A TANDEM QUEUE WITH TWO MARKOVIAN INPUTS AND RETRIAL CUSTOMERS

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A tandem retrial queue consisting of two stations is studied. The first station has a single server. An input flow at the first station is described by the Markovian Arrival Process (MAP). If a customer from this flow meets the busy server, it goes to the orbit of infinite size and tries its luck later on exponentially distributed random time. The service time distribution at the first station is assumed to be general. After service at this station the customer proceeds to the multi-server second station. If this customer meets a free server at the second station, it starts service immediately; else the customer leaves the system forever. Besides the customers proceeding from the first station, an additional MAP flow of customers arrives at the second station directly, not entering the first station. A customer from this flow is lost if there is no available server at the second station. The service time by a second station server is exponentially distributed. We derive the stationary distribution of the system states at embedded epochs and at an arbitrary time, calculate the main performance measures. Numerical results are presented.

Keywords: tandem retrial queue, Markovian Arrival Process, multi-server second station, asymptotically quasi-Toeplitz Markov chain

1. Introduction

Queueing networks are widely used in capacity planning and performance evaluation of computer and communication systems, service centres, manufacturing lines and other systems where customers, jobs, packets, etc. are subjected to a successive processing. Some examples of their application to real systems can be found in [1]. Tandem queues are good mathematical models of telecommunication systems and networks. So, their investigation is important for applications.

The theory of tandem queues is well developed, see, e.g., [2–4]. However, most of papers are devoted to the exponential queueing models. Over the last two decades efforts of many investigators in tandem queues were directed to weakening the distribution assumptions on the service times and arrival pattern. In particular, the arrival process should be able to capture correlation and burstiness since real traffic in modern communication networks exhibits these features. One of the most suitable models of such an arrival process is the Markovian Arrival Process (MAP). The MAP has the advantage of being almost as computationally tractable as a stationary Poisson process, as well as it is well suited when correlation in the input flow cannot be ignored.

As far as we know the tandem queues where the effect of retrials is taken into account were investigated only in [5] and [6]. Note that retrial queues allow for the phenomenon that a customer who can not get service immediately upon arrival returns to the system after a random time. The MAP/PH/1 → ∞PH/1/K + 1 tandem retrial queue was considered in [5]. The system with a stationary Poisson arrival process, general service time distribution at both stations and the constant retrial rate was analysed in [6].

The model considered in the present paper is more general comparing to the ones in [5], [6]. We deal with a tandem retrial queue under the assumption that customers arrive according to the MAP at both stations of the tandem. The operation of a second station of the tandem is described by a multi-server queue. The service time distribution at the first station is general. We assume that both stations have no buffer space.

The rest of the paper is organized as follows. In section 2, the mathematical model is described. In section 3, the results concerning the stationary distribution of the embedded Markov chain at service completion epochs at the first station are presented. The stationary state distribution at an arbitrary time is calculated in section 4. The system performance measures are derived in Section 5. Section 6 contains numerical results illustrated the behaviour of the performance measures depending on system parameters. Finally, section 7 concludes the paper.
2. The Mathematical Model

We consider a tandem queue consisting of two stations in series. The first station has a single server and a general service time distribution function $B(t)$ with the finite first moment $b = \int_0^\infty t dB(t)$.

The input flow of customers entering the first station is described by the $MAP$. This process is coded as $MAP^{(1)}$ and is defined by means of the underlying process $\nu_t, t \geq 0$, which is an irreducible continuous time Markovian chain with the state space $\{0, \ldots, W\}$ where $W$ is some finite integer. Arrivals occur only at the epochs of the process $\nu_t, t \geq 0$, transitions. The intensities of transitions accompanied by an arrival of $k$ customers are combined into the matrices $D_k, k = 0, 1, \ldots$, of size $(W + 1) \times (W + 1)$.

The matrix generating function of these matrices is $D(1) = D_0 + D_1 z, |z| \leq 1$. The matrix $D(1)$ is the infinitesimal generator of the process $\nu_t, t \geq 0$.

The stationary distribution vector $\theta$ of this process satisfies the equations $\theta D(1) = \theta, \theta e = 1$. Here and in the sequel $\theta$ is a zero row vector and $e$ is a column vector consisting of 1's.

The average intensity $\lambda$ (mean rate) in the $MAP^{(1)}$ is defined by $\lambda = (\theta D(1) e)$. The coefficient of variation $c_{\text{var}}$ of intervals between customer arrivals is defined by $c_{\text{var}} = 2 \lambda (D_0)^{-1} e - 1$.

The coefficient of correlation $c_{\text{cor}}$ of the successive intervals between customer arrivals is given by $c_{\text{cor}} = (\lambda (D_0)^{-1} D_1 (D_0)^{-1} e - 1) / c_{\text{var}}^2$.

For more information about $MAP$ and related research see, e.g., [7].

If a customer meets a free first station server upon arrival, it automatically starts service. Else this customer goes to so called orbit and from the orbit tries its luck later on after a random amount of time. The times between retrials made by all customers are exponentially distributed with some parameter $\alpha_i$ when the number of customers in the orbit is equal to $i, i > 0, \alpha_0 = 0$. We do not fix the explicit dependence of the intensities $\alpha_i$ on $i$ assuming only that $\lim_{i \to \infty} \alpha_i = \infty$. Note that such dependence describes the classic retrial strategy ($\alpha_i = i\alpha, \alpha > 0$) and the linear strategy ($\alpha_i = i\alpha + \gamma, \alpha > 0, \gamma > 0$) as special cases.

After service at the first station a customer proceeds to the second station which is represented by $N$ independent identical servers. If this customer meets a free server at the second station, it starts service immediately; else the customer leaves the system forever. The service time by a second station server is exponentially distributed with the parameter $\mu > 0$.

Besides the customers proceeding from the first station an additional $MAP$ flow of customers arrives at the second station directly, not entering the first station. This $MAP$ is coded as $MAP^{(2)}$ and is defined by the underlying process $\eta_t, t \geq 0$, with the state space $\{0, 1, \ldots, \nu\}$ and the matrix generating function $A(z) = A_0 + A_1 z, |z| \leq 1$.

The mean rate of the $MAP^{(2)}$ is $h = \mathcal{G} A(1)$ where $\mathcal{G}$ is the unique solution to the system $\mathcal{G} A(1) = 0, \mathcal{G} e = 1$.

We assume that a customer from the $MAP^{(2)}$ is lost if there is no available server at the second station.

Let us introduce some notation:

- $I(O)$ is an identity (zero) matrix of appropriate dimension. When needed the dimension of the matrix will be identified with a suffix;
- $\otimes$ and $\oplus$ are symbols of the Kronecker product and sum of matrices, see, e.g., [8];
- $\nu = \nu + 1; \nu = \nu + 1$.
\[ P(n, t), n \geq 0, \] are coefficients of the matrix expansion \[ e^{D(t)\nu} = \sum_{n=0}^{\infty} P(n, t)z^n, |z| \leq 1. \] The \((v, v')\)th entry of the matrix \(P(n, t)\) defines the probability that \(n\) customers arrive in the \(MAP^{(1)}\) during the interval \((0, t]\) and the state of the underlying process \(V_n\) at the epoch \(t\) is \(v'\) given \(V_0 = v, v, v' = 0, W\).

- \(Q\) and \(\tilde{Q}\) are square matrices of dimension \((N + 1)\bar{W}\) and \((N + 1)\bar{F}\) respectively:

\[
Q = \begin{pmatrix}
0 & 1 & 0 & \ldots & 0 \\
0 & 0 & 1 & \ldots & 0 \\
0 & 0 & 0 & \ldots & 1 \\
0 & 0 & 0 & \ldots & 1
\end{pmatrix} \otimes I_{pF}, \quad \tilde{Q} = \begin{pmatrix}
0 & 1 & 0 & \ldots & 0 \\
0 & 0 & 1 & \ldots & 0 \\
0 & 0 & 0 & \ldots & 1 \\
0 & 0 & 0 & \ldots & 1
\end{pmatrix} \otimes I_{F}.
\]

3. The Stationary Distribution of the Embedded Markov Chain

Let \(t_n\) be the \(n\)-th service completion epoch at the first station, \(n \geq 1\).

Consider the process \(\xi_n = \{i_n, r_n, \eta_n, V_n\}, n \geq 1\), where \(i_n\) is the number of customers in the orbit at the epoch \(t_n - 0, i_n \geq 0\); \(r_n\) is the number of busy servers at the second station at the epoch \(t_n - 0, r_n = 0, N\); \(\eta_n\) and \(V_n\) are the states of the \(MAP^{(1)}\) and \(MAP^{(2)}\) arrival processes respectively at the epoch \(t_n, \eta_n = 0, V, v_n = 0, W\).

The process \(\xi_n, n \geq 1\), is a four-dimensional Markov chain with one countable and three finite state space components.

Enumerate the states of the chain \(\xi_n, n \geq 1\), in the lexicographic order and form the square matrices \(P_{i, l}, i, l \geq 0\), of size \((N + 1)\bar{W}\) of transition probabilities from the states having the value \(i\) of the first component to the states having the value \(l\) of this component.

**Lemma 1.** The non-zero transition probability matrices \(P_{i, l}\) are defined as follows:

\[
P_{i, l} = Q(C_{\alpha}\Omega_{l-i+1} + C_{\nu}(I_{(N+1)\bar{F}} \otimes D_{l})\Omega_{l-i}), l \geq \max\{i - 1, 0\}, i \geq 0,
\]

where \(C_i = \int_0^\infty e^{-\alpha t}e^{(C \otimes D_{\nu})t}dt = (\alpha I - C \otimes D_{\nu})^{-1}, i \geq 0,\) \(\Omega_n = \int_0^\infty e^{C t} \otimes P(n, t)dB(t), n \geq 0,\) \(\Omega_{-1} = O,\)

\[
C = \begin{pmatrix}
A_0 & A_1 & O & \ldots & O \\
\mu I_{F} & A_0 - \mu I_{F} & A_1 & \ldots & O \\
O & O & O & \ldots & A_0 - (N - 1)\mu I_{F} \\
O & O & O & \ldots & N\mu I_{F} - A(1) - N\mu I_{F}
\end{pmatrix}.
\]

**Proof.** Formula (1) becomes clear if we take into account the meaning of the matrices which occur in the right-hand side of (1). The matrix \(Q\) is used to install the number of busy servers at the second station just after the service completion at the first station. Given the number of busy servers at the second station has installed after the service completion at the first station, the process \(\{r_t, \eta_t\}\) which describes the evolution of the number of busy servers and the arrival process at the second station, behaves like Quasi-Birth-and-Death process until the next service completion epoch. The generator \(C\) of this process has form (2).

The \((r, \eta, v; r', \eta', v')\)th entry of the matrix \(C_{\eta}\) defines the probability that, given \(i\) customers stay in the orbit and the state of the process \(\{r_t, \eta_t, v_t\}\) is \((r, \eta, v)\) after the service completion epoch at the first station,
the next service at this station will be initiated by a customer from the orbit and the state of the process \( \{ r, \eta, v \} \) will be \( ( r', \eta', v') \) at the beginning of the service.

The matrix \( C(I_{i+1}) \bigotimes D_1 \) has the analogous probabilistic sense with the only difference that the next service at the first station is initiated by a customer arriving in the MAP\(^{(1)}\).

The \( (r, \eta, v; r', \eta', v') \) th entry of the matrix \( \Omega_n \) defines the probability that \( n \) customers arrive in the \( MAP\(^{(1)}\) \) and the process \( \{ r, \eta, v \} \) transits from the state \( ( r, \eta, v ) \) to the state \( ( r', \eta', v') \) during the service time at the first station.

Taking into account the above explanations and using the total probability formula we readily obtain expression (1) for transition probability matrices.

**Corollary 1.** The Markov chain \( \xi_n, n \geq 1 \), belongs to the class of asymptotically quasi-Toeplitz Markov chain.

**Proof.** It is seen from (1) that transition probability matrices \( P_{i,l} \) depend on \( i \) and \( l \) and this dependence can not be reduced to the dependence on the difference \( l - i \) only. It means that the Markov chain \( \xi_n, n \geq 1 \), is a level dependent one. At the same time, the dependence of \( i \) vanishes as \( i \to \infty \) and the matrices \( P_{i,l} \) approach to matrices that depend on the value \( i \) and \( l \) only via the difference \( l - i \). It implies that the chain under consideration belongs to the class of asymptotically quasi-Toeplitz Markov chains, see [9].

So, the further investigation of the process \( \xi_n, n \geq 1 \), will be based on the results given in [9].

Let us denote

\[
\tilde{Y}_k = \lim_{i \to \infty} P_{i+k-l}, k \geq 0.
\] (3)

The matrices \( \tilde{Y}_k, k \geq 0 \), can be considered as transition probability matrices of some Markov chain, say, \( \zeta_n, n \geq 1 \), with the same state space as the chain \( \xi_n, n \geq 1 \). The chain \( \zeta_n, n \geq 1 \), is called as limiting chain relative to the chain \( \xi_n, n \geq 1 \). The limiting chain \( \zeta_n, n \geq 1 \), is a level independent one, moreover it belongs to the class of quasi-Toeplitz Markov chains, see [9].

Denote by \( \tilde{Y}(z) \) the generating function of the matrices \( \tilde{Y}_k, k \geq 0 \).

**Corollary 2.** The generating function of the limiting chain \( \zeta_n, n \geq 1 \), transition probability matrices has the following form

\[
\tilde{Y}(z) = Q \int_0^\infty e^{Cr} \bigotimes e^{D(z)r} dB(t).
\] (4)

**Proof.** Taking into account Lemma 1 and using (3) we get expression (4) for generating function \( \tilde{Y}(z) \).

**Theorem 1.** The sufficient condition for ergodicity of the Markov chain \( \xi_n, n \geq 1 \), is the fulfilment of the inequality

\[
\rho = \lambda b_1 < 1.
\] (5)

**Proof.** The matrix \( \tilde{Y}(1) \) is an irreducible one. So, it follows from [9] that the sufficient condition for ergodicity of the chain \( \xi_n, n \geq 1 \), is the fulfilment of the inequality

\[
x \tilde{Y}'(1)e < 1,
\] (6)

where \( x \) is the unique solution to the system

\[
x \tilde{Y}(1) = x, \ x e = 1.
\] (7)

Let the vector \( x \) be of the form \( x = \delta \bigotimes \Theta \), where \( \delta \) is a solution to the system

\[
\delta Q \int_0^\infty e^{Cr} dB(t) = \delta, \ \delta e = 1.
\] (8)
Substituting this vector \( x \) and \( \tilde{Y}(1) \) given by (4) into (7) we get the following relations:

\[
x\tilde{Y}(1) = (\delta \otimes \Theta) \int_0^\infty e^{Ct} \otimes e^{D(t)} dB(t) = \int_0^\infty \delta Q e^{Ct} \otimes \Theta e^{D(t)} dB(t) = \delta Q e^{Ct} dB(t) \otimes \Theta = \delta \otimes \Theta = x.
\]

So, the vector \( x \) is a solution of (7).

The vector \( \delta \) is the unique solution to system (8) since the matrix \( \int_0^\infty e^{Ct} dB(t) \otimes \Theta \) is irreducible stochastic. This implies that the vector \( x = \delta \otimes \Theta \) is the unique solution to (7). Substituting this vector and \( \tilde{Y}(1) \), calculated by means (4), into inequality (6) we reduce this inequality to form (5) using obvious transformations.

The value \( \rho \) in (5) is the system load. Inequality (5) becomes intuitively clear if we take into account that the value \( \lambda b_1 \) is the mean number of customers entering the first station during the service of a customer by the first server.

In what follows we suppose that inequality (5) is fulfilled.

Denote the stationary state probabilities of the Markov chain \( \xi_n, n \geq 1 \), by \( \pi(i, r, \eta, \nu), i \geq 0, r = 0, N, \eta = 0, V, \nu = 0, W \).

Let also \( \pi_i \) be the row vector of probabilities \( \pi(i, r, \eta, \nu) \) listed in the lexicographic order of arguments \( (r, \eta, \nu), i \geq 0 \).

Denote by \( \Pi(z) = \sum_{i=0}^\infty \pi_i z^i, |z| \leq 1 \), the generating function of these vectors.

To compute the vectors \( \pi_i, i \geq 0 \), we use the numerically stable algorithm elaborated for asymptotically quasi-Toeplitz Markov chains, see [9]. It is based on censoring technique and asymptotic properties of the chain under consideration.

The algorithm consists of the following steps:

1. Calculate the matrix \( G \) as the minimal nonnegative solution of the matrix equation \( G = \tilde{Y}(G) \). The matrix \( G \) is calculated by the iterative method.

2. For pre-assigned sufficiently large integer \( i_0 \) calculate the matrices \( G_{i_0-1}, G_{i_0-2}, \ldots, G_0 \) using the equation of the backward recursion \( G_i = P_{i+1,i} + \sum_{n=i+1}^{\infty} P_{i,n} G_{n-1}G_{n-2} \ldots G_1, i = i_0 - 1, i_0 - 2, \ldots, 0 \), with the boundary condition \( G_i = G, i \geq i_0 \).

3. Calculate the matrices \( \bar{P}_{i,j} = P_{i,j} + \sum_{n=i+1}^{\infty} P_{i,n} G_{n-1}G_{n-2} \ldots G_1, i \geq 1, i \geq 0 \), where \( G_i = G, i \geq i_0 \).

4. Calculate the matrices \( F_i \) using the recurrent formulas \( F_i = (\bar{P}_{0,i} + \sum_{i=1}^{l-1} F_{i-l} \bar{P}_{i-l})(I - \bar{P}_{i,0})^{-1}, l \geq 1 \).

5. Calculate the vector \( \pi_0 \) as the unique solution to the system \( \pi_0(i - \bar{P}_{0,0}) = 0, \pi_0 \sum_{j=0}^\infty F_{i} e = 1 \).

6. Calculate the vectors \( \pi_i \) as follows: \( \pi_i = \pi_0 F_i, l \geq 1 \).

4. The Stationary Distribution at an Arbitrary Time

Define the process of the system states at an arbitrary time as \( \xi_t = (i_t, r_t, \eta_t, \nu_t), t \geq 0 \), where \( i_t \) is the number of customers at the first station (in the orbit and in service), \( r_t \) is the number of busy servers at the second station, \( \eta_t \) and \( \nu_t \) are the states of the MAP\(^{(1)}\) and MAP\(^{(2)}\) arrival processes respectively at time \( t, t \geq 0 \).
The process $\xi_t, t \geq 0$, is non-Markovian. But the stationary distribution of this process can be related to the stationary distribution of the embedded Markov chain $\xi_n, n \geq 1$, using the results for Markov renewal and semi-regenerative processes (see [10]).

Let 
\[
p(i, r, \eta, v) = \lim_{t \to \infty} P[\xi_t = i, r_t = r, \eta_t = \eta, v_t = v], i \geq 0, r = 0, N, \eta = 0, F, v = 0, W,
\]
be the steady-state probabilities of the process $\xi_t, t \geq 0$. Let also $p_i$ be the vector of probabilities $p(i, r, \eta, v)$ listed in the lexicographic order of components $(r, \eta, v), i \geq 0$.

**Theorem 2.** The steady-state probability vectors $p_i, i \geq 0$, of the process $\xi_t, t \geq 0$, are related to the stationary probability vectors $\pi_i, i \geq 0$, of the embedded Markov chain $\xi_n, n \geq 1$, as follows:

\[
p_0 = \tau^{-1} \pi_0 Q C_0, \tag{10}
\]
\[
p_i = \tau^{-1}\{p_i Q C_i + \sum_{j=0}^{i-1} \pi_j Q C_j \tilde{\Omega}_{i-j} + \sum_{j=0}^{i-1} \pi_j Q C_j (I_{(N+1)r} \otimes D_j) \tilde{\Omega}_{i-j-1}\}, i \geq 1, \tag{11}
\]

where $\tilde{\Omega}_n = \int_0^\infty e^{\Omega} \otimes P(n, r)(1 - B(t))dt, n \geq 0$, and $\tau$ is the mean value of inter-departure time at the first station, $\tau = b_1 + \sum_{i=0}^{N} \pi_i Q C_i e$.

**Proof.** The process $\xi_t, t \geq 0$, is a semi-regenerative one with the embedded Markov renewal process $\{\xi_n, t_n\}, n \geq 1$. By [10], limits (9) exist if the process $\{\xi_n, t_n\}$ is an irreducible aperiodic process and the value $\tau$ of the mean inter-departure time at the first station is finite. It is easily verified that ergodicity of the Markov chain $\xi_n, n \geq 1$, implies that all these conditions hold true. So, limits (9) exist if inequality (5) is satisfied.

Formulas (10), (11) for the steady-state probability vectors $p_i, i \geq 0$, are derived using the limiting theorem for semi-regenerative processes given in [10].

**Corollary 3.** The generating function $P(z) = \sum_{i=0}^{\infty} p_i z^i, |z| \leq 1$, of the stationary distribution of the process $\xi_t, t \geq 0$, is related to the generating function $\Pi(z)$ of the stationary distribution of the embedded Markov chain $\xi_n, n \geq 1$, as follows: $P(z)[C \oplus D(z)] = \tau^{-1} \Pi(z)[zI - Q]$.

5. **Performance Measures**

Having the stationary distributions $\pi_i, i \geq 0$, and $p_i, i \geq 0$, been calculated, we can find different stationary performance measures of the system under consideration:

- Mean number of customers at the first station at the service completion epoch at this station $L = \Pi'(1)e$.
- Mean number of customers at the first station at an arbitrary time $\tilde{L} = P'(1)e$.
- Mean number of busy servers at the second station at the service completion epoch at the first station $N_{bus} = \Pi(1)(I_{N+1} \otimes e_{pp}) \text{diag}\{r, r = 0, N\}e$.
- Mean number of busy servers at the second station at an arbitrary time $\tilde{N}_{bus} = P(1)(I_{N+1} \otimes e_{pp}) \text{diag}\{r, r = 0, N\}e$.
- Probability that an arbitrary customer from the MAP$^{(2)}$ will be lost $P_{\text{lost}}^{(2)} = \hat{h}^{-1} P(l)(\hat{e}_{N+1} \otimes A_P \otimes e_{pp})$, where $\hat{e}_{N+1}$ is a column vector of size $N + 1$ having 1 as the last entry and zeroes as the rest entries.
• Probability that an arbitrary customer from the MAP will be lost
  \[ P_{\text{loss}}^{(1)} = 1 - \lambda^{-1} \left[ \mu \tilde{N}_{\text{busy}} - h(1 - P_{\text{loss}}^{(2)}) \right] \]

• Probability of immediate access to the first station server
  \[ P_{\text{imm}} = (\lambda \tau)^{-1} \sum_{i=0}^{\infty} \pi_i QC_i (e_{(N+1)} \tau \otimes D_1 e) \]

• Probability that a customer from the MAP will be successfully served at the both stations without visiting the orbit
  \[ P_{\text{success}}^{(1)} = (\lambda \tau)^{-1} \sum_{i=0}^{\infty} \pi_i QC_i (I_{(N+1)} \tau \otimes D_1 e) \int_{0}^{\infty} e^{r} dB(t) \tilde{e}. \]

6. Numerical Examples

The aim of the numerical examples is to demonstrate an impact of correlation in the input flows arriving at both stations on the performance measures of the queue under consideration.

**Experiment 1.** In this experiment we analyse the influence of the correlation in the MAP entering the first station on the key performance measures. The effect of correlation is investigated for different values of system load \( \rho \).

For this purpose we consider three MAP processes defined by the matrices \( D_0 \) and \( D_1 \). All these MAPs have the same average intensity \( \lambda = 1 \) and different coefficients of correlation.

The MAP has the coefficient of correlation \( c_{\text{cor}} = 0.2 \) and is characterized by the matrices

\[
D_0 = \begin{pmatrix}
-1.349076 & 1.09082 \times 10^{-6} \\
1.09082 \times 10^{-6} & -0.043891
\end{pmatrix},
D_1 = \begin{pmatrix}
1.340137 & 0.008939 \\
0.0244854 & 0.0194046
\end{pmatrix}.
\]

The MAP has the coefficient of correlation \( c_{\text{cor}} = 0.1 \) and is defined by the matrices

\[
D_0 = \begin{pmatrix}
-1.17494 & 0.34832 \times 10^{-6} \\
0.34832 \times 10^{-6} & -0.025736
\end{pmatrix},
D_1 = \begin{pmatrix}
1.171346 & 0.0036006 \\
0.0200534 & 0.0056824
\end{pmatrix}.
\]

These processes have the same coefficient of variation \( c_{\text{var}} = 3.5 \).

The MAP is a Poisson process with \( D_0 = -1, D_1 = 1 \). It has the coefficient of correlation \( c_{\text{cor}} = 0 \) and the coefficient of variation \( c_{\text{var}} = 1 \).

The MAP arrived at the second station is identical to the MAP. It has the coefficient of correlation \( c_{\text{cor}} = 0.1 \) and the mean rate \( h = 1 \).

The service time distribution at the first station is assumed to be Erlangian of order 3 with the intensity 15. The mean service time \( b_1 = 0.2 \) and the squared coefficient of variation \( c_{\text{var}} = 1/3 \).

We consider the classical retrial strategy \( \alpha_i = i \alpha, \alpha = 5, i \geq 0 \). The number of servers at the second station \( N = 10 \), the mean service rate \( \mu = 0.5 \).

Let us vary the mean rate \( \lambda \) for all arrival processes in the interval \([0.5, 4.5]\) by multiplying the matrices \( D_0, D_1 \) by some positive constant. Any desired value of \( \lambda \) can be obtained while \( c_{\text{cor}} \) does not change. Note also that in this experiment the system load \( \rho \) takes values from 0.1 to 0.9.

Figures 1, 2 illustrate the dependence of the mean number of customers \( L \) and \( \tilde{L} \) at the first station, the mean number of busy servers \( N_{\text{busy}} \) and \( \tilde{N}_{\text{busy}} \) at the second station on the mean rate \( \lambda \).

Figures 3, 4 show the dependence of the loss probabilities \( P_{\text{loss}}^{(1)} \) and \( P_{\text{loss}}^{(2)} \), the probabilities \( P_{\text{success}}^{(1)} \) and \( P_{\text{imm}} \) on the mean rate \( \lambda \).
Figure 1. $L$ and $\bar{L}$ as functions of the mean rate in the arrival process $MAP^{(1)}$

Figure 2. $N_{busy}$ and $\bar{N}_{busy}$ as functions of the mean rate in the arrival process $MAP^{(1)}$

Figure 3. $P_{loss}^{(1)}$ and $P_{loss}^{(2)}$ as functions of the mean rate in the arrival process $MAP^{(1)}$

Figure 4. $P_{success}^{(1)}$ and $P_{success}^{(2)}$ as functions of the mean rate in the arrival process $MAP^{(1)}$
Basing on the figures one can conclude that the mean number of customers at the first station, the number of busy servers at the second station, the loss probability of customers from both arrival flows increase, while the probability of immediate access to the first station server and the probability of successful service at both stations without visiting the orbit decrease when the input intensity $\lambda$ (and the system load $\rho$) grows. It confirms the evident fact that increase of the arrival intensity makes worse the quality of service in the system.

More important conclusion is that, under the same value of the arrival rate $\lambda$, the increase of correlation in the arrival process essentially affects the value of the system performance measures, including measures relating to the second station of the system. Thus, assumption that the input flow is not correlated and can be approximated by means of stationary Poisson process with the same intensity while actually it is correlated can imply huge errors in prediction of system operation.

**Experiment 2.** In this experiment we are interesting in how the coefficient of correlation in the arrival process $MAP^{(2)}$ impacts on the system performance measures.

We assume that the arrival process at the first station is identical to $MAP^{(1)}_2$ with the average intensity $\lambda$ = 1.

Here we consider the $MAP^{(1)}_1$, $MAP^{(1)}_2$, and $MAP^{(1)}_3$ presented in the first experiment as the $MAP^{(2)}_1$, $MAP^{(2)}_2$, and $MAP^{(2)}_3$ input flows at the second station. These processes have the mean rate $\mu = 1$ and different coefficients of correlation: $c_{cor} = 0.2$, $c_{cor} = 0.1$, and $c_{cor} = 0$.

The rest system parameters are the same as in the first experiment.

Let us vary the mean rate $\mu$ for all arrival processes in the interval [0.5, 8]. The system load $\rho$ does not change in this experiment.

Figures 5, 6 show the dependence of the loss probabilities $P_{loss}^{(1)}$ and $P_{loss}^{(2)}$, the mean number of busy servers $\bar{N}_{busy}$ at the second station and the probability $P_{success}^{(1)}$ on the mean rate $\mu$ in the arrival process $MAP^{(2)}$.

![Figure 5. $P_{loss}^{(1)}$ and $P_{loss}^{(2)}$ as functions of the mean rate in the arrival process $MAP^{(2)}$](image1)

![Figure 6. $\bar{N}_{busy}$ and $P_{success}^{(1)}$ as functions of the mean rate in the arrival process $MAP^{(2)}$](image2)
It is clear from these figures that the loss probabilities $P^{(1)}_{\text{loss}}$ and $P^{(2)}_{\text{loss}}$, the mean number of busy servers $\bar{N}_{\text{busy}}$ at the second station increase, while the probability $P^{(1)}_{\text{success}}$ of successful service at both stations without visiting the orbit decreases when the intensity of the additional arrival process at the second station grows. Note that numerical calculations confirm that system characteristics relating to the first station, such as the mean number of customers at the first station and the probability of immediate access to the first station, do not depend on the intensity $h$.

As well as in the first experiment, based on the presented figures, we have to conclude that the system performance measures are sensitive with respect to the correlation in arrival process. One more interesting fact is that, under $h < 5$, the probabilistic measures $P^{(1)}_{\text{loss}}$ and $P^{(1)}_{\text{success}}$ become worse when the coefficient of correlation increases, but, when $h$ becomes greater than 5, the behaviour of these probabilities is reversed, i.e. the increase of the correlation has positive influence on $P^{(1)}_{\text{loss}}$ and $P^{(1)}_{\text{success}}$.

Conclusions

In this paper, the tandem queue with two Markovian inputs and retrial customers is investigated. The processes of system states at embedded epochs and an arbitrary time are studied. The condition for stationary distribution existence of these processes is derived and the algorithms for calculating the steady state probabilities are presented. Expressions for the probability of immediate access, the loss probabilities, the probability of successful service and other important performance characteristics of the system are obtained. The dependence of the system performance measures on the correlation in the input flows at both stations is numerically illustrated.

The results can be exploited for capacity planning, performance evaluations and optimisation of real-life tandem queues and two-node networks with the random multiple access to the first station as well as for validation of general networks decomposition algorithms in case of correlated bursty traffic.

References


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