l\, \eta_{Mn}; \quad l,\, n = 0,\, 1,\, 2,\, \cdots,\, N-2], \end{aligned} \tag{63}$$

and the matrix $A(t) = A^{(\alpha)}(t)$ of unknown coefficients is

$$A(t) = [a_{mn}(t); \quad m = 0,\, 1,\, \cdots,\, M-2, \;\; n = 0,\, 1,\, \cdots,\, N-2]. \tag{64}$$

This approach of truncating the exact infinite ultraspherical expansion for $u(x,\, y,\, t)$ by dropping the equations for the highest modes from equation Eq.45 and determining them directly from the boundary conditions is amount to Lanczos tau-method. For solving system Eq.48 under the initial conditions

$$\left.\begin{array}{l} A(0) = [a_{mn}(0); \quad m = 0,\, 1,\, \cdots, M-2, \;\; n = 0,\, 1,\, \cdots,\, N-2], \quad \text{and} \\ \\ \dot{A}(0) = [\dot{a}_{mn}(0); \quad m = 0,\, 1,\, \cdots,\, M-2, \;\; n = 0,\, 1,\, \cdots,\, N-2], \end{array}\right\}, \tag{65}$$

we refer to the kroneker matrix algebra (see Graham [23] and Loan [24]). Define the tensor product of two matrices H and T as

$$H \otimes T = [H_{mn}\, T; \quad m = 0,\, 1,\, \cdots,\, M-2, \;\; n = 0,\, 1,\, 2,\, \cdots,\, N-2], \tag{66}$$

and their tensor sum as

$$H \oplus T = H \otimes I_{N-1} + I_{M-1} \otimes T, \tag{67}$$

where I is the identity matrix. Introducing the so-called block vectors

$$\underline{a}(t) \equiv [\underline{a}_0(t),\, \underline{a}_1(t),\, \cdots,\, \underline{a}_{N-2}(t)]^T, \tag{68}$$

where

$$A(t) \equiv [\underline{a}_0(t)\; \underline{a}_1(t)\; \cdots,\, \underline{a}_{N-2}(t)]^T, \tag{69}$$

and

$$vec\,A(t) = \begin{pmatrix} \underline{a}_0(t) \\ \underline{a}_1(t) \\ \cdots \\ \underline{a}_{N-2}(t) \end{pmatrix}, \tag{70}$$

we can reduce system Eq.48 into the following matrix differential equation

$$vec\,\ddot{A}(t) = G\;vec\,A(t), \quad \text{with} \quad G = H \oplus T^T. \tag{71}$$

Again this is a linear nonhomogeneous 2^{nd} order system of ordinary differential equations with constant coefficients, similar to the system Eq.17, and must be solved by using each of the methods described for the one-dimensional case under the following initial conditions

$$\begin{aligned} A(0) &= [a_{mn}(0) = f_{mn}; \quad m = 0,\,1,\,\cdots,\,M-2,\; n = 0,\,1,\,\cdots,\,N-2], \quad \text{and} \\ \dot{A}(0) &= [\dot{a}_{mn}(0) = g_{mn}; \quad m = 0,\,1,\,\cdots,\,M-2,\; n = 0,\,1,\,\cdots,\,N-2]. \end{aligned} \tag{72}$$

where f_{mn} and g_{mn} are constants given in equation Eq.43.

5.1 Numerical results and comparisons

Consider problem Eq.39 with $c_1 = c_2 = 0$, $f(x,\,y) = \sin(\frac{\pi(x+1)}{2})\sin(\frac{\pi(y+1)}{2})$ and $g(x,\,y) = 0$, $-1 \le x,\,y \le 1$. Taking $M = N = 8$, then the matrices $A(0) = A^{(\alpha)}(0)$, $H = H^{(\alpha)}$ and $T = T^{(\alpha)}$ of system Eq.48 for $\alpha = 0$ and $\alpha = \frac{1}{2}$ are given by

$$A^{(0)}(0) = \begin{pmatrix} 0.891141 & -0.471438 & 0.0264246 & -0.000563282 \\ -0.471438 & 0.249404 & -0.0139793 & 0.000297992 \\ 0.0264246 & -0.0139793 & 0.000783557 & -0.0000167027 \\ -0.000563282 & 0.000297992 & -0.0000167027 & 3.56045E-7 \end{pmatrix}, \tag{73}$$

$$H^{(0)} = \begin{pmatrix} -256 & -504 & -448 & -296 \\ -240 & -480 & -432 & -288 \\ -192 & -384 & -384 & -264 \\ -112 & -224 & -224 & -224 \end{pmatrix}, \tag{74}$$

$$T^{(0)} = \begin{pmatrix} -256 & -240 & -192 & -112 \\ -504 & -480 & -384 & -224 \\ -448 & -432 & -384 & -224 \\ -296 & -288 & -264 & -224 \end{pmatrix}, \tag{75}$$

and

$$A^{(\frac{1}{2})}(0) = \begin{pmatrix} 0.405285 & -0.437412 & 0.0329635 & -0.000846995 \\ -0.437412 & 0.472086 & -0.0355766 & 0.000914137 \\ 0.0329635 & -0.0355766 & 0.00268106 & -0.0000688896 \\ -0.000846995 & 0.000914137 & -0.0000688896 & 1.77011E-6 \end{pmatrix}, \tag{76}$$

$$H^{(\frac{1}{2})} = \begin{pmatrix} -36 & -33 & -26 & -15 \\ -165 & -165 & -130 & -75 \\ -234 & -234 & -234 & -135 \\ -195 & -195 & -195 & -195 \end{pmatrix}, \tag{77}$$

$$T^{(\frac{1}{2})} = \begin{pmatrix} -36 & -165 & -234 & -195 \\ -33 & -165 & -234 & -195 \\ -26 & -130 & -234 & -195 \\ -15 & -75 & -135 & -195 \end{pmatrix}. \tag{78}$$

System Eq.52 can now be solved for $A(t)$ by using any of the methods discussed previously. For need of space we will not write down here the matrices of nonzero coefficients values $a_{mn}(t)$ of the matrix G due to different methods. Instead we are going to tabulate the solution values $u(x,\,y,\,t)$ of the two-dimensional wave equation due to various methods directly

Table 9: Solution values for two-dimensional wave equation at $t = 0.2$ by using the EVM

		$\alpha = 0$			$\alpha = \frac{1}{2}$		
x	y	$u_{num.}$	u_{exact}	Error	$u_{num.}$	u_{exact}	error
± 0.8	± 0.8	0.0862208	0.0862209	$6.39949E-8$	0.0862209	0.0862209	$2.09725E-8$
± 0.8	± 0.4	0.225729	0.225729	$2.73095E-8$	0.225729	0.225729	$1.45424E-8$
± 0.8	0.0	0.279016	0.279017	$1.67445E-7$	0.279017	0.279017	$8.14518E-8$
± 0.4	± 0.8	0.225729	0.225729	$2.73095E-8$	0.225729	0.225729	$1.45424E-8$
± 0.4	± 0.4	0.590967	0.590967	$2.95633E-7$	0.590967	0.590967	$2.19893E-7$
± 0.4	0.0	0.730475	0.730475	$1.54201E-8$	0.730475	0.730475	$1.14981E-8$
0.0	± 0.8	0.279016	0.279017	$1.67445E-7$	0.279017	0.279017	$8.14518E-8$
0.0	± 0.4	0.730475	0.730475	$1.54201E-8$	0.730475	0.730475	$1.14981E-8$
0.0	0.0	0.902916	0.902917	$4.13567E-7$	0.902916	0.902917	$3.07541E-7$

Table 10: Solution values for two-dimensional wave equation at $t = 0.2$ by using the VIM

		$\alpha = 0$			$\alpha = \frac{1}{2}$		
x	y	$u_{num.}$	u_{exact}	Error	$u_{num.}$	u_{exact}	error
± 0.8	± 0.8	0.0862208	0.0862209	$9.30337E-8$	0.0862209	0.0862209	$2.09725E-8$
± 0.8	± 0.4	0.225729	0.225729	$3.61571E-8$	0.225729	0.225729	$1.45425E-8$
± 0.8	0.0	0.279016	0.279017	$2.43613E-7$	0.279017	0.279017	$8.1452E-8$
± 0.4	± 0.8	0.225729	0.225729	$3.61571E-8$	0.225729	0.225729	$1.45425E-8$
± 0.4	± 0.4	0.590967	0.590967	$4.48333E-7$	0.590967	0.590967	$2.19893E-7$
± 0.4	0.0	0.730475	0.730475	$3.33968E-8$	0.730475	0.730475	$1.14982E-8$
0.0	± 0.8	0.279016	0.279017	$2.43613E-7$	0.279017	0.279017	$8.1452E-8$
0.0	± 0.4	0.730475	0.730475	$3.33968E-8$	0.730475	0.730475	$1.14982E-8$
0.0	0.0	0.902916	0.902917	$6.02438E-7$	0.902916	0.902917	$3.07542E-7$

Table 11: Solution values for two-dimensional wave equation at $t = 0.2$ by using PM

		$\alpha = 0$			$\alpha = \frac{1}{2}$		
x	y	$u_{num.}$	u_{exact}	Error	$u_{num.}$	u_{exact}	error
± 0.8	± 0.8	0.0862208	0.0862209	$6.39949E-8$	0.0862209	0.0862209	$2.09725E-8$
± 0.8	± 0.4	0.225729	0.225729	$2.73096E-8$	0.225729	0.225729	$1.45425E-8$
± 0.8	0.0	0.279016	0.279017	$1.67445E-7$	0.279017	0.279017	$8.1452E-8$
± 0.4	± 0.8	0.225729	0.225729	$2.73096E-8$	0.225729	0.225729	$1.45425E-8$
± 0.4	± 0.4	0.590967	0.590967	$2.95633E-7$	0.590967	0.590967	$2.19893E-7$
± 0.4	0.0	0.730475	0.730475	$1.542E-8$	0.730475	0.730475	$1.14982E-8$
0.0	± 0.8	0.279016	0.279017	$1.67445E-7$	0.279017	0.279017	$8.1452E-8$
0.0	± 0.4	0.730475	0.730475	$1.542E-8$	0.730475	0.730475	$1.14982E-8$
0.0	0.0	0.902916	0.902917	$4.13567E-7$	0.902916	0.902917	$3.07542E-7$

Table 12: Solution values for two-dimensional wave equation at $t = 0.2$ by using RKM

		$\alpha = 0$			$\alpha = \frac{1}{2}$		
x	y	$u_{num.}$	u_{exact}	Error	$u_{num.}$	u_{exact}	error
± 0.8	± 0.8	0.0862208	0.0862209	$6.39514E-8$	0.0862209	0.0862209	$2.09374E-8$
± 0.8	± 0.4	0.225729	0.225729	$2.7214E-8$	0.225729	0.225729	$1.46362E-8$
± 0.8	0.0	0.279016	0.279017	$1.6731E-7$	0.279017	0.279017	$8.13389E-8$
± 0.4	± 0.8	0.225729	0.225729	$2.7214E-8$	0.225729	0.225729	$1.46362E-8$
± 0.4	± 0.4	0.590967	0.590967	$2.95835E-7$	0.590967	0.590967	$2.20143E-7$
± 0.4	0.0	0.730475	0.730475	$1.57164E-8$	0.730475	0.730475	$1.18003E-8$
0.0	± 0.8	0.279016	0.279017	$1.6731E-7$	0.279017	0.279017	$8.13389E-8$
0.0	± 0.4	0.730475	0.730475	$1.57164E-8$	0.730475	0.730475	$1.18003E-8$
0.0	0.0	0.902916	0.902917	$4.13143E-7$	0.902916	0.902917	$3.07177E-7$

Table 13: Solution values for two-dimensional wave equation at $t = 0.2$ by using the AAM

		$\alpha = 0$			$\alpha = \frac{1}{2}$		
x	y	$u_{num.}$	u_{exact}	Error	$u_{num.}$	u_{exact}	error
± 0.8	± 0.8	0.0862208	0.0862209	$6.39949E-8$	0.0862209	0.0862209	$2.09725E-8$
± 0.8	± 0.4	0.225729	0.225729	$2.73095E-8$	0.225729	0.225729	$1.45424E-8$
± 0.8	0.0	0.279016	0.279017	$1.67445E-7$	0.279017	0.279017	$8.14518E-8$
± 0.4	± 0.8	0.225729	0.225729	$2.73095E-8$	0.225729	0.225729	$1.45424E-8$
± 0.4	± 0.4	0.590967	0.590967	$2.95633E-7$	0.590967	0.590967	$2.19893E-7$
± 0.4	0.0	0.730475	0.730475	$1.54201E-8$	0.730475	0.730475	$1.14981E-8$
0.0	± 0.8	0.279016	0.279017	$1.67445E-7$	0.279017	0.279017	$8.14518E-8$
0.0	± 0.4	0.730475	0.730475	$1.54201E-8$	0.730475	0.730475	$1.14981E-8$
0.0	0.0	0.902916	0.902917	$4.13567E-7$	0.902916	0.902917	$3.07541E-7$

from equation Eq.42 for $x,\ y = \pm 0.8,\ \pm 0.4$ and 0.0 at $t = 0.2$ for $\alpha = 0$ and $\alpha = \frac{1}{2}$ in Tables 9 up to 13, respectively.
Results due to the analytical approximation method serves as the exact solution of the system Eq.52 in its present form resulting by considering only $M = N = 8$ in the solution expansion. It is to be noted here that the exact solution of the two-dimensional wave equation under consideration is given by

$$u(x,\ y,\ t) = \sin(\frac{\pi\,(x+1)}{2})\ \sin(\frac{\pi\,(y+1)}{2})\ \cos(\frac{\pi\,t}{\sqrt{2}}). \tag{79}$$

Comparison of results with the exact solution shows the efficiency and accuracy of these methods. It is clear from the Tables that the maximum absolute errors lie within a tolerance value of 10^{-7} in all cases. Thus the present numerical results obtained through out this work compare favorably with those of the analytical solutions. For more accurate results we should increase the approximation orders M and N in the ultraspherical expansion solutions. The abbreviations $u_{num.}$ and u_{exact} stand for numerical and exact values of the solutions. In spite that our programs can generate results for any $\alpha \in \left(-\frac{1}{2},\ \infty\right)$ we will not consider values for α other than 0 or $\frac{1}{2}$ only for need of space, otherwise the paper will become very lengthy.

6 Conclusion

In this work we have presented ultraspherical spectral methods based on Chebyshev and Legendre polynomial series expansions. Several newly improved efficient methods have been applied to solve the resulting systems of ordinary differential equations.

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