Alternating Analysis and Synthesis Filters: A New Pseudo-QMF Bank

Fernando Cruz-Roldán, Pedro Amo-López, Pilar Martín-Martín, and Francisco López-Ferreras

Dep. de Teoría de la Señal y Comunicaciones, Escuela Politécnica, Alcalá University, 28871 Alcalá de Henares, Spain
E-mail: fernando.cruz@uah.es, pedro.amo@uah.es, p.martin@uah.es, francisco.lopez@uah.es

1. INTRODUCTION

Figure 1 shows an $M$-channel maximally decimated filter bank with a parallel structure. These systems have been thoroughly studied throughout the past two decades and there is plenty of literature that heavily describes their properties, design techniques, and applications (see, e.g., [1] for a list of references). The
system comprises an analysis stage and a synthesis stage. The analysis bank has \( M \) filters \( H_k(z) \) which split the input signal \( x[n] \) in \( M \) frequency bands, whose corresponding temporal signals \( x_k[n] \) are processed at an intermediate stage. At the receiving end, the \( M \) subband signals \( v_k[n] \) are interpolated and recombined using a set of synthesis filters \( F_k(z) \). In the filter bank, without an intermediate processing stage, the reconstructed signal can be obtained as

\[
\hat{X}(z) = \frac{1}{M} \cdot X(z) \cdot \sum_{k=0}^{M-1} F_k(z) \cdot H_k(z) + \frac{1}{M} \cdot \sum_{\ell=1}^{M-1} X(zW_\ell^M) \cdot \sum_{k=0}^{M-1} F_k(z) \cdot H_k(zW_\ell^M).
\]

This output signal \( \hat{X}(z) \) may be affected by the following errors: amplitude distortion, phase distortion, and aliasing. In perfect reconstruction (PR) filter banks none of these errors is originated and the output signal is equal to the input signal except for a delay and a constant scale factor. Nevertheless, the designing complexity of PR systems is substantial. Pseudo-QMF banks can be an alternative to perfect reconstruction systems, to avoid the highly nonlinear optimization of the prototype filter coefficients. In cosine-modulated filter banks, analysis and synthesis filters are cosine-modulated versions of lowpass prototype filters. Thus, the design of the whole filter bank thus comes down to the design of the prototype filters. These systems were proposed under the name of pseudo-QMF banks by Nussbaumer [2] and Rothweiler [3]. Later on, Chu developed various approaches to design pseudo-QMF cosine-modulated filter banks [4].

More recently, Nguyen proposed the near perfect reconstruction (NPR) pseudo-QMF banks [5]. This method yields to efficient systems, because amplitude distortion is very small and the aliasing error is comparable to the stopband attenuation of the prototype filter, but it is required optimization of a nonlinear cost function to design the prototype filter. In [6] we can find a design method that simplifies the design process of the prototype filter—it is reduced to the optimization of the cutoff frequency in Kaiser window designs, but amplitude distortion and aliasing errors of the filter bank are increased.

There are two methods in which the need to optimize the prototype coefficients is eliminated. First, in controlled in-band aliasing filter bank the analysis and synthesis filters are generated from a prototype filter with linear-phase Type I or Type II [7, 8]. Designing this filter bank is very easy, but amplitude distortion can be excessive in certain regions of the
spectrum. Secondly, under the spectral factorization approach to pseudo-QMF design [9, 10], the arbitrary-length prototype filter is obtained as a spectral factor of a $2M$th band filter. The main drawback of this method is amplitude distortion, which can be excessive in low and high frequencies, depending on the initial phase factor $\phi_0$ chosen.

This article presents a new pseudo-QMF bank known as alternating analysis and synthesis filters (AASF). This new system presents the following advantages:

1. Filter bank design is very simple: analysis and synthesis filters are obtained applying a modulation to a linear-phase Type I or Type II FIR prototype filter (with real coefficients) [11], for which the design stage is limited to the design of only a filter.
2. Amplitude distortion introduced is very low.
3. The filter bank does not present phase distortion.
4. Aliasing error is slightly superior to the stopband attenuation of the prototype filter designed.

The paper is organized as follows. In Section 2 we show the guidelines proposed to design the prototype filter and the cosine-modulation scheme which allows analysis and synthesis filters to be obtained. Section 3 describes the errors present in any pseudo-QMF bank and the designing conditions that must be imposed on cosine-modulation phase factors in the AASF bank to reduce such errors. Section 4 contains several examples that are useful in justifying the validity of the method.

The following notation is used in this paper: $\omega$ is used as the frequency variable, whereas the term “normalized frequency” is used to denote $f = \omega/\pi$. Boldfaced letters denote matrices (upper case) and vectors (lower case). The tilde mark is defined as $\tilde{H}(z) = H(z^{-1})$. In order to simplify expressions we will use the term $W_M = e^{-j2\pi/M}$. To obtain the relation between input and output signals of an $M$-channel filter bank, first we define the modulation matrices of the analysis bank $H(z)$ and those of the input signal $X(z)$ components, as well as the synthesis bank vector $f(z)$, as follows:

$$X(z) = \begin{bmatrix} X(z) \\ X(zW_M) \\ \vdots \\ X(zW_M^{M-1}) \end{bmatrix}$$

$$H(z) = \begin{bmatrix} H_0(z) & H_0(zW_M) & \cdots & H_0(zW_M^{M-1}) \\ H_1(z) & H_1(zW_M) & \cdots & H_1(zW_M^{M-1}) \\ \vdots & \vdots & \ddots & \vdots \\ H_{M-1}(z) & H_{M-1}(zW_M) & \cdots & H_{M-1}(zW_M^{M-1}) \end{bmatrix}$$

$$f(z) = [F_0(z) \ F_1(z) \ \cdots \ F_{M-1}(z)].$$

The term $H_k(z)$ denotes the $k$th column of the modulation matrix of the analysis bank $H(z)$. 
2. DESIGN OF THE ANALYSIS AND SYNTHESIS FILTERS

A. Prototype Filter Design

In the new AASF system, we do not need to optimize a highly nonlinear cost function to obtain the filter coefficients or to execute spectral factorization algorithms. The real-coefficients prototype filter must be designed satisfying these conditions:

- The filter $P(z)$ of arbitrary length must be FIR linear-phase Type I or Type II.
- The 3-dB cutoff frequency must be located in $\omega_c = \pi/2M$, where $M$ is the number of channels of the filter bank.
- $|P(e^{j\omega})| \approx 0$ for $|\omega| > \pi/M$.

B. Choice of Modulation Type

Let $P(z) = \sum_{n=0}^{N-1} p[n] \cdot z^{-n}$ be the linear-phase prototype filter. We define

$$S_k(z) = a_k \cdot P(zW((k+1/2)/2M)) + a_k^* \cdot P(zW^{-(k+1/2)}) \quad 0 \leq k \leq (M - 1),$$

(1)

where $a_k = e^{j\phi_k}$. From the function above, analysis filters $H_k(z)$ and synthesis filters $F_k(z)$, $0 \leq k \leq (M - 1)$, are obtained as

$$H_k(z) = \begin{cases} S_k(z) & k \text{ even} \\ z^{-(N-1)} \cdot \tilde{S}(z) & k \text{ odd} \end{cases}, \quad (2)$$

$$F_k(z) = z^{-(N-1)} \cdot \tilde{H}(z). \quad (3)$$

This choice ensures that the pseudo-QMF bank is free from phase distortion. The relations in the time domain which enable us to find the impulse response to the aforementioned filters are

$$h_k[n] = \begin{cases} s_k[n] & k \text{ even} \\ s_k[N-1-n] & k \text{ odd} \end{cases} \quad \begin{cases} 0 \leq k \leq (M - 1) \\ 0 \leq n \leq (N - 1) \end{cases}, \quad (4)$$

where

$$s_k[n] = 2 \cdot p[n] \cdot \cos \left( \frac{k + 1}{2} \frac{\pi}{M} n + \phi_k \right) \quad \begin{cases} 0 \leq n \leq N - 1 \\ 0 \leq k \leq M - 1 \end{cases}. \quad (5)$$

3. ERRORS INTRODUCED IN THE PSEUDO-QMF BANK

In this section, we will study each one of the errors present in pseudo-QMF banks and ways to reduce them in the AASF bank.

A. Amplitude Distortion

Let $T_0(z)$ be the overall distortion transfer function defined as

$$T_0(z) = \frac{1}{M} \cdot f(z) \cdot H(z). \quad (6)$$
There is amplitude distortion when the magnitude response for overall distortion transfer function \(|T_0(e^{j\omega})|\) is not a constant; i.e.,

\[
|T_0(e^{j\omega})| \neq c \quad c \in \mathbb{R}^+.
\]

In order to reduce amplitude distortion, it is advisable to obtain a closed expression to characterize \(T_0(z)\). As the AASF bank presents a cosine-modulation scheme similar to that of the pseudo-QMF bank with controlled in-band aliasing [10], the function \(T_0(z)\) is obtained similarly in both banks:

\[
T_0(z) = \frac{1}{M} \sum_{k=0}^{M-1} F_k(z) \cdot H_k(z) = \frac{z^{-(N-1)}}{M} \sum_{k=0}^{M-1} \tilde{H}_k(z) \cdot H_k(z)
\]

\[
= \frac{z^{-(N-1)}}{M} \sum_{k=0}^{M-1} \left( a_k^* \cdot \tilde{P}(z \cdot W_2^{(k+1/2)}) + a_k \cdot \tilde{P}(z \cdot W_2^{-(k+1/2)}) \right) \cdot \left( a_k \cdot P(z \cdot W_2^{(k+1/2)}) + a_k^* \cdot P(z \cdot W_2^{-(k+1/2)}) \right).
\]

If prototype filter \(P(z)\) is a linear-phase Type I or Type II filter,

\[
\tilde{P}(z) = z^{(N-1)} \cdot P(z)
\]

is satisfied or in an equivalent way

\[
\tilde{P}(z W_2^{(k+1/2)}) = (z W_2^{(k+1/2)})(N-1) \cdot P(z W_2^{(k+1/2)}).
\]

This relation (7) is very useful in obtaining a closed expression for \(T_0(z)\) as well as in reducing aliasing, as we will show next. It can be proved (see [7] for more details) that \(T_0(z)\) is characterized by

\[
T_0(z) \approx \frac{1}{M} \cdot (S_T(z) + S_0(z) + S_{M-1}(z)),
\]

where

\[
S_T(z) = \sum_{k=0}^{2M-1} P^2(z W_2^{(k+1/2)}) \cdot W_2^{(k+1/2)(N-1)}
\]

\[
S_0(z) = P(z W_2^{1/2}) \cdot P(z W_2^{-1/2}) \cdot 2 \cos\left(\frac{1}{2} \cdot \frac{\pi}{M} \cdot (N - 1) + 2\phi_0\right)
\]

\[
S_{M-1}(z) = P(z W_2^{(M-1/2)}) \cdot P(z W_2^{-(M-1/2)}) \cdot 2 \cos\left(\left( M - 1 + \frac{1}{2} \right) \cdot \frac{\pi}{M} \cdot (N - 1) + 2\phi_{M-1}\right).
\]

Terms \(S_0(z)\) and \(S_{M-1}(z)\) are cancelled when the initial phase factor \(\phi_0\) and final phase factor \(\phi_{M-1}\) respectively verify

\[
\phi_0 = \frac{\pi}{4} \cdot \left( \pm (2 \cdot r + 1) - \frac{1}{M} \cdot (N - 1) \right) \quad r \in \mathbb{Z}
\]

\[
\phi_{M-1} = \frac{\pi}{4} \cdot \left( \pm (2 \cdot r + 1) - \left( 2 - \frac{1}{M} \right) \cdot (N - 1) \right) \quad r \in \mathbb{Z}.
\]
If expressions (8) and (9) are simultaneously verified, the overall, distortion transfer function $T_0(z)$ is characterized by

$$T_0(z) \approx \frac{1}{M} \sum_{k=0}^{2M-1} P^2(z W_{2M}^{(k+1/2)}) \cdot W_{2M}^{(k+1/2)(N-1)}.$$  \hspace{1cm} (10)

In such cases, the maximum amplitude distortion present in the output signal is reduced.

**B. Phase Distortion**

Phase distortion occurs when the phase response of the overall distortion function $T_0(z)$ is not linear. If the prototype filter $p[n]$ is linear-phase or a spectral factor of another linear-phase $2M$th band filter [5, 9], Eq. (4), which enables us to obtain analysis and synthesis filters, guarantees the phase linearity of $T_0(z)$. Thus, in AASF systems there is no phase distortion at all.

**C. Aliasing**

Let $T_{\ell}(z)$ be the transfer functions due to aliasing components $X(z W_{M}^{\ell})$:

$$T_{\ell}(z) = \frac{1}{M} \sum_{k=0}^{M-1} F_k(z) \cdot H_k(z W_{M}^{\ell}) = \frac{1}{M} \cdot f(z) \cdot H^{\ell+1}(z), \hspace{1cm} \ell = 1, \ldots, M - 1. \hspace{1cm} (11)$$

We can affirm the presence of aliasing when for any value of $\ell$ it is satisfied that $T_{\ell}(z) \neq 0$. If in Eq. (11) we express analysis and synthesis filters in terms of the prototype $P(z)$ using expressions (1)–(3), and considering that $P(z)$ is a linear-phase Type I or Type II filter which satisfies Eq. (7), we have

$$M \cdot T_{\ell}(z) = \sum_{k=0}^{M-1} z^{-(N-1)}$$

$$\cdot \left( a_k^2 (z W_{2M}^{(k+1/2)})^{(N-1)} \cdot P(z W_{2M}^{(k+1/2)}) \cdot P(z W_{2M}^{(k+1/2)+2\ell}) \right) + a_k^2 (z W_{2M}^{(k+1/2)})^{(N-1)} \cdot P(z W_{2M}^{(k+1/2)+2\ell}) \cdot P(z W_{2M}^{(k+1/2)+2\ell})$$

$$\cdot \left( z W_{2M}^{-1/(k+1/2)} \right)^{(N-1)} \cdot P(z W_{2M}^{-1/(k+1/2)}) \cdot P(z W_{2M}^{-1/(k+1/2)+2\ell})$$

$$+ \sum_{k=0}^{M-1} \left( z W_{M}^{\ell} \right)^{-1} (N-1)$$

$$\cdot \left( a_k^2 (z W_{2M}^{(k+1/2)+2\ell)})^{(N-1)} \cdot P(z W_{2M}^{(k+1/2)+2\ell}) \cdot P(z W_{2M}^{(k+1/2)+2\ell}) \right) + a_k^2 (z W_{2M}^{(k+1/2)+2\ell)})^{(N-1)} \cdot P(z W_{2M}^{(k+1/2)+2\ell}) \cdot P(z W_{2M}^{(k+1/2)+2\ell})$$

$$\cdot \left( z W_{2M}^{-1/(k+1/2)+2\ell}) \right)^{(N-1)} \cdot P(z W_{2M}^{-1/(k+1/2)+2\ell}) \cdot P(z W_{2M}^{-1/(k+1/2)+2\ell})$$
prove that Cruz-Roldán et al. corresponding output signal is Second, we have to analyze which terms of signal

\[ P(z) = \sum_{k=0}^{M-1} t_k \cdot P(e^{j\omega} \cdot W_2^{(k+1)/2}) \cdot P(e^{j\omega} \cdot W_2^{-(k+1)/2}) \]

When prototype filter \( P(z) \) is designed as proposed in Section 2, it is easy to prove that

\[ P(e^{j\omega} \cdot W_2^{(k+1)/2}) \cdot P(e^{j\omega} \cdot W_2^{-(k+1)/2}) \approx 0 \]

\[ P(e^{j\omega} \cdot W_2^{-(k+1)/2}) \cdot P(e^{j\omega} \cdot W_2^{(k+1)/2}) \approx 0. \]

Therefore, the frequency response of the aliasing transfer functions \( T_\ell(z) \) are

\[ T_\ell(e^{j\omega}) \approx \frac{1}{M} \cdot \sum_{k=0}^{M-1} \left( t_k \cdot P(e^{j\omega} \cdot W_2^{(k+1)/2}) \cdot P(e^{j\omega} \cdot W_2^{-(k+1)/2}) \right. \]

\[ + \left. t_k^* \cdot P(e^{j\omega} \cdot W_2^{-1}) \right), \quad (12) \]

where

\[ t_k = \begin{cases} 
  a_k^2 \cdot W_2^{(k+1)/2}(N-1) & \text{k even} \\
  a_k^2 \cdot W_2^{-(k+1)/2}(N-1) & \text{k odd}
\end{cases} \]

and \( 1 \leq \ell \leq (M-1) \). This expression enables us to analyze total aliasing due to all subband components, but it is not easy to suppress such error. Nevertheless, if the aim is to cancel the most significant components of aliasing, it is better to analyze separately the contribution to total aliasing of each channel \( k \) or subband component. In [9, 10] aliasing is evaluated such that the prototype filter is a spectral factor of a valid 2Mth band filter; these works also indicate the conditions required so that, through the election of \( \phi_k \) angles, the most significant terms of aliasing are cancelled. This article describes a parallel study where the prototype filter is linear-phase Type I or Type II.

In order to find the condition which enables us to minimize aliasing, we must take several steps. First of all, analysis filters can be expressed in terms of the prototype \( P(z) \) as

\[ H_k(z) = \begin{cases} 
  a_k \cdot P(zW_2^{(k+1)/2}) + a_k^* \cdot P(zW_2^{-(k+1)/2}) & \text{k even} \\
  z^{-(N-1)} \cdot \left( a_k^* \cdot (z \cdot W_2^{(k+1)/2})^*(N-1) \cdot P(zW_2^{(k+1)/2}) \right. \]

\[ + \left. a_k \cdot (z \cdot W_2^{-(k+1)/2})^*(N-1) \cdot P(zW_2^{-(k+1)/2}) \right) & \text{k odd}
\end{cases} \]

Second, we have to analyze which terms of signal \( Y_k(z) \) are going to be the most significant. If the prototype filter is appropriately designed, there will only be significant aliasing due to the channels adjacent to that under consideration. Let us consider as an example a four-channel bank \( M = 4 \) and \( k = 2 \). The corresponding output signal is

\[ Y_2(z) = \frac{1}{4} \cdot F_2(z) \cdot H_2(z) \cdot X(z) + \frac{1}{4} \cdot F_2(z) \cdot \sum_{\ell=1}^{3} H_2(zW_4^\ell) \cdot X(zW_4^\ell). \]
The analysis filter of the second channel will be characterized by

\[ H_2(z) = a_2 \cdot P(zW_8^{5/2}) + a_2^* \cdot P(zW_8^{-5/2}). \]

Given that \( k \) is an even number, every component of aliasing can be expressed as

\[ Y_{2,al}(z) = \frac{1}{4} \cdot F_2(z) \cdot \left( H_2(zW_4) \cdot X(zW_4) + H_2(zW_4^2) \cdot X(zW_4^2) + H_2(zW_4^3) \cdot X(zW_4^3) \right). \]

In an equivalent way

\[ Y_{2,al}(z) = \frac{1}{4} \cdot F_2(z) \cdot \left( \left( a_2 \cdot P(zW_8^{-7/2}) + a_2^* \cdot P(zW_8^{-1/2}) \right) \cdot X(zW_4) + \left( a_2 \cdot P(zW_8^{-3/2}) + a_2^* \cdot P(zW_8^{3/2}) \right) \cdot X(zW_4^2) + \left( a_2 \cdot P(zW_8^{1/2}) + a_2^* \cdot P(zW_8^{7/2}) \right) \cdot X(zW_4^3) \right). \]

Figures 2a and 2b represent \( P(zW_8^{5/2}) \) and \( P(zW_8^{-5/2}) \)—corresponding to the analysis filters \( H_2(z) \)—along with all their shifted versions related to aliasing components \( X(zW_4^\ell) \), \( 1 \leq \ell \leq 3 \). If we consider the frequency response of the synthesis filter \( F_2(z) \) (Fig. 2c) it can be easily deduced that the most significant
FIG. 3. Cancellation of the most significant aliasing components for a four-channel pseudo-QMF bank.

Aliasing components are located in low- and high-frequency regions of pass bands and transition bands of product $H_2(z) \cdot F_2(z)$:

$$A_{2,\text{low}}(z) = F_2(z) \cdot (a_2 \cdot P(z W_8^{-3/2}) \cdot X(z W_4^{-2}) + a_2^* \cdot P(z W_8^{3/2}) \cdot X(z W_4^{2}))$$

$$A_{2,\text{high}}(z) = F_2(z) \cdot (a_2 \cdot P(z W_8^{-7/2}) \cdot X(z W_4^{-3}) + a_2^* \cdot P(z W_8^{7/2}) \cdot X(z W_4^{3})).$$

One way to cancel these significant aliasing terms consists of designing components $A_{k,\text{high}}(z)$ and $A_{k+1,\text{low}}(z)$ overlapped, so that they present the same absolute value but opposite signs, i.e.,

$$A_{k,\text{high}}(z) = -A_{k+1,\text{low}}(z), \quad 0 \leq k \leq 2.$$

Considering every filter of the four-channel bank in the preceding example, this strategy is represented in Fig. 3. Double aliasing component in high and low frequencies occurs in subbands with index $k = 1, 2$, because for $k = 0$ there is only aliasing due to the higher channel, while for $k = 3$ there is only one term due to the lower component.
In general, we can express the most significant components of aliasing in high and low frequency regions of channel $k$ as follows. For $k$ even,

$$A_{k, \text{low}}(z) = F_k(z) \cdot \left( a_k \cdot P \left( z W_{2M}^{-(k-1/2)} \right) \right) \cdot X \left( z W_{M}^{-k} \right) + a_k^* \cdot P \left( z W_{2M}^{-(k-1/2)} \right) \cdot X \left( z W_{M}^{+k} \right)$$

$$A_{k, \text{high}}(z) = F_k(z) \cdot \left( a_k \cdot P \left( z W_{2M}^{-(k+1/2)} \right) \right) \cdot X \left( z W_{M}^{-k} \right) + a_k^* \cdot P \left( z W_{2M}^{-(k+1/2)} \right) \cdot X \left( z W_{M}^{+k} \right).$$

For $k$ odd,

$$A_{k, \text{low}}(z) = F_k(z) \cdot \left( a_k^* \cdot W_{2M}^{(k+1/2)(N-1)} \cdot P \left( z W_{2M}^{-(k-1/2)} \right) \cdot X \left( z W_{M}^{-k} \right) + a_k \cdot W_{2M}^{-(k+1/2)(N-1)} \cdot P \left( z W_{2M}^{-(k-1/2)} \right) \cdot X \left( z W_{M}^{k} \right) \right)$$

$$A_{k, \text{high}}(z) = F_k(z) \cdot \left( a_k^* \cdot W_{2M}^{(k+1/2)(N-1)} \cdot P \left( z W_{2M}^{-(k+1/2)} \right) \cdot X \left( z W_{M}^{-k} \right) + a_k \cdot W_{2M}^{-(k+1/2)(N-1)} \cdot P \left( z W_{2M}^{-(k+1/2)} \right) \cdot X \left( z W_{M}^{k} \right) \right).$$

As already stated, the most significant terms of aliasing are canceled if we design the aliasing components $A_{k, \text{high}}(z)$ and $A_{k+1, \text{low}}(z)$ in such a way that

$$A_{k, \text{high}}(z) = -A_{k+1, \text{low}}(z), \quad 0 \leq k \leq M - 2.$$

If we take as an example even values of $k$, each of the stated terms can be expressed as

$$A_{k, \text{high}}(z) = F_k(z) \cdot \left( a_k \cdot P \left( z W_{2M}^{-(k+1/2)} \right) \right) \cdot X \left( z W_{M}^{-k} \right) + a_k^* \cdot P \left( z W_{2M}^{-(k+1/2)} \right) \cdot X \left( z W_{M}^{k} \right)$$

$$A_{k+1, \text{low}}(z) = F_{k+1}(z) \cdot \left( a_{k+1} \cdot W_{2M}^{(k+1/2)(N-1)} \cdot P \left( z W_{2M}^{-(k+1/2)} \right) \right) \cdot X \left( z W_{M}^{-k} \right) + a_{k+1}^* \cdot W_{2M}^{-(k+1/2)(N-1)} \cdot P \left( z W_{2M}^{-(k+1/2)} \right) \cdot X \left( z W_{M}^{k} \right).$$

Therefore, two conditions must be satisfied:

$$a_k \cdot F_k(z) \cdot P \left( z W_{2M}^{-(k+1/2)} \right) = -a_{k+1}^* \cdot F_{k+1}(z) \cdot W_{2M}^{(k+1/2)(N-1)}$$

$$a_k^* \cdot F_k(z) \cdot P \left( z W_{2M}^{(k+1/2)} \right) = -a_{k+1} \cdot F_{k+1}(z) \cdot W_{2M}^{-(k+1/2)(N-1)}$$

For $k$ even, synthesis filters can be expressed in terms of the prototype filter $P(z)$ as

$$F_k(z) = a_k^* \cdot W_{2M}^{(k+1/2)(N-1)} \cdot P \left( z W_{2M}^{(k+1/2)} \right) + a_k \cdot W_{2M}^{-(k+1/2)(N-1)} \cdot P \left( z W_{2M}^{-(k+1/2)} \right)$$

$$F_{k+1}(z) = a_{k+1} \cdot P \left( z W_{2M}^{(k+1/2)} \right) + a_{k+1}^* \cdot P \left( z W_{2M}^{-(k+1/2)} \right).$$

Substituting expressions (15) and (16) into condition (13), and considering that

$$P \left( e^{j\omega} \cdot W_{2M}^{(k+1/2)} \right) \cdot P \left( e^{j\omega} \cdot W_{2M}^{-(k+1/2)} \right) \approx 0,$$

$$P \left( e^{j\omega} \cdot W_{2M}^{-(k+1/2)} \right) \cdot P \left( e^{j\omega} \cdot W_{2M}^{(k+1/2)} \right) \approx 0,$$
we get

\[ a_k^2 W_{2M}^{-2k(N-1)} = -a_{k+1}^2 W_{2M}^{2k+1(N-1)}. \]  

(17)

Since \( a_k = e^{j\phi_k} \), operating in (17) we obtain

\[ \phi_{k+1} = \pm (2m + 1) \cdot \frac{\pi}{2} - (k + 1) \cdot (N - 1) \cdot \frac{\pi}{M} - \phi_k, \quad 0 \leq k \leq (M - 2), \]  

(18)

where \( m \) is any integer and \((N - 1)\) is the prototype filter order. This condition enables us to cancel the most significant components of aliasing. If we consider condition (14) or any of those which stem from imposing an odd value for \( k \), the result is the same.

We can conclude that, given a filter bank of arbitrary length characterized by Eq. (4) and any initial phase factor \( \phi_0 \), if the rest of the cosine-modulation phase terms \( \phi_k \) satisfy Eq. (18), the most significant terms of aliasing present in the filter bank output signal are canceled and, therefore, the aliasing error is reduced.

\section*{D. Special Case}

One possible choice which satisfies simultaneously conditions (8), (9), and (18) is that in which the factors \( \phi_k \) satisfy

\[ \phi_k = \frac{\pi}{4} \cdot \left( 1 - (2k + 1) \cdot (N - 1) \cdot \frac{1}{M} \right), \quad 0 \leq k \leq (M - 1), \]  

(19)

where \((N - 1)\) is the order of the prototype filter and \( M \) the number of channels. Relation (19) guarantees that if filter \( P(z) \) is correctly designed, maximum amplitude distortion is reduced and the most significant components of aliasing are canceled. Substituting Eq. (19) into Eq. (5), we get

\[ s_k[n] = 2 \cdot p[n] \cdot \cos \left( \left( k + \frac{1}{2} \right) \frac{\pi}{M} \left( n - \frac{N - 1}{2} \right) + \frac{\pi}{4} \right) \]  

\( 0 \leq n \leq N - 1 \)  

(20)

\[ 0 \leq k \leq M - 1. \]

The impulse responses of analysis and synthesis filters can be expressed as

\[ h_k[n] = \begin{cases} 2 \cdot p[n] \cdot \cos \left( \left( k + \frac{1}{2} \right) \frac{\pi}{M} \left( n - \frac{N - 1}{2} \right) + \frac{\pi}{4} \right) & k \text{ even} \\ 2 \cdot p[N - 1 - n] \cdot \cos \left( \left( k + \frac{1}{2} \right) \frac{\pi}{M} \left( \frac{N - 1}{2} - n \right) + \frac{\pi}{4} \right) & k \text{ odd} \end{cases} \]  

(20)

\[ f_k[n] = h_k[N - 1 - n]. \]  

(21)

Using the facts that \( p[n] = p[N - 1 - n] \) and \( \cos(-x) = \cos(x) \), we can rewrite expressions (20) and (21) as

\[ h_k[n] = 2 \cdot p[n] \cdot \cos \left( (2k + 1) \frac{\pi}{2M} \left( n - \frac{N - 1}{2} \right) + (-1)^k \cdot \frac{\pi}{4} \right) \]  

\[ f_k[n] = 2 \cdot p[n] \cdot \cos \left( (2k + 1) \frac{\pi}{2M} \left( n - \frac{N - 1}{2} \right) - (-1)^k \cdot \frac{\pi}{4} \right), \]

which is a particular case of conventional pseudo-QMF cosine-modulated filter banks [5, 9].
4. DESIGN EXAMPLES

We now demonstrate the efficiency of the previously described system by presenting three design examples.

**EXAMPLE 1.** In this example, a four-channel AASF bank is designed. Length-33 prototype filter has been designed using a Kaiser window ($\beta = 6.1$). A 3-dB cutoff frequency of prototype filter is situated at $\pi / 8$, and the magnitude response of the resultant filter is represented in Fig. 4a. The magnitude variation of function $T_0(e^{j\omega})$ does not surpass 0.0063, as shown in Fig. 4b. With respect to aliasing, function $T_a(e^{j\omega}) = \left(\sum_{\ell=1}^{M-1}|T_{\ell}(e^{j\omega})|^2\right)^{1/2}$ of the resultant bank is represented in Fig. 4c. Its maximum aliasing distortion is $E_a = -64.0513$ dB. Figure 4d shows the normalized energy representation in the output signals.

![FIG. 4. Example 1: (a) Magnitude response plot for $P(z)$; (b) magnitude response plot for the overall distortion transfer function $T_0(z)$; (c) magnitude response plot for the aliasing function $T_a(z)$; (d) normalized energy representation in the output signals $y_k[n]$ when the input signal $x[n]$ is a series of pure tones of different frequencies.](image-url)
EXAMPLE 2. We design a prototype filter of order 128 using a Kaiser window ($\beta = 10.07$) to obtain an eight-channel AASF bank. The cutoff frequency at 3 dB of the prototype filter is located in $\pi/16$. Magnitude responses of $P(e^{j\omega})$, $T_0(e^{j\omega})$, and $T_{al}(e^{j\omega})$ are plotted in Figs. 5a–5c, respectively. Figure 5d shows the normalized energy representation in the output signals $y_k[n]$ (0 ≤ $k$ ≤ 3) when the input signal $x[n]$ is a series of pure tones of different frequencies.

EXAMPLE 3. In this example, a 32-channel AASF bank is designed. Length-321 prototype filter was designed using a Blackman window. Table 1 includes results obtained for all examples. Phase factors $\phi_k$ have been chosen to
reduce maximum amplitude distortion and simultaneously to cancel the most significant aliasing components.

Table 2 shows results obtained for similar eight-channel pseudo-QMF banks, designed using the spectral factorization approach (SFA) [9, 10], the NPR [5], and the Kaiser window approach (KWA) [6]. The prototype filter for the SFA bank has been obtained by applying a spectral factorization to the corresponding valid filter, the last designed using the eigenfilter technique [12]. \( \phi_0 \) value of the cosine modulation has been chosen so that maximum amplitude distortion is reduced. In the case of the NPR bank, the prototype filter has been taken from [13]. The prototype filter for the KWA bank has been designed using the program available as indicated in [6]. The NPR bank produces the lowest figure for amplitude distortion, while the KWA bank has produced a lower level of aliasing.

Two different signals are applied as entries to each of the banks designed: an audio signal and an electrogram signal. The results concerning PRD, PSNR, and MaxError [14] can be also found in Tables 1 and 2. The best behavior was registered in the NPR bank. Nevertheless, the resultant figures in AASF and KWA banks are very close to those of NPR system and it should be noted that

### Table 1

<table>
<thead>
<tr>
<th>Input signal</th>
<th>Prototype filter order</th>
<th>( E_a ) (dB)</th>
<th>( R_{pp} )</th>
<th>PRD</th>
<th>PSNR</th>
<th>MaxError</th>
</tr>
</thead>
<tbody>
<tr>
<td>Audio</td>
<td>32</td>
<td>4</td>
<td>-64.0513</td>
<td>0.0063</td>
<td>0.1053</td>
<td>64.4219</td>
</tr>
<tr>
<td>Audio</td>
<td>128</td>
<td>8</td>
<td>-100.588</td>
<td>0.0034</td>
<td>0.0578</td>
<td>69.6406</td>
</tr>
<tr>
<td>Audio</td>
<td>320</td>
<td>32</td>
<td>-71.22</td>
<td>0.0164</td>
<td>1.5101</td>
<td>41.2907</td>
</tr>
<tr>
<td>Electrogram</td>
<td>32</td>
<td>4</td>
<td>-64.0513</td>
<td>0.0063</td>
<td>0.3987</td>
<td>63.6868</td>
</tr>
<tr>
<td>Electrogram</td>
<td>128</td>
<td>8</td>
<td>-100.588</td>
<td>0.0034</td>
<td>0.1242</td>
<td>73.9181</td>
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<tr>
<td>Electrogram</td>
<td>320</td>
<td>32</td>
<td>-71.22</td>
<td>0.0164</td>
<td>0.8819</td>
<td>57.1961</td>
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</table>

### Table 2

<table>
<thead>
<tr>
<th>Input signal</th>
<th>Pseudo-QMF bank</th>
<th>( E_a ) (dB)</th>
<th>( R_{pp} )</th>
<th>PRD</th>
<th>PSNR</th>
<th>MaxError</th>
</tr>
</thead>
<tbody>
<tr>
<td>Audio</td>
<td>NPR</td>
<td>-67.9258</td>
<td>2.1e-13</td>
<td>0.0389</td>
<td>73.0653</td>
<td>5.32e-4</td>
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<tr>
<td>Audio</td>
<td>SFA</td>
<td>-83.4071</td>
<td>3.2086</td>
<td>6.8244</td>
<td>28.1905</td>
<td>0.1141</td>
</tr>
<tr>
<td>Audio</td>
<td>KWA</td>
<td>-111.018</td>
<td>0.0043</td>
<td>0.0308</td>
<td>75.0901</td>
<td>0.0011</td>
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<tr>
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<td>NPR</td>
<td>-67.9258</td>
<td>2.1e-13</td>
<td>0.0242</td>
<td>87.8627</td>
<td>1.87e-4</td>
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<tr>
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<td>32.8707</td>
<td>0.1111</td>
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<tr>
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<td>0.0043</td>
<td>0.1885</td>
<td>70.2967</td>
<td>0.001</td>
</tr>
</tbody>
</table>

**Notes.** Prototype filter order is 128. NPR, near perfect reconstruction; SFA, spectral factorization approach; and KWA, Kaiser window approach.
### TABLE 3

Errors Introduced by Different Pseudo-QMF Banks and Design Characteristics

<table>
<thead>
<tr>
<th>Pseudo-QMF bank</th>
<th>Prototype filter design method</th>
<th>Spectral factorization</th>
<th>Phase distortion</th>
<th>Amplitude distortion</th>
<th>Aliasing</th>
<th>Comments</th>
</tr>
</thead>
<tbody>
<tr>
<td>SFA</td>
<td>Any technique to design spectral factor of linear-phase 2Mth band FIR filters</td>
<td>YES</td>
<td>NO</td>
<td>It can be high in the vicinity of $\omega = 0$ and $\omega = \pi$</td>
<td>Slightly higher than stopband attenuation of the prototype filter</td>
<td>$\phi_0$ must be carefully chosen. Sometimes, a spectral factorization algorithm must be applied to obtain the prototype filter.</td>
</tr>
<tr>
<td>NPR</td>
<td>Through optimization</td>
<td>NO</td>
<td>NO</td>
<td>Almost null</td>
<td>Similar to stopband attenuation of the prototype filter</td>
<td>Prototype filter design is very complex. Nevertheless, the bank is virtually a PR one.</td>
</tr>
<tr>
<td>KWA</td>
<td>Through optimization, using a Kaiser window</td>
<td>NO</td>
<td>NO</td>
<td>Very low</td>
<td>Slightly higher than stopband attenuation of the prototype filter</td>
<td>Prototype filter design is not complex and the errors introduced by the bank are trivial.</td>
</tr>
<tr>
<td>AASF</td>
<td>Any linear-phase FIR technique capable to control the position of $\omega_c$</td>
<td>NO</td>
<td>NO</td>
<td>Very low</td>
<td>Slightly higher than stopband attenuation of the prototype filter</td>
<td>It is necessary to locate the prototype filter 3-dB cutoff frequency exactly at $\omega_c = \pi/2M$.</td>
</tr>
</tbody>
</table>
the prototype filters’ design in the first two methods is much more simple than in the NPR systems.

Table 3 shows a comparative summary of these pseudo-QMF banks. The new AASF system does not introduce phase distortion, while amplitude distortion is similar and below its value with other methods which require the utilization of a highly nonlinear optimization algorithm or the calculation of a spectral factor to obtain prototype filter coefficients.

CONCLUDING REMARKS

We have presented a new pseudo-QMF bank whose main advantage is ease in designing the prototype filter $P(z)$. Coefficients of $P(z)$ are obtained using any technique to design linear-phase Type I or Type II FIR filters capable of locating a 3-dB cutoff frequency in $\pi/2M$. Closed expressions for overall distortion transfer functions $T_0(z)$ and the frequency response of the aliasing transfer functions $T_\ell(z)$ are obtained. It has also been shown that it is possible to cancel the most significant components of aliasing if phase factors $\phi_k$ of cosine modulation are properly chosen; furthermore, a solution for terms $\phi_k$ has been proposed which permits the simultaneous reduction of aliasing and amplitude distortion. Design examples show that very good characteristics can be obtained as well.

REFERENCES