On the Performance of Orthogonal Space-Time Block Codes over Independent, Nonidentical Rayleigh/Ricean Fading Channels

Jun He and Pooi Yuen Kam
Department of Electrical and Computer Engineering
National University of Singapore, Republic of Singapore, S117576
Email: {hejun,elekampy}@nus.edu.sg

Abstract—We consider the bit error performance of orthogonal space-time block codes (STBC) over independent, non-identically distributed, Rayleigh/Ricean, block fading channels with perfect channel state information. With symbol-by-symbol detection, we derive the expressions for the exact bit error probability (BEP) in both Rayleigh and Ricean fading channels. The results are applicable to any number of transmit and receive antennas for which orthogonal STBC’s are defined. A simple but insightful upper bound on the BEP is also obtained. From the expressions of the BEP, we find that in Rayleigh channels, the nonidentical distributions degrade the performance. However, for Ricean case, the nonidentical distributions can have different effects on the performance.

I. INTRODUCTION

The use of multiple transmit antennas with space-time codes [1] is a well known technique to improve the performance of wireless communication systems. In practice, space-time block codes (STBC) [2], [3] are commonly used due to their simple decoder structures. The performance of STBC has been studied in many works, e.g. [4], [5], etc.. Most of the previous works assume that the channels between the transmit and receive antennas are independent and identically distributed (i.i.d.). However, the assumption of identical channel statistics is not always true in real environments, especially in multiple-input multiple-output (MIMO) systems. Several factors may introduce the imbalance between channels. For example, in a MIMO system, the antenna spacing needs to be sufficiently large to reduce the correlation between channels. Therefore, the channels may involve very different propagation environments. In practice, directional antennas are commonly used at a base station. The different pointing directions of the antennas may also cause nonidentical channel gains. As a third example, one may consider the cooperative diversity scenario, where the antennas are not co-located and some distributed STBC [6] may be used. Then it is natural to expect that the channels are nonidentically distributed. Therefore, it is of great practical and theoretical interest to examine the effects of nonidentical channels upon the performance of STBC.

Some existing works have analyzed the performance of single-input multiple-output (SIMO) systems over independent, nonidentical, Rayleigh fading channels [7]–[9]. To the best of our knowledge, nonidentical channels have only been addressed in MIMO systems by [10] and [11] recently. Tao et al. considered the optimal detector and its error performance of differential STBC over independent and semi-identically distributed, block Rayleigh fading channels in [10], where the semi-identically distributed channels refer to the case that the channel gains associated with a common receive antenna are identically distributed, but the ones associated with a common transmit antenna are not. In [11], Li et al. examined the pairwise error probability of space-time trellis codes over independent, nonidentically distributed, fast Rayleigh fading channels. A new pilot power allocation scheme is also proposed based on the performance result.

In this paper, we analyze the bit error probability (BEP) of orthogonal STBC in nonidentically distributed, Rayleigh/Ricean channels. Both the upper bound and the exact BEP results are obtained with closed form. The results are applicable to any number of transmit and receive antennas for which orthogonal STBC’s are defined. With the analytical performance results, we examine the different effects of nonidentical channel parameters upon the performance of STBC. The results show that the nonidentical channel distributions degrade the performance in Rayleigh channels, which is similar to the observation in SIMO systems over Rayleigh fading channels [7]–[9]. But in Ricean channels, the nonidentical distributions can have different effects on the performance.

The rest of this paper is organized as below. Section II describes the system model and the symbol-by-symbol (SBS) receiver structure. In section III, we analyze the bit error performance and obtain the closed-form BEP results together with a simple upper bound for both Rayleigh and Ricean fading channels. Section IV examines the effects of different levels of the unbalanced channel parameters on the BEP of the orthogonal STBC. Section V provides numerical examples. A summary is given in section VI.

II. SYSTEM MODEL AND RECEIVER STRUCTURE

We consider a point to point communication system with $N_T$ transmit and $N_R$ receive antennas. The space-time block code, $S$, is a $P \times N_T$ matrix, where each row of $S$ is transmitted through $N_T$ transmit antennas at same symbol period, and the transmission of one block covers $P$ continuous symbol periods. It has a linear complex orthogonal design, and can be...
represents as [12]

\[ S = \sum_{l=1}^{L} (s_l A_l + s_l^* B_l) \]  

Here \( A_l, B_l \) are \( P \times N_T \) matrices with constant complex entries, and \( L \) is the number of symbols transmitted in one block. Thus, each entry of \( S \) is a linear combination of the data symbols \( s_l, l = 1, \cdots, L \) and their conjugates \( s_l^* \), where each \( s_l \) is from a certain complex signal constellation. We assume here \( M \)-ary phase-shift keying (MPSK) modulation with average transmitted energy per symbol \( E_s \). For orthogonal STBC, \( S \) satisfies

\[ S^H S = D \]

where \( D \) is a diagonal matrix [3]. For an arbitrary signal constellation, it requires that

\[ A_l^H A_k + B_k^H B_l = \delta_{lk} \text{diag} [\lambda_{1,l}, \cdots, \lambda_{N_T,l}] \]  

(3)

\[ A_l^H B_k + A_k^H B_l = 0 \]

(4)

where \( \{\lambda_{i,l}\}_{i=1}^{N_R} \) are positive numbers. The received signal, \( R \), is a \( P \times N_R \) matrix, which is given by

\[ R = SH + N \]

(5)

Here, \( N \) is a \( P \times N_R \) noise matrix, whose entries are i.i.d. circularly complex Gaussian random variables with mean zero and variance \( N_o/2 \) per dimension. \( H = [h_{ij}] \) is a \( N_T \times N_R \) channel matrix, where the entries \( h_{ij} \) are independent, circularly complex Gaussian random variables, each with deterministic mean \( m_{ij} \) and variance \( 2\sigma^2 \). Since the channels are nonidentical, each channel can have a different mean and variance. We assume the channel matrix \( H \) is perfectly known at the receiver, and the receiver uses the SBS detector. For those orthogonal STBC’s that satisfy condition (2), the SBS detector is equivalent to the maximum likelihood block-by-block detector and gives the best performance achievable.

III. BIT ERROR PERFORMANCE ANALYSIS

With MPSK modulation, we have \( s_t = \sqrt{E_s} e^{j\phi} \), and the detector makes its decision \( \hat{s}_t \) on \( s_t \) as [12]

\[ \hat{s}_t = \arg \max_{s_k} \text{Re} [z_k e^{-j\phi_k}] \]

(6)

Here, we have

\[ z_k = \text{Tr} [R^H B_k H + H^H A_k^H R] = x_k + u_k \]

(7)

where

\[ x_k = \sum_{l=1}^{L} \left[ s_l^* \text{Tr} [H^H A_l^H B_l H + H^H A_l H B_l^H] + s_l \text{Tr} [H^H A_l^H A_l H + H^H A_l^H B_l^H B_l H] \right] \]

(8)

\[ u_k = \text{Tr} [N^H B_k H + H^H A_k N] \]

(9)

By applying (3) and (4) to (8), we can see that the expression for \( x_k \) reduces to

\[ x_k = s_k \sum_{i=1}^{N_T} \sum_{j=1}^{N_R} \lambda_{i,k} |h_{ij}|^2 \]

(10)

Conditioned on the transmitted signal \( s_k \) and the channel matrix \( H, x_k \) can be seen from (10) to be a constant. Similarly, \( u_k \) can be seen from (9) to be a conditional, circularly complex Gaussian random variable with mean zero and variance \( N_0 \sum_{i=1}^{N_T} \sum_{j=1}^{N_R} \lambda_{i,k} |h_{ij}|^2 \). It then follows easily from (7) that \( z_k \) is a conditional, circularly complex Gaussian random variable with mean \( s_k \sum_{i=1}^{N_T} \sum_{j=1}^{N_R} \lambda_{i,k} |h_{ij}|^2 \) and variance \( N_0 \sum_{i=1}^{N_T} \sum_{j=1}^{N_R} \lambda_{i,k} |h_{ij}|^2 \).

For equally likely symbols, we can assume \( s_k = \sqrt{E_s} \), and the conditional BEP can be computed from the probability

\[ P(\text{Re}[z_k e^{-j\phi}] < 0 | s_k = \sqrt{E_s}, H) \]

(11)

Applying Craig’s alternative form of Q-function [14], we can rewrite conditional BEP as

\[ P(\text{Re}[z_k e^{-j\phi}] < 0 | s_k = \sqrt{E_s}, H) = Q \left( \frac{2E_s}{N_o} \cos^2 \alpha \sum_{i=1}^{N_T} \sum_{j=1}^{N_R} \lambda_{i,k} |h_{ij}|^2 \right) \]

(12)

where \( \gamma = E_s/N_o \) is the input signal-to-noise ratio (SNR) per symbol. To average the above conditional error probability, we take the expectation of the integrand in (12) over the entries of \( H = [h_{ij}] \), using the following lemma obtained from [15, eqn. 7.76].

Lemma: If \( x \) is a real Gaussian random variable with mean \( m_x \) and variance \( \sigma^2_x \), we have

\[ E[\exp(wx^2)] = \frac{\exp \left( \frac{wm_x^2}{1-2w\sigma_x^2} \right)}{\sqrt{1-2w\sigma_x^2}} \]

(13)

where \( w \) is any complex constant with real part less than \( 1/2\sigma^2 \).

Since \( |h_{ij}|^2 = |\text{Re}[h_{ij}]|^2 + |\text{Im}[h_{ij}]|^2 \), and \( \text{Re}[h_{ij}] \) is independent of \( \text{Im}[h_{ij}] \), we can apply the above lemma to (12). Defining \( \mu_i = \gamma \lambda_{i,k} \cos^2 \alpha \), we obtain

\[ P(\text{Re}[z_k e^{-j\phi}] < 0 | s_k = \sqrt{E_s}) = \frac{1}{\pi} \int_0^\pi \prod_{i=1}^{N_T} \prod_{j=1}^{N_R} \exp \left( \frac{-\mu_i |h_{ij}|^2}{\sin^2 \theta + 2\sigma_x^2 \mu_i / \sin^2 \theta} \right) d\theta \]

(14)

The above expression (14) for the exact BEP is explicit. However, its evaluation still involves numerical integration. The dependence of the BEP on the system parameters can shown more explicitly using a bound. A simple upper bound
on the BEP can be obtained by setting $\theta = \pi/2$ in equation (14), giving
\[
P(\text{Re}[z_{ke^{-j\omega}]} < 0|s_k = \sqrt{E_k}) < \frac{1}{2} \prod_{i=1}^{N_T} \prod_{j=1}^{N_R} e^{-\frac{\mu_i|m_{ij}|^2}{2 + 2\sigma_{ij}^2\mu_i}} (15)
\]
The product terms show that the total diversity order of the STBC is $N_T N_R$.

For the special case of Rayleigh fading, we have the means $m_{ij} = 0$ for all $i$ and $j$. The results (14) and (15) reduce, respectively, to
\[
P(\text{Re}[z_{ke^{-j\omega}]} < 0|s_k = \sqrt{E_k}) = \frac{1}{\pi} \int_0^{\frac{\pi}{2}} \prod_{i=1}^{N_T} \prod_{j=1}^{N_R} \left(1 + \frac{2\sigma_{ij}^2\mu_i}{\sin^2 \theta}\right)^{-1} d\theta \quad (16)
\]
and
\[
P(\text{Re}[z_{ke^{-j\omega}]} < 0|s_k = \sqrt{E_k}) < \frac{1}{2} \prod_{i=1}^{N_T} \prod_{j=1}^{N_R} (1 + 2\sigma_{ij}^2\mu_i)^{-1} (17)
\]
Equations (14) and (16) give the BEP for a single symbol $s_k$. As there are $L$ symbols in one block, the average BEP for one block is obtained from the average probability $\Gamma(\alpha)$, which is given by
\[
\Gamma(\alpha) = \frac{1}{L} \sum_{k=1}^{L} P(\text{Re}[z_{ke^{-j\omega}]} < 0|s_k = \sqrt{E_k}) (18)
\]
For BPSK, the BEP is given by $P_b^B = \Gamma(\alpha = 0)$, and for QPSK with gray coding, by $P_b^Q = \Gamma(\alpha = \frac{\pi}{4})$ [13].

IV. EFFECTS OF NONIDENTICAL CHANNEL PARAMETERS

For an orthogonal STBC, we have $\lambda_{ik} = \lambda$, a constant for all $i$, and each $\mu_i = \gamma \lambda \cos^2 \alpha$ is also a constant. We define the received SNR for each transmit-receive pair as $\gamma_{ij} = E[|h_{ij}|^2]$. We first consider Rayleigh channels, so the total received SNR, $\gamma_{Ray}$, is given by
\[
\gamma_{Ray} = \sum_{i=1}^{N_T} \sum_{j=1}^{N_R} \gamma_{ij} = \sum_{i=1}^{N_T} \sum_{j=1}^{N_R} 2\sigma_{ij}^2 \gamma (19)
\]
We can obtain a lower bound from (16) as
\[
P(\text{Re}[z_{ke^{-j\omega}]} < 0|s_k = \sqrt{E_k}) \geq \frac{1}{\pi} \int_0^{\frac{\pi}{2}} \left(1 + \frac{A_1}{\sin^2 \theta}\right)^{-N_T N_R} d\theta (20)
\]
where
\[
A_1 = \frac{\lambda \cos^2 \alpha}{N_T N_R} \sum_{i=1}^{N_T} \sum_{j=1}^{N_R} 2\sigma_{ij}^2 \gamma = \frac{\lambda \cos^2 \alpha}{N_T N_R} \gamma_{Ray} (21)
\]
The equality sign holds when all channel variances $2\sigma_{ij}^2$ are equal. Therefore, for a fixed total received SNR $\gamma_{Ray}$, the nonidentical channel distribution can be seen to degrade the bit error performance.

In Ricean channels, the total received SNR $\gamma_{Ric}$ consists of the direct line of sight (LOS) component and scattered component, and is given by
\[
\gamma_{Ric} = \sum_{i=1}^{N_T} \sum_{j=1}^{N_R} \gamma (2\sigma_{ij}^2 + |m_{ij}|^2) = \sum_{i=1}^{N_T} \sum_{j=1}^{N_R} 2\sigma_{ij}^2 (1 + K_{ij}) (22)
\]
where $K_{ij} = \frac{|m_{ij}|^2}{2\sigma_{ij}^2}$ is the Ricean $K$-factor of the channel from $i$-th transmit antenna to $j$-th receive antenna. There are three cases of interest.

A. Nonidentical channel variances, identical Ricean $K$-factors

First, we assume all channels have the same Ricean $K$-factors, i.e., $K_{ij} = K_o$ for all channels, where $K_o$ is a constant. However, the channel variances are nonidentical. Therefore, these channels can be seen as scaled versions of one another, in that the LOS component $|m_{ij}|^2$ has to bear a fixed relationship with the scattered component $2\sigma_{ij}^2$. We can obtain a lower bound from (14) as
\[
P(\text{Re}[z_{ke^{-j\omega}]} < 0|s_k = \sqrt{E_k}) \geq \frac{1}{\pi} \int_0^{\frac{\pi}{2}} \exp \left(-\frac{N_T N_R K_o A_1}{\sin^2 \theta + A_2}\right) d\theta (23)
\]
where, again, the equality sign holds when all channel variances $2\sigma_{ij}^2$ are equal. Therefore, it is similar to the Rayleigh channel case in that for a fixed total received SNR $\gamma_{Ric}$ and identical Ricean $K$-factors, the nonidentical channel distributions degrade the bit error performance. For the special case of $K_o = 0$, it reduces to the Rayleigh channel case (20). This result also shows that for STBC over identical channels, the transmit powers should be equally assigned to the transmit antennas, in order to obtain the best performance.

B. Nonidentical Ricean $K$-factors, identical Channel variances

In this case, we assume all channels have the same channel variances, i.e., $2\sigma_{ij}^2 = 2\sigma^2$ for all channels, where $2\sigma^2$ is a constant. However, the Ricean $K$-factors are different for each channel. Now, we have (14) reduces to
\[
P(\text{Re}[z_{ke^{-j\omega}]} < 0|s_k = \sqrt{E_k}) = \frac{1}{\pi} \int_0^{\frac{\pi}{2}} \exp \left(-\frac{A_2 \sum_{i=1}^{N_T} \sum_{j=1}^{N_R} K_{ij}}{\sin^2 \theta + A_2}\right) d\theta (24)
\]
where $A_2 = 2\sigma^2 \gamma \lambda \cos^2 \alpha$. From (24), it is obvious that the BEP only depends on $\sum_{i=1}^{N_T} \sum_{j=1}^{N_R} K_{ij}$ and not on each $K_{ij}$ individually. Therefore, for a fixed total received SNR $\gamma_{Ric}$ and identical channel variances, the nonidentical Ricean $K$-factors do not affect the bit error performance. For the special case of $\sigma^2 = 0$, it reduces to Gaussian channel case.
C. Nonidentical Channel variances, identical channel means

Now we assume all channels have the same channel means, i.e., \( m_{ij} = m \) for all channels, where \( m \) is a constant. However, the channel variances are different for each channel. In Rayleigh channels, the nonidentical channel variances always degrade the performance. However, it is not true in the Ricean case. If the channel means are identical and nonzero, the unbalanced channel variances can either degrade or enhance the performance. It may be hard to get this observation from the analytical BEP expression (14), so we will illustrate this result in details with numerical examples in the next section.

V. NUMERICAL RESULTS AND DISCUSSION

For the purpose of illustration, we consider a MIMO system with two transmit and one receive antenna. We use Alamouti’s code [2] with QPSK modulation. There, the code matrix is given by

\[
S = \begin{bmatrix} s_1 & s_2 \\ -s_2^* & s_1^* \end{bmatrix}, \quad A_1 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \quad A_2 = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}
\]

\[
B_1 = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}, \quad B_2 = \begin{bmatrix} 0 & 0 \\ -1 & 0 \end{bmatrix}
\]

(25)

For this case, it is easy to see that \( \lambda_{i,l} = 1 \), for all \( i \) and \( l \). We define here the following parameters:

\[
\eta = \frac{2\sigma_1^2}{2\sigma_{11}^2 + 2\sigma_{21}^2}, \quad \theta = \frac{|m_{11}|^2}{|m_{11}|^2 + |m_{21}|^2}
\]

\[
\zeta = \frac{|m_{11}|^2 + |m_{21}|^2}{2\sigma_1^2 + |m_{11}|^2 + 2\sigma_{21}^2 + |m_{21}|^2}
\]

(26)

(27)

Here \( \eta \) is the fraction of the scattered component received at the first channel, \( \theta \) is the fraction of the LOS component received at the first channel, and \( \zeta \) is the ratio of the total LOS components to the total received SNR.

We first consider Rayleigh channels with \( m_{ij} = 0 \) for all \( i \) and \( j \), and the total received SNR \( \gamma \sum_{i=1}^{N_T} \sum_{j=1}^{N_R} 2\sigma_{ij}^2 (1 + K) \) is set to \( 4\gamma \) for convenience. Fig. 1 plots the exact BEP (16) and BEP upper bound (17) with \( \eta = 50\% \), 15\% and 5\%, respectively. The results show that the upper bound (17) on the BEP is tight, and within 2 dB from the exact BEP (16). They also show that the nonidentical channel distributions degrade the performance of STBC in Rayleigh fading channels, e.g., for a BEP of \( 10^{-4} \), the unbalanced channel variances (\( \eta = 5\% \)) cause a lose in SNR of about 4 dB, compared with the identical channel case (\( \eta = 50\% \)). In Fig. 2, we consider Ricean channels with identical Ricean K-factors and nonidentical channel variances. The total received SNR \( \gamma \sum_{i=1}^{N_T} \sum_{j=1}^{N_R} 2\sigma_{ij}^2 (1 + K) \) is, again, set to \( 4\gamma \) for convenience, where \( \gamma = 15 \) dB. We plot the exact BEP (14) with Ricean K-factor = 3, 2, 1, 0.5 and 0, respectively. Fig. 2 shows that for all values of Ricean K-factor, if the total receive SNR is fixed, the best bit error performance is achieved when \( \eta = 0.5 \). In other words, the unbalanced channel variances, again, degrade the performance of STBC.

Fig. 3 considers Ricean channels with identical channel variances and nonidentical Ricean K-factors. With the same fixed total receive SNR given in the last example, we plot the exact BEP (14) with \( \zeta = 25\% \), 50\% and 75\%, respectively. We can see that for identical channel variances, the nonidentical channel means (or the nonidentical Ricean K-factors) do not affect the bit error performance. When \( \zeta \) increases from 25\% to 75\%, we can see from Fig. 3 that the bit error performance also increases. This is similar to the single channel case in that the increase of LOS component improves the quality of the channel, thus reducing the BEP.
In the last example, we consider Ricean channels with identical channel means and nonidentical channel variances. With the same fixed total received SNR, we compare the BEP for nonidentical channel variances ($\eta = 10\%$) and identical channel variances (50%) in Fig. 4. It shows that, when $\zeta$ is small, the bit error performance of the identical channel case is better. But when $\zeta$ increases, e.g. $\zeta > 0.25$, the bit error performance of the nonidentical channel case is better. Therefore, we can conclude that the bit error performance of the nonidentical channel case is not always worse than that of the identical channel case in Ricean channels. If the channel means are identical, the unbalanced channel variances can degrade or enhance the performance, depending on the ratio of the total LOS components to the total received SNR.

VI. Conclusion

We analyze the bit error performance for orthogonal STBC over independent and nonidentically distributed channels. The exact BEP and a simple upper bound on the BEP are obtained for BPSK and QPSK modulation. The results show that the nonidentical channel distribution degrades the bit error performance of STBC in Rayleigh channels. In Ricean channels, if the channel variances are identical, the unbalanced Ricean $K$-factors do not affect the bit error performance. If the Ricean $K$-factors are identical, the unbalanced channel variances can degrade the performance of STBC. However, if the channel means are identical, the unbalanced channel variances can either degrade or enhance the performance of STBC, depending on the ratio of the LOS component to the total received SNR.

References