TURING DEGREES AND THE WORD AND CONJUGACY PROBLEMS FOR FINITELY PRESENTED GROUPS

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1. *Introduction*

The unsolvability of the conjugacy problem for finitely presented groups was first shown by Novikov [**26**]. Shortly thereafter the corresponding result for the word problem was proved by Novikov [**27**] and Boone [**6**] (see also Boone [**7**], Higman [**20**] and Britton [**9**]). Friedberg [**19**] and Mucnik [**25**] revitalised the theory of recursively enumerable (r.e.) Turing degrees by showing that there are degrees which are incomparable. A trivial corollary of this is that there are degrees which lie strictly between $\mathbf{0}$ and $\mathbf{1}$. It was then natural to ask whether or not each such degree contains a word problem and a conjugacy problem (of finitely presented groups). Fridman [**17, 18**] gave an affirmative answer for the word problem. The same result was proved also by Clapham [**10**], Bokut' [**3**] and Boone [**8**]. The question for the conjugacy problem was settled, also in the affirmative, by Bokut' [**4**], Collins [**13**] and Miller [**22**] (for finitely generated recursively presented groups—see also Miller [**24**]).

Let G be a recursively presented group whose word problem has degree $\mathbf{a}$ and conjugacy problem has degree $\mathbf{b}$. Naturally $\mathbf{a}$, $\mathbf{b}$ are recursively enumerable. Moreover $\mathbf{a} \leq_T \mathbf{b}$ since a word of G is equal to the identity if and only if it is conjugate to it. The question now arises of whether or not the converse is true. The fact that we can have $\mathbf{a} \neq \mathbf{b}$ was shown by Fridman [**16**] who proved that there is a finitely presented group with solvable word problem but unsolvable conjugacy problem. Miller [**23**] showed that the full converse holds for finitely generated recursively presented groups (see also Miller [**24**]). Later on Collins [**15**] was able to extend his analysis from [**13**] to prove the following.

THEOREM 1.1 (Collins) *Let* $\mathbf{a}$, $\mathbf{b}$ *be arbitrary r.e. Turing degrees with* $\mathbf{a} \leq_T \mathbf{b}$. *Then there is a finitely presented group whose word problem has degree* $\mathbf{a}$ *and whose conjugacy problem has degree* $\mathbf{b}$.

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Many of the proofs in this area are fairly lengthy and a fair amount of effort has been directed towards their simplification. It was as part of this effort that Britton [**9**] produced his famous lemma on HNN extensions which soon became a standard tool for future work. A significant simplification was also produced by Aanderaa and Cohen [**1, 2**] who defined a new class of machines called *modular machines*. These are numerically encoded versions of Turing machines with the advantage that it is much easier to 'embedd' them faithfully into groups. Aanderaa and Cohen used modular machines to produce two remarkably easy examples of finitely presented groups with unsolvable word problem. They used one of these examples to prove a theorem of Collins [**14**] which states that for each r.e. unbounded truth-table degree there is a finitely presented group whose word problem has that degree (see [**21**] for a proof of the same result using the other group defined by Aanderaa and Cohen). They also produced a proof of Higman's embedding theorem [**20**] and showed that their embedding preserves the unbouded truth-table degree of the word problem (the existence of such a degree preserving embedding was first shown by Clapham [**11**] for Turing degrees and then by Valiev [**28**] for truth-table degrees).

In [**21**] the modular machine approach was combined with the notion of 'groups with a standard basis' which is due to Bokut' [**3**] (see also [**5**]). This was used to give a fairly unified treatment of results such as those described in the previous paragraph (concentrating for the most part on Turing degrees) as well as some other results. This paper uses the same approach to provide a new proof of Collins' theorem.

In his proof Collins used the groups studied by Boone [**8**] and Britton [**9**] with an extra generator, a change of a relation and two new relations. We follow a similar procedure using the groups introduced by Aanderaa and Cohen [**1**] (see also §5 of [**21**]). The reason for the change of the relations is that it introduces an extra kind of symmetry which greatly helps the analysis of conjugacy in the new groups.

Let M be an extended modular machine with quadruples (a_i, b_i, c_i, R) for $i \in I$ and (a_j, b_j, c_j, L) for $j \in J$ and with $(0, 0)$ being a terminal configuration (see §2 for a definition).

Put

$$
\begin{aligned}
K_0 &= \langle\, y, t \mid \quad \rangle, \\
K_1 &= \langle\, K_0, x \mid\ yx = xy \,\rangle, \\
K_2 &= \langle\, K_1, r_i, l_j \mid\ x^m r_i = r_i x^{m^2}, y^m r_i = r_i y, t(a_i, b_i) r_i = r_i t(c_i, 0), \\
&\qquad\qquad\qquad x^m l_j = l_j x, y^m l_j = l_j y^{m^2}, t(a_j, b_j) l_j = l_j t(0, c_j); \\
&\qquad\qquad\qquad i \in I, j \in J \,\rangle, \\
K_3 &= \langle\, K_2, h \mid\ r_i h = h r_i, l_j h = h l_j;\ i \in I, j \in J \,\rangle, \\
K_4 &= \langle\, K_3, k \mid\ r_i k = k r_i, l_j k = k l_j, t^{-1} h t k = k t^{-1} h t;\ i \in I, j \in J \,\rangle,
\end{aligned}
$$

where $t(\alpha, \beta) \equiv x^{-\alpha} y^{-\beta} t x^{\alpha} y^{\beta}$ and $m > 1$ is given by the modular machine M. Note that K_0, K_1, K_2 are the same as the groups G_0, G_1, G_2 of **[21]** where it is shown that they form a Britton tower. Since h, k induce the identity isomorphism it follows that $K_0, \ldots, K_4$ is a Britton tower. We shall occasionally refer to K_4 by the more suggestive notation $K(M)$. The main bulk of this paper is devoted to showing:

THEOREM 1.2 *The word problem for $K(M)$ is Turing equivalent to the special halting problem for M.*

THEOREM 1.3 *The conjugacy problem for $K(M)$ is Turing equivalent to the confluence problem for M provided that M has solvable loop problem.*

Thus Collins' theorem will follow provided that there is an extended modular machine M with special halting problem of degree **a**, confluence problem of degree **b** and solvable loop problem. The fact that such an M exists is discussed in the next section.

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2. *Decision problems for modular machines*

Despite the title of this section, most of it will be devoted to general machines. This is due to the fact that our results apply to the general situation, without any essential changes to the proofs. For ease of reference, we state some basic definitions most of which can also be found in **[12]**. A (deterministic) *machine* M consists of

1. a countable set of *configurations* (together with a numbering),

2. a recursive subset of configurations called the *terminal configurations*,

3. a recursive function on configurations which defines the machine's *basic moves* and is written $C \Rightarrow C'$.

For example a *modular machine* M normally has $\mathbf{N}^2$ as its set of configurations. M also has

1. integers $m > n > 0$,

2. finitely many quadruples
$$\begin{aligned} &(a_i, b_i, c_j, R), \qquad i \in I, \\ &(a_j, b_j, c_j, L), \qquad j \in J, \end{aligned}$$
such that $0 \leq a_k, b_k < m$ and $0 \leq c_k < m^2$ for all $k \in I \cup J$.

   Moreover, at most one quadruple begins with any given pair of integers (this ensures that M is *deterministic*).

The basic move function of M is defined as follows. Given a configuration (α, β) put $(\alpha, \beta) = (um + a, vm + b)$ where $0 \leq a, b < m$. If M has a quadruple (a, b, c, R) then $(um + a, vm + b) \Rightarrow (um^2 + c, v)$. If M has quadruple (a, b, c, L) then $(um + a, vm + b) \Rightarrow (u, vm^2 + c)$. Otherwise (α, β) is terminal. Clearly the set of terminal configurations is recursive. An *extended modular machine* uses $\mathbf{Z}^2$ as its set of configurations but otherwise it is the same as an ordinary modular machine.

The letter m together with the sets of quadruples (a_i, b_i, c_j, R) for $i \in I$ and (a_j, b_j, c_j, L) for $j \in J$ will always have the above meaning for a modular machine M. The number n is used for defining input and output functions for M. We do not define these as they will not be needed and so we shall use n as a general variable.

Returning to general machines, if $C_0 \Rightarrow C_1 \Rightarrow \cdots \Rightarrow C_n$ then we write $C_0 \rightarrow C_n$. The *halting problem* for M is
$$H(M) = \{\, C \mid C \rightarrow C' \text{ where } C' \text{ is terminal} \,\}.$$

The *word*, or *derivability*, *problem* for M is
$$D(M) = \{\, (C, C') \mid C \rightarrow C' \,\}.$$

The *confluence problem* for M is

$$K(M) = \{\, (C, C') \mid \exists C'' \text{ s.t. } C \to C'' \text{ and } C' \to C'' \,\}.$$

The *loop problem* for M is

$$L(M) = \{\, C \mid \exists C' \text{ s.t. } C \to C' \text{ and } C' \to C' \text{ in a positive number of steps} \,\}.$$

We say that M *halts from* C if $C \in H(M)$ and it *loops from* C if $C \in L(M)$. We also say that C, C' *conflow* if $(C, C') \in K(M)$.

It is frequently convenient to single out a special terminal configuration C_0 of M and define the *special halting problem* of M to be

$$H_0(M) = \{\, C \mid (C, C_0) \in D(M) \,\}.$$

For modular machines we take $C_0 = (0, 0)$, provided that $(0, 0)$ is terminal (otherwise $H_0(M) = \emptyset$). For single-tape Turing machines defined by quintuples (or quadruples) we normally take C_0 to consist of the empty tape and a special halting state q_0.

Any realistic machine works by obeying instructions and every basic move $C \Rightarrow C'$ can be given a label which is the instruction used to produce C' from C. In the case of a Turing machine the labels are the quintuples (or quadruples) which define it. For a modular machine the labels are its quadruples. From now on we shall assume that each basic move of a machine M has a label attached to it.

A computation $C_0 \Rightarrow C_1 \Rightarrow \cdots \Rightarrow C_n$ determines a unique path Π consisting of the labels of each basic move. We use $C_0 \xrightarrow{\Pi} C_n$ to represent such a computation sequence. We denote the *length* of Π by $l(\Pi)$. For an arbitrary configuration C and a path Π we put $C\Pi = C'$ if there is a computation $C \xrightarrow{\Pi} C'$, otherwise $C\Pi$ is undefined. We put $\Sigma \leq \Pi$ if there is a path Δ such that $\Pi = \Sigma\Delta$. Note that if $C\Pi = C\Pi'$ (where both sides are defined) then either $\Pi \leq \Pi'$ or $\Pi' \leq \Pi$. Moreover if equality does not hold then M loops from C. For two pairs (Σ, Σ') and (Π, Π') we put $(\Sigma, \Sigma') \leq (\Pi, \Pi')$ if there is a path Δ such that $\Pi = \Sigma\Delta$ and $\Pi' = \Sigma'\Delta$.

A pair (C, C') of configurations is said to *conflow via* a pair of paths (Π, Π') if $C\Pi, C'\Pi'$ are both defined and $C\Pi = C'\Pi'$. A pair of paths (Π, Π') is said to be a *mass confluence* for the configurations (C_i, C_i') for $0 \leq i \leq n$ if each (C_i, C_i') conflows via (Π, Π') for $0 \leq i \leq n$. The *mass confluence problem* for M is the problem of deciding for arbitrary configurations (C_i, C_i') for $0 \leq i \leq n$ whether or not they mass conflow (note that n is also arbitrary).

Lemma 2.1 *Let (C, C') conflow via (Σ, Σ') and via (Π, Π'). Then it follows that M loops from C and C' or $(\Sigma, \Sigma') \leq (\Pi, \Pi')$ or $(\Pi, \Pi') \leq (\Sigma, \Sigma')$.*

Proof. We may assume without loss of generality that $\Pi = \Sigma\Delta$ for some path Δ. Then $C\Sigma\Delta$ is defined and $C'\Sigma'\Delta = C\Sigma\Delta = C\Pi = C'\Pi'$. Thus either $(\Sigma, \Sigma') \leq (\Pi, \Pi')$ or M loops from C' (and hence C). □

Lemma 2.2 *Let M have solvable loop problem and let (C, C') be a pair of configurations which conflow. Suppose that M does not loop from C (and hence C'). Then there is a minimal confluence (Σ, Σ') for (C, C') which can be computed.*

Proof. Let (Π, Π') be any confluence for (C, C'). If we choose a confluence (Σ, Σ') for (C, C') with $l(\Sigma) + l(\Sigma')$ minimal the result then follows from lemma 2.1. □

Note that in the preceding lemma we require M to have solvable loop problem in order to ensure that the set $P = \{ (C, C') \mid M \text{ does not loop from } C, C' \}$ is recursively enumerable. The lemma then states that there is a partial recursive function f from pairs of configurations to pairs of paths such that if $(C, C') \in P$ conflow then $f(C, C')$ is a minimal confluence for (C, C').

Lemma 2.3 *Suppose that M loops from C, D. Then*

1. *we can recursively find paths $\Pi_0, \ldots, \Pi_r, \Pi_{r+1} \ldots, \Pi_{r+s}, \Sigma_{r+1} \ldots, \Sigma_{r+s}$ such that any computation from C is either by a path Π_i for $0 \leq i \leq r$ or by a path $\Pi_i \Sigma_i^n$ for $r+1 \leq i \leq r+s$ and $n \geq 0$.*

2. *We can recursively find paths $\Gamma_0, \ldots, \Gamma_p, \Gamma_{p+1}, \ldots, \Gamma_{p+q}, \Delta_{p+1}, \ldots, \Delta_{p+q}$ such that the set*

$$\{ \Gamma \mid C\Gamma \text{ and } D\Gamma \text{ are both defined} \}$$

is just

$$\{ \Gamma_0, \ldots, \Gamma_p \} \cup \{ \Gamma_i \Delta_i^n \mid p+1 \leq i \leq p+q \text{ and } n \geq 0 \}.$$

Moreover $(C\Gamma, D\Gamma) = (C\Gamma', D\Gamma')$ if and only if $\Gamma = \Gamma'$ or $\Gamma = \Gamma_i \Delta_i^t, \Gamma' = \Gamma_i \Delta_i^u$ for some i with $p+1 \leq i \leq p+q$ and $t, u \geq 0$. In particular the set

$$\{ (C\Gamma, D\Gamma) \mid C\Gamma, D\Gamma \text{ are both defined for some path } \Gamma \}$$

is finite and recursively computable from C, D.

Proof. For the first part let $C = C_0, C_1, \ldots, C_r, C_{r+1}, \ldots, C_{r+s+1} = C_{r+1}$ be the configurations of the computation from C up to the first loop. The result follows if we choose Π_i for $0 \le i \le r+s$ to be the path such that $C_i = C_0\Pi_i$ and Σ_i for $r+1 \le i \le r+s$ to be the minimal non-empty path such that $C_i = C_i\Sigma_i$.

For the second part we use the notation introduced in the previous paragraph. We can check which, if any, of the Π_i for $0 \le i \le r$ apply to D and put these amongst $\Gamma_0, \ldots, \Gamma_p$. Now for each $i > p$ consider the paths $\Pi_i\Sigma_i^n$ for all $n \ge 0$. Note that there are only finitely many configurations reachable from D and we can find out what they are. If $D\Pi_i$ is not defined then $D\Pi_i\Sigma_i^n$ is not defined for any $i > p$ and $n \ge 0$. Assume now that $D\Pi_i$ is defined. There are two possibilities to consider.

(1) There is an $n \ge 0$ with $D\Pi_i\Sigma_i^n$ defined and $D\Pi_i\Sigma_i^{n+1}$ not defined. Here we obtain finitely many paths Γ such that $C\Gamma, D\Gamma$ are both defined and $\Gamma = \Pi_i\Sigma_i^t$ for some $t \ge 0$. We put all these amongst $\Gamma_0, \ldots, \Gamma_p$. Note that in this case we cannot have $D\Pi_i\Sigma_i^t = D\Pi_i\Sigma_i^u$ for $u > t$.

(2) There exist $u \ge 0$ and $t < u$ such that $D\Pi_i\Sigma_i^t = D\Pi_i\Sigma_i^u$. Let u be minimal. Clearly $D\Pi_i\Sigma_i^k$ is defined for all k and $D\Pi_i\Sigma_i^k = D\Pi_i\Sigma_i^{k-(u-t)}$ for $k \ge u$. Here we have finitely many different endpoints $\{ D\Pi_i\Sigma_i^n \mid 0 \le n < u \}$ and paths of type $\Lambda_j\Theta_j^k$ for all $k \ge 0$ where $\Theta_j = \Sigma_i^{u-t}$ and the Λ_j are the elements of $\{ \Pi_i\Sigma_i^n \mid 0 \le n < u \}$. We put the Λ_j amongst $\Gamma_{p+1}, \ldots, \Gamma_{p+q}$ and the Θ_j amongst $\Delta_{p+1}, \ldots, \Delta_{p+q}$.

The rest of the lemma follows easily. □

Note that the preceding lemma can be extended to any number of configurations.

We shall say that a machine is *strongly labelled* if whenever $C_0 \Rightarrow C'$, $C_1 \Rightarrow C'$ and both basic moves have the same label then $C_0 = C_1$. If we label modular machines by quadruples, as indicated above, then they are strongly labbelled.

THEOREM 2.1 *Suppose that M is strongly labelled and has solvable loop problem. Then the mass confluence problem for M is Turing equivalent to the confluence problem for M.*

Proof. The confluence problem for any machine M is certainly Turing reducible to its mass confluence problem since the pair (C, C') conflows if and only if it mass conflows.

We show the opposite reduction for two pairs of configurations $(C, C'), (D, D')$. The general case is similar.

First we decide (with respect to an oracle for the confluence problem of M) whether or not $(C, C'), (D, D')$ both conflow. If either pair fails to conflow then the answer to the problem is 'no'. Form now on we assume that both pairs conflow and consider two cases.

Case 1. M does not loop from at least one of C or D.

We may assume without loss of generality that M does not loop on C. By lemma 2.2 we can recursively find a minimal confluence (Σ, Σ') for (C, C'). Now if $(C, C'), (D, D')$ mass conflow via (Π, Π') then by lemma 2.1 there is a path Δ such that $\Pi = \Sigma\Delta$ and $\Pi' = \Sigma'\Delta$. An easy induction shows that if $D\Sigma\Delta = D'\Sigma'\Delta$ then $D\Sigma = D'\Sigma'$. (Note that we need M to be strongly labelled for the induction to go through.) Thus $(C, C'), (D, D')$ mass conflow if and only if they do so via (Σ, Σ').

Case 2. M loops from C and D.

Note that M also loops from C' and D'. By lemma 2.2 we can recursively find the finite sets

$$\{\, (C\Gamma, D\Gamma) \mid \Gamma \text{ is any path such that } C\Gamma, D\Gamma \text{ are both defined} \,\}$$

and

$$\{\, (C'\Gamma', D'\Gamma') \mid \Gamma' \text{ is any path such that } C'\Gamma', D'\Gamma' \text{ are both defined} \,\}.$$

Now $(C, C'), (D, D')$ mass conflow if and only if these two sets have non-empty intersection.
□

THEOREM 2.2 *Let M be a strongly labelled machine with solvable loop problem. Suppose that we are given*

1. *pairs of configurations (C_i, C'_i) for $0 \le i \le n$,*
2. *a recursive function σ from configurations to integers which is additive, i.e. $\sigma(\Pi_1\Pi_2) = \sigma(\Pi_1) + \sigma(\Pi_2)$ for all paths Π_1, Π_2,*
3. *an integer m,*
4. *pairs of integers f_j, f'_j for $0 \le j \le n$.*

Then the problem of deciding whether or not there is a mass confluence (Π, Π') for the (C_i, C'_i) such that $m^{\sigma(\Pi)} f_j = m^{\sigma(\Pi')} f'_j$ for $0 \le j \le n$ is Turing reducible to the confluence problem for M.

Proof. We give a reduction to the mass confluence problem for M. An application of theorem 2.1 completes the result.

Clearly we may assume that $m \neq 0$ and we may also disregard any pair f_j, f'_j which is zero. Note also that if any f_j (respectively f'_j) is zero then the answer to the problem is 'no' unless f'_j (respectively f_j) is also zero. Thus we may assume that none of the given integers is zero. Now if there is to be a 'yes' answer then we must have $f_i/f'_i = f_j/f'_j$ for $0 \leq i, j \leq n$ so suppose that this is the case. It therefore follows that we may replace the f_j, f'_j with just two integers f, f'.

We prove the theorem for two pairs of configurations $(C, C'), (D, D')$. The general case is similar. First we check (with repect to an oracle for the mass confluence problem of M) whether or not $(C, C'), (D, D')$ mass conflow. If they do not then the answer to the problem is 'no'. From now on we assume that they do mass conflow and consider two cases.

Case 1. M does not loop from at least one of C or D.

It was shown in Case 1 of theorem 2.1 that we can recursively find a mass confluence (Σ, Σ') for $(C, C'), (D, D')$ such that if (Π, Π') is any other mass confluence for the two pairs then $\Pi = \Sigma\Delta$ and $\Pi' = \Sigma'\Delta$ for some path Δ. But now $m^{\sigma(\Pi)}f = m^{\sigma(\Pi')}f'$ if and only if $m^{\sigma(\Sigma)}f = m^{\sigma(\Sigma')}f'$.

Case 2. M loops from C and D.

Let $\Pi_0, \ldots, \Pi_r, \Pi_{r+1} \ldots, \Pi_{r+s}, \Sigma_{r+1} \ldots, \Sigma_{r+s}$ be paths for C, D constructed as in the second part of lemma 2.3 and let $\Gamma_0, \ldots, \Gamma_p, \Gamma_{p+1} \ldots, \Gamma_{p+q}, \Delta_{p+1} \ldots, \Delta_{p+q}$ be paths for C', D'. It follows from lemma 2.3 that we can recursively find finitely many pairs of paths of form $(\Pi_i, \Gamma_j), (\Pi_i\Sigma_i, \Gamma_j), (\Pi_i, \Gamma_j\Delta_j)$ or $(\Pi_i\Sigma_i, \Gamma_j\Delta_j)$ such that the following are all the mass confluences for $(C, C'), (D, D')$:

1. (Π_i, Γ_j).

2. $(\Pi_i\Sigma_i^t, \Gamma_j)$ for all $t \geq 0$.

3. $(\Pi_i, \Gamma_j\Delta_j^u)$ for all $u \geq 0$.

4. $(\Pi_i\Sigma_i^t, \Gamma_j\Delta_j^u)$ for all $t, u \geq 0$.

We can merge all four cases into one by considering the following problem: given paths $\Pi, \Gamma, \Sigma, \Delta$ decide whether or not the equation

$$m^{\sigma(\Pi)+t\sigma(\Sigma)} f = m^{\sigma(\Gamma)+u\sigma(\Delta)} f'$$

has integer solutions $t, u \geq 0$. Furthermore we are allowed to specify that $t = 0$ or $u = 0$.

If we do specify that $t = 0$ or $u = 0$ then the question is trivially decidable. For the general case a solution exists only if f/f' is a power of m. If this is so then we wish to know if an equation of form $at - bu = c$, with $a, b \geq 0$, has solutions in non-negative integers t, u. If any one of a, b, c is zero the problem is clearly decidable. Otherwise the equation has an *integer* solution t_0, u_0 if and only if $(a, b) \mid c$. If this is so then we can find a non-negative solution $t_1 = t_0 + bx, u_1 = u_0 + ax$ for some integer x. □

Finally we concentrate on modular machines. As mentioned above we take the label of a basic move $C \Rightarrow C'$ to be the quadruple which is used to produce C' from C. The recursive function σ of theorem 2.2 is defined by

$$\sigma(\Pi) = \#(R\text{-quadruples in } \Pi) - \#(L\text{-quadruples in } \Pi).$$

We shall need the following result from **[12]**.

THEOREM 2.3 (Cohen) *Let* $\mathbf{a}$, $\mathbf{b}$ *be any two r.e. Turing degrees with* $\mathbf{a} \leq_T \mathbf{b}$*. Then there is an extended modular machine* M *with terminal configuration* $(0, 0)$ *such that*

1. *the special halting problem for* M *has degree* $\mathbf{a}$*,*

2. *the confluence problem for* M *has degree* $\mathbf{b}$*.*

Unfortunately, as mentioned in §1, we need M to have solvable loop problem. Following the analysis of **[12]** it is easy to see that the modular machines constructed there have solvable loop problem provided they are obtained from Turing machines whose loop problem for unmarked tape is solvable. The discussion of normal and non-normal configurations in **[12]** shows that this holds if each Turing machine is constructed from a large scale machine whose loop problem is solvable. However, if we construct a large scale machine with halting problem of degree $\mathbf{a}$, confluence problem of degree $\mathbf{b}$ and solvable derivability problem by fitting together two of the machines mentioned in **[12]**, we find that the machine never loops. This is because in any computation which does not halt one register becomes arbitrarily large. Fortunately we do not use the derivability problem and so we may take it to be solvable.

For the rest of the paper M will denote a modular machine which satisfies theorem 2.3 and has solvable loop problem.

3. *The word problem degree*

We use Bokut' normal forms to prove Theorem 1.2. These forms were introduced by Bokut' [**3**] under the concept of a 'group with standard basis'. A more detailed account is given by him in [**5**]. A brief, but for our purposes adequate account, will be found in §3 of [**21**].

The existence of Bokut' normal forms for K_2 is established in §5 of [**21**]. We include the forbidden subwords here for ease of reference.

K_1 : $y^\delta x^\epsilon$.

K_2 : $x^m y^s r_i,\ x^{-1} y^s r_i,\ x^{m^2} y^s r_i^{-1},\ x^{-1} y^s r_i^{-1},$
$y^m r_i,\ y^{-\epsilon} r_i^\epsilon,\ y^{-1} r_i^{-1},$
$t^\epsilon x^{a_i} y^{b_i} r_i,\ t^\epsilon x^{c_i} r_i^{-1},$
$y^m l_j,\ y^{-1} l_j,\ y^{m^2} l_j^{-1},\ y^{-1} l_j^{-1},$
$x^m y^s l_j,\ x^{-\epsilon} y^s l_j^\epsilon,\ x^{-1} y^s l_j^{-1},$
$t^\epsilon x^{a_j} y^{b_j} l_j,\ t^\epsilon x^{c_j} l_j^{-1},$ for all $i \in I$, $j \in J$ and $s \in \mathbf{Z}$.

(We shall use $\delta, \epsilon, \zeta, \eta$, possibly with subscripts, as variables for ± 1 throughout the paper.) Note that the two sets of forbidden subwords are recursive and so the normal forms for K_2 are recursively computable. In order to extend the normal forms to K_4 we need the following:

$$t^{-1}(\alpha,\beta)ht(\alpha,\beta)k = kt^{-1}(\alpha,\beta)ht(\alpha,\beta) \iff (\alpha,\beta) \in H_0(M). \tag{3.1}$$

Using the relations of K_2 we see that, for all $i \in I$,

$$r_i^{-1} t(\alpha m + a_i, \beta m + b_i) r_i = t(\alpha m^2 + c_i, \beta), \quad \text{for all } (\alpha,\beta) \in \mathbf{Z}^2.$$

Moreover,

$$(\alpha m + a_i, \beta m + b_i) \in H_0(M) \iff (\alpha m^2 + c_i, \beta) \in H_0(M).$$

Similar remarks apply for the l_j. It now follows that if S is any $\{ r_i, l_j \}$-word such that $S^{-1} t S = t(\alpha,\beta)$ then $(\alpha,\beta) \in H_0(M)$ (recall that $(0,0)$ is terminal for M). Therefore

$$A(k) = \langle\, t^{-1}(\alpha,\beta)ht(\alpha,\beta), r_i, l_j; (\alpha,\beta) \in H_0(M), i \in I, j \in J \,\rangle.$$

Thus we simply have to show that

$$t^{-1}(\alpha,\beta)ht(\alpha,\beta) \in A(k) \Rightarrow (\alpha,\beta) \in H_0(M).$$

Suppose that $t^{-1}(\alpha,\beta)ht(\alpha,\beta) \in A(k)$ and write this word in the generators of $A(k)$ which are given in the presentation of K_4. Using h as a stable letter we see by Britton's lemma that

$$t^{-1}(\alpha,\beta)ht(\alpha,\beta) = S_0 t^{-1} h t S_1$$

where S_0, S_1 are freely reduced $\{ r_i, l_j \}$-words. We may rewrite the Britton tower $K_0, \ldots, K_4$ so that the r_i, l_j come after h as stable letters (see the tower $P_0, \ldots, P_4$ of §5). Britton's lemma now implies that $S_0 \equiv S_1^{-1}$ and so

$$t^{-1}(\alpha,\beta)ht(\alpha,\beta) = t^{-1}(\alpha',\beta')ht(\alpha',\beta')$$

for some $(\alpha',\beta') \in H_0(M)$. Using h as a stable letter we see that $t^{-1}(\alpha',\beta')t(\alpha,\beta) \in A(h) = \langle r_i, l_j; i \in I, j \in J \rangle$ and so $(\alpha,\beta) = (\alpha',\beta')$.

We now adjoin the set of all equations in (3.1) to the relations of K_4. The following gives our choice of distinguished letters.

K_3 : $\underline{r}_i h = h \underline{r}_i,\ \underline{l}_j h = h \underline{l}_j.$

K_4 : $\underline{r}_i k = k \underline{r}_i,\ \underline{l}_j k = k \underline{l}_j,$
$\quad t^{-1}(\alpha,\beta)\underline{h}t(\alpha,\beta)k = kt^{-1}(\alpha,\beta)\underline{h}t(\alpha,\beta),\quad$ for all $(\alpha,\beta) \in H_0(M)$.

This gives the following forbidden subwords (where $V(a,b,\ldots)$ denotes any $\{ a,b,\ldots \}$-word and $\mathcal{C}(W)$ denotes the Bokut' normal form of W).

K_3 : $r_i\mathcal{C}(V(x^{m^2}, y, t(c_i,0)))h^\epsilon,\ r_i^{-1}\mathcal{C}(V(x^m, y^m, t(a_i,b_i)))h^\epsilon,$
$\quad l_j\mathcal{C}(V(x, y^{m^2}, t(0,c_j)))h^\epsilon,\ l_j^{-1}\mathcal{C}(V(x^m, y^m, t(a_j,b_j)))h^\epsilon.$

K_4 : same as for K_3 with h replaced by k, and
$\quad h^\delta V(r_i,l_j)t(\alpha,\beta)k^\epsilon,\quad$ where $V(r_i,l_j)$ is freely reduced and $(\alpha,\beta) \in H_0(M)$.

The existence of normal forms for K_3 and K_4 follows easily from length arguments. We now prove that these forms are unique.

To prove uniqueness for K_3 let $Rh^\epsilon, Th^\epsilon \in C_3$ for $R,T \in C_2$ and suppose that there is a $V \in \langle r_i, l_j \rangle$ such that $R = TV$ where V is freely reduced. Suppose that $V \not\equiv 1$ for a contradiction.

TV cannot be in normal form for otherwise $\mathcal{C}(TVh^\epsilon)$ does not end in h^ϵ which contradicts the fact that $Rh^\epsilon \equiv \mathcal{C}(TVh^\epsilon)$. Let p^ζ be the first symbol in V and suppose that it is cancelled in reducing TV to normal form. Thus $T \equiv T_0 p^{-\zeta} T_1$ where T_1 is $\{ r_i, l_j \}$-free and is in $A(p^\zeta)$. But this contradicts the assumption that Th^ϵ is normal.

Thus p^ζ is not cancelled and so the last letter in V, q^η say, is not cancelled either. Hence R ($\equiv \mathcal{C}(TV)$) ends in $q^\eta X$ where $X \in A(q^\eta)$. This contradicts the assumption that Rh^ϵ is normal.

Hence K_3 has a set C_3 of unique Bokut' normal forms. Moreover these normal forms are recursively computable since the set of forbidden subwords for K_3 is recursive and normal forms for K_2 are recursively computable.

To prove uniqueness for K_4 let $Rk^\epsilon, Tk^\epsilon \in C_4$ where $R, T \in C_3$ and suppose that there is a $D \in A(k)$ such that $R = TD$. We write D as a word in the generators $\{ r_i, l_j, t^{-1}(\alpha,\beta)ht(\alpha,\beta) \mid (\alpha,\beta) \in H_0(M) \}$. Since h induces the identity isomorphism we may assume that D has no h-pinches and so no h-symbols are cancelled in reducing D to normal form. If D is h-free then an argument similar to that for K_3 gives $D = 1$. So suppose that D involves h and $D \neq 1$ for a contradiction.

Suppose that the first h-symbol in D is cancelled in reducing TD to normal form. Let $D \equiv Lt^{-1}(\alpha,\beta)h^\zeta D'$ where L is an $\{ r_i, l_j \}$-word. Then we must have $T \equiv T_0 h^{-\zeta} T_1$ where T_1 is h-free and $T_1 L t^{-1}(\alpha,\beta) \in A(h^\zeta) = \langle r_i, l_j \rangle$. So $T_1 L t^{-1}(\alpha,\beta) = S$ for some $\{ r_i, l_j \}$-word S. Now $T_1 = S t(\alpha,\beta) L^{-1}$ and after removing any $\{ r_i, l_j \}$-pinches we have $T_1 = S_0 t(\alpha',\beta') S_1$ where S_0, S_1 are freely reduced $\{ r_i, l_j \}$-words and $(\alpha',\beta') \in H_0(M)$. If $S_1 \equiv 1$ then $T_1 \equiv \mathcal{C}(S_0 t(\alpha',\beta')) \equiv S_0 t(\alpha',\beta')$ and Tk^ϵ is not normal contrary to assumption. Thus $S_1 \not\equiv 1$, say $S_1 \equiv S_1' p^\eta$. Then $T_1 \equiv \mathcal{C}(S_0 t(\alpha',\beta') S_1' p^\eta)$ ends in $p^\eta V$ for some $V \in A(p^\eta)$ and once again Tk^ϵ cannot be normal.

Thus the first h-symbol in D is not cancelled and so the last one is not cancelled either. Let $D \equiv D' h^\eta t(\alpha,\beta) S$ where S is a freely reduced $\{ r_i, l_j \}$-word. Thus $\mathcal{C}(TD) \equiv T' h^\eta \mathcal{C}(X t(\alpha,\beta) S)$ for some word T' and some $X \in A(h^\eta) = \langle r_i, l_j \rangle$. But this means that $Rk^\epsilon \equiv \mathcal{C}(TD) k^\epsilon$ is not normal, contrary to assumption.

This completes the proof that K_4 has unique Bokut' normal forms. Note that these forms are $H_0(M)$-recursively computable since the set of forbidden subwords for K_4 is $H_0(M)$-recursive and normal forms for K_3 are recursively computable.

We have thus shown that the word problem for $K(M)$ is reducible to $H_0(M)$. The converse follows from (3.1). This completes the proof of Theorem 1.2

We end this section by defining a variant of the normal forms obtained above which will be useful later on. A normal $\{ x, y, t \}$-word which ends in $x^\alpha y^\beta$ can be written uniquely in the form

$$x^{\alpha_0}y^{\beta_0}t^{\epsilon_1}(\alpha_1,\beta_1)\cdots t^{\epsilon_n}(\alpha_n,\beta_n)$$

where $t^{\epsilon_i}(\alpha_i,\beta_i) \neq t^{-\epsilon_{i+1}}(\alpha_{i+1},\beta_{i+1})$ for $1 \leq i \leq n-1$ and $\alpha_n = \alpha, \beta_n = \beta$. In order to obtain the new normal form for a word W, denoted by $\mathcal{C}^*(W)$, we $H_0(M)$-compute $\mathcal{C}(W)$ and then put each $\{x,y,t\}$-subword of $\mathcal{C}(W)$ into the form given above. Clearly $\mathcal{C}^*(W)$ is unique and $H_0(M)$-computable.

4. *General definitions and lemmas*

Let E^* be an HNN extension of E with stable letters $\{p_v\}$. We shall write (E^*, E, p) for this and $A_E(p_v)$, $A_E(p_v^{-1})$ for the associated subgroups of p_v. By '$?(U \sim_{E^*} V)$' we shall mean 'the problem of determining whether or not U is conjugate to V in E^*' (where U, V are given). A word W of E^* is:

- *pinch-reduced w.r.t.* (E^*, E, p) if it is not equal to a word with fewer p-symbols (i.e. W has no p-pinches),
- *cyclically pinch-reduced w.r.t.* (E^*, E, p) if all cyclic permutations of W are pinch-reduced w.r.t. (E^*, E, p). (A cyclic permutation of W is any word YX where $W \equiv XY$. Naturally W is a cyclic permutation of itself. Clearly W is conjugate to each of its cyclic permutations.)

We shall omit the qualifying phrase 'w.r.t. (E^*, E, p)' whenever there can be no confusion.

Let $l_p(W)$ denote the number of p-symbols in W and consider a sequence $W = W_0, W_1, W_2, \ldots$ such that for each i the word W_{i+1} is equal to a cyclic permutation of W_i and $l_p(W_{i+1}) < l_p(W_i)$. Clearly such a sequence must eventually halt in a cyclically pinch-reduced word $\widetilde{W}$ which we call a *cyclic pinch-reduction* of W. Note that $W \sim_{E^*} \widetilde{W}$ and so $U \sim_{E^*} V$ if and only if $\widetilde{U} \sim_{E^*} \widetilde{V}$ for all $U, V \in E^*$.

The *p-projection* of W, denoted by $\pi_p(W)$, is the formal word obtained by deleting all symbols of W other than p-symbols. Two words U, V of E^* are *p-parallel* if $\pi_p(U) \equiv \pi_p(V)$ and *p-circumparallel* if $\pi_p(U)$ is a cyclic permutation of $\pi_p(V)$. Britton's lemma implies that if two pinch-reduced words are equal then they are p-parallel. The analogous statement for conjugacy is contained in Collins' lemma.

Lemma 4.1 (Collins' lemma) *Let U, V be cyclically pinch-reduced words of E^* not both p-free. Then $U \sim_{E^*} V$ if and only if U, V are p-circumparallel and there exist U_0, V_0 such that*

1. *U_0, V_0 are cyclic permutations of U, V respectively,*
2. *U_0, V_0 both have p_v^ϵ as initial (respectively final) symbol,*
3. *there is an $X \in A_E(p_v^\epsilon)$ (respectively $X \in A_E(p_v^{-\epsilon})$) such that $X^{-1}U_0X = V_0$.*

Let U, V be two words of E^* not both p-free. We say that V is a *right* (respectively *left*) *conjugate* of U by W w.r.t. (E^*, E, p) if $W^{-1}UW = V$ and in removing all p-pinches from $W^{-1}UW$ the p-symbols of W in left to right (respectively of W^{-1} in right to left) order are the rightmost (respectively leftmost) symbols of the words $p_v^{-\epsilon}Xp_v^\epsilon$ which are the successive pinches.

Lemma 4.2 *Let U, V be cyclically pinch-reduced words of E^* neither p-free and W a pinch-reduced word such that $W^{-1}UW = V$. Then V is either a left or a right conjugate of U by W w.r.t. (E^*, E, p). If W is not p-free then these cases are mutually exclusive.*

Lemma 4.3 *Let U, V be cyclically pinch-reduced words of E^* neither p-free and W a pinch-reduced word of E^*. If V is a left (respectively right) conjugate of U by W w.r.t. (E^*, E, p) then U is a right (respectively left) conjugate of V by W^{-1} w.r.t. (E^*, E, p).*

Most of the preceding definitions and all the lemmas are taken from [**13**], Part I. The proofs of the lemmas are fairly straightforward applications of Britton's lemma and are carried out by means of induction on the number of p-symbols of some appropriate word.

Suppose we want to show that the conjugacy problem for E^* is of degree at most $\mathbf{d}$ and we have a $\mathbf{d}$-recursive procedure for obtaining cyclic pinch-reductions of words in E^*. It follows from above that in examining $?(U \sim_{E^*} V)$ for words U, V of E^*, at least one of which is not p-free, we may assume that

$$U \equiv p_{v_1}^{\epsilon_1}X_1 \cdots p_{v_s}^{\epsilon_s}X_s,$$
$$V \equiv p_{v_1}^{\epsilon_1}Y_1 \cdots p_{v_s}^{\epsilon_s}Y_s,$$

where X_i, Y_i are p-free for $1 \leq i \leq s$ and U, V are cyclically pinch-reduced. Moreover we need only look at elements of $A_E(p_{v_1}^{\epsilon_1})$ in trying to find a conjugating word. (A dual statement also holds where U, V both end in a p-symbol.) Another useful fact is that if $U, V \in E$ and each stable letter of E^* induces the identity isomorphism then $U \sim_{E^*} V$ if and only if $U \sim_E V$. We shall use these observations in the rest of the paper without further comment.

We proceed to prove another two general lemmas which will be helpful in analysing conjugacy in $K(M)$. A *prefix* of a word W is any word X for which there is a word Y such that $W \equiv XY$.

LEMMA 4.4 *Let U, V be cyclically pinch-reduced words of E^* with $\pi_p(U) \equiv S_0 \not\equiv 1$. Suppose V is a left conjugate of U by S w.r.t. (E^*, E, p) where S is a freely reduced p-word. Then $S \equiv S_0^f S_1$ where S_1 is a prefix of S_0 and $f \geq 0$.*

Proof. We use induction on $l_p(S)$. If $l_p(S) = 0$ there is nothing to prove. So suppose that $S \equiv p_u^\delta T$ and put $U \equiv X p_v^\epsilon U_1$ where X is p-free. Now $S^{-1}US \equiv T^{-1} p_u^{-\delta} X p_v^\epsilon U_1 p_u^\delta T$ and $p_u^{-\delta} X p_v^\epsilon$ is a pinch which is equal to a p-free word X^*. Thus $\delta = \epsilon$, $u = v$ and $S^{-1}US = T^{-1} X^* U_1 p_v^\epsilon T$. Now $X^* U_1 p_v^\epsilon$ is cyclically pinch-reduced and lemma 4.2 implies that V is a left conjugate of it by T. Put $S_0 \equiv p_v^\epsilon R$ so that $\pi_p(X^* U_1 p_v^\epsilon) \equiv R p_v^\epsilon$. Induction now implies that $T \equiv (R p_v^\epsilon)^f R_1$ where $f \geq 0$ and R_1 is a prefix of $R p_v^\epsilon$ (which we may assume to be a prefix of R). Thus $S = p_v^\epsilon T$ and there can be no cancellation between p_v^ϵ and T for otherwise U would not be cyclically pinch-reduced. Therefore $S \equiv p_v^\epsilon T \equiv (p_v^\epsilon R)^f p_v^\epsilon R_1$ and the result follows. □

Note that if in the preceding lemma $f > 0$ then $\pi_p(U)$ is itself cyclically pinch-reduced. We now define two reduction procedures which lead us to our last general lemma.

ρ-reduction w.r.t. (E^*, E, p). Let $W \equiv W_1 p^\epsilon X W_2$ where X is non-empty and is p-free. If $X \in A_E(p^{-\epsilon})$ then $W_1 X^* W_2'$ is a primitive ρ-reduction of W, where X^* is the image of X under the isomorphism induced by $p^{-\epsilon}$ and W_2' is $p^\epsilon W_2$ freely reduced.

Clearly a sequence of primitive ρ-reductions must eventually terminate in a word $W''S$ where W'' does not end in a p-symbol and S is a freely reduced p-word. We shall use $\rho(W)$ as a variable for words such as $W''S$ which are obtained from W by a terminating sequence of primitive ρ-reductions (it will always be clear which Britton tower is being used).

λ-reduction w.r.t. (E^*, E, p). Dual to ρ-reduction. This time we obtain a word SW''. We shall denote such words by $\lambda(W)$.

Lemma 4.5 *Let U, V be cyclically pinch-reduced words of E^* with $\pi_p(U) \equiv S_0 \not\equiv 1$. Let S be a freely reduced p-word with $l_p(S) \geq l_p(S_0)$. Then V is a left conjugate of U by S w.r.t. (E^*, E, p) if and only if*

1. *$S \equiv S_0^f S_1$ where S_1 is a prefix of S_0 and $f > 0$,*
2. *$\lambda(U) \equiv S_0 U'$ where U' is p-free,*
3. *$\rho(S_1 V S_1^{-1}) = V' S_0$ where V' is p-free,*
4. *$V' = S_0^{-f+1} U' S_0^{f-1}$.*

Proof. If the four conditions hold then V is certainly a left conjugate of U by S w.r.t. (E^*, E, p). For the direct implication we have:

Part 1: follows from lemma 4.4.

Part 2: suppose $\lambda(U) = S_2 X p^\epsilon U_0$ where X is a p-free non-empty word and S_2 is a freely reduced p-word. Then by part 1 we have $S \equiv S_2 p^\epsilon S_3$ and so $S^{-1}US = S_3^{-1} p^{-\epsilon} X p^\epsilon U_0 S$. It follows that $p^{-\epsilon} X p^\epsilon$ is a pinch contrary to the assumption that $S_2 X p^\epsilon U_0$ is λ-reduced.

Part 3: we have $V = S^{-1}US = S_1^{-1} S_0^{-f} S_0 U' S_0^f S_1 = S_1^{-1}(S_0^{-f+1} U' S_0^{f-1}) S_0 S_1$. Since U, V are cyclically pinch-reduced Collins' lemma implies that $l_p(V) = l_p(U) = l_p(S_0)$. Thus the number of p-pinches in the final word above must be at least f. From the assumption that S is freely reduced and part 1 it follows that $S_1^{-1} S_0^{-f+1}$ and $S_0^{f-1} S_0 S_1$ are freely reduced. Hence there is a p-free word V' which is equal to $S_0^{-f+1} U' S_0^{f-1}$.

Part 4: proved in part 3. □

Suppose that we have words U, V neither p-free which are cyclically pinch-reduced and wish to decide (possibly w.r.t. an oracle) whether or not there is a p-word S such that $S^{-1}US = V$. By lemma 4.3 there is no loss of generality if we assume that V is a left conjugate of U by S. Now lemma 4.4 implies that if S exists at all then it is of form $S_0^f S_1$ where $f \geq 0$, $S_0 \equiv \pi_p(U)$ and S_1 is a prefix of S_0. If $l_p(S) < l_p(S_0)$ then there are only finitely many choices for S and our problem reduces to the word problem for E^*. Otherwise lemma 4.5 implies that our problem reduces to looking for some prefix S_1 of S_0 such that $\lambda(U) \equiv S_0 U', \rho(S_1 V S_1^{-1}) \equiv V' S_0$ where U', V' are p-free and deciding the question:

$$\text{is there a } g \geq 0 \text{ such that } S_0^{-g} U' S_0^g = V'? \tag{4.1}$$

Thus, provided we have suitable procedures for λ- and ρ-reduction, we may assume that U, V can be replaced with p-free words U', V' and consider the question (4.1). In fact λ- and ρ-reduction will be recursive or $H_0(M)$-recursive whenever we use them. We shall use the reduction to question (4.1) in the rest of the paper without further comment.

5. *Preliminary remarks*

First of all we note that all the reductions introduced in §4 are $H_0(M)$-recursive for $K_0, \ldots, K_4$. This follows from the fact that $K(M)$ has $H_0(M)$-recursive Bokut' normal forms. Furthermore given any word W and a stable letter q in the Britton tower $K_0, \ldots, K_4$ we can $H_0(M)$-decide whether or not $W \in A(q^\epsilon)$ for this is so if and only if $\mathcal{C}(q^{-\epsilon} W q^\epsilon W^{-1}) \equiv 1$. Moreover if $W \in A(q^\epsilon)$ then there is an $H_0(M)$-recursive procedure which returns a word in the generators of $A(q^\epsilon)$ and which is equal to W. This procedure simply enumerates in turn all words in the appropriate generators and tests each one for equality with W. When the test succeeds the current word is returned.

The following are easily seen to be Britton towers which define the group $K(M)$.

$$H_0 = K_0,\ H_1 = K_1,\ H_2 = K_2,$$
$$H_3 = \langle\, H_2, k \mid\ r_i k = k r_i, l_j k = k l_j;\ i \in I, j \in J \,\rangle,$$
$$H_4 = \langle\, H_3, h \mid\ r_i h = h r_i, l_j h = h l_j, t k t^{-1} h = h t k t^{-1};\ i \in I, j \in J \,\rangle.$$

$$P_0 = K_0,\ P_1 = K_1,$$
$$P_2 = \langle\, P_1, h \mid\ \ \rangle,$$
$$P_3 = \langle\, P_2, k \mid\ t^{-1} h t k = k t^{-1} h t \,\rangle,$$
$$\begin{aligned} P_4 = \langle\, P_3, r_i, l_j \mid\ & x^m r_i = r_i x^{m^2}, y^m r_i = r_i y, t(a_i, b_i) r_i = r_i t(c_i, 0), \\ & h r_i = r_i h, k r_i = r_i k, \\ & x^m l_j = l_j x, y^m l_j = l_j y^{m^2}, t(a_j, b_j) l_j = l_j t(0, c_j), \\ & h l_j = l_j h, k l_j = l_j k; i \in I, j \in J \,\rangle. \end{aligned}$$

If q is a stable letter then, for example, we shall use $A_H(q^\epsilon)$ to denote the associated subgroup of q^ϵ in the tower $H_0, \ldots, H_4$. For the rest of the paper we put $\{\, p_v \,\} = \{\, r_i, l_j \,\}$ so that a phrase such as 'W is p-free' will mean 'W is $\{\, r_i, l_j \,\}$-free'. The symbol S, possibly with subscripts and superscripts, will be used as a variable for freely reduced p-words throughout. We also put

$$\sigma(S) = \#(r_i, l_j^{-1} \text{ symbols in } S) - \#(r_i^{-1}, l_j \text{ symbols in } S),$$

(cf. the definition of σ on sequences of quadruples of a modular machine given near the end of §2). Note that if $S^{-1} x^\alpha y^\beta S = x^{\alpha'} y^{\beta'}$ then $\alpha' = m^\sigma \alpha$ and $\beta' = m^{-\sigma} \beta$.

Let U, V be two words of $K(M)$. In considering $?(U \sim_{K(M)} V)$ we can regard U, V as elements of any of our three Britton towers which define $K(M)$. It is therefore desirable to have a procedure for cyclically pinch-reducing words of $K(M)$ w.r.t. $(P_4, P_3, p), (H_4, H_3, h)$ and (K_4, K_3, k) simultaneously and which uses an oracle whose degree is at most that of the confluence problem for M. In fact we define a procedure which is $H_0(M)$-recursive.

Let $\{ q_v \}$ be one of $\{ p_v \}, \{ h \}, \{ k \}$. For $q \in \{ q_v \}$ let $A[q^\epsilon]$ denote the associated subgroup of q^ϵ in the Britton tower which has q as one of its top stable letters. Given $\{ q_v \}$ construct a sequence $W \equiv W_0, W_1, W_2, \ldots$ such that for each $i > 0$ there is a cyclic permutation $W_i' q^{-\epsilon} X q^\epsilon$ of W_{i-1} where $q \in \{ q_v \}$, X is $\{ q_v \}$-free, $X \in A[q^\epsilon]$ and $W_i \equiv W_i' \mathcal{C}(q^{-\epsilon} X q^\epsilon)$.

If $\{ q_v \} = \{ k \}$ then it is clear that the procedure eventually produces a word which is cyclically pinch-reduced w.r.t. (K_4, K_3, k) since the Bokut' normal form of a k-pinch produces a word which is k-free. However the situation when $\{ q_v \}$ is $\{ p_v \}$ or $\{ h \}$ is not immediately clear since, for example, an r_i-pinch in the tower $P_0, \ldots, P_4$ is not necessarily an r_i-pinch in the tower $K_0, \ldots, K_4$ where Bokut' normal forms were defined. Note however that for each $q \in \{ p_v, h, k \}$ we have that $l_q(\mathcal{C}(U)) \le l_q(U)$. Suppose that X is $\{ q_v \}$-free, $X \in A[q^\epsilon]$ and let X^* be the image of X under the isomorphism induced by q^ϵ. Now $\mathcal{C}(q^{-\epsilon} X q^\epsilon) \equiv \mathcal{C}(X^*)$ so that $l_q(\mathcal{C}(q^{-\epsilon} X q^\epsilon)) = l_q(\mathcal{C}(X^*)) = 0$. Thus the procedure given above produces a cyclically pinch-reduced word in all cases. As a consequence of this discussion a membership test of form '$X \in A[q^\epsilon]$' can be replaced by the condition '$\mathcal{C}(q^{-\epsilon} X q^\epsilon)$ is q-free'. We may now cyclically pinch-reduce W w.r.t. $(P_4, P_3, p), (H_4, H_3, h)$ and (K_4, K_3, k) by constructing a sequence $W \equiv \widetilde{W}_{-1}, \widetilde{W}_0, \widetilde{W}_1, \ldots$ such that for each $i \ge 0$

1. $\widetilde{W}_{3i}$ is obtained from $\widetilde{W}_{3i-1}$ by putting $\{ q_v \} = \{ p_v \}$ in the procedure given above;

2. $\widetilde{W}_{3i+1}$ is obtained from $\widetilde{W}_{3i}$ by putting $\{ q_v \} = \{ h \}$ in the procedure given above;

3. $\widetilde{W}_{3i+2}$ is obtained from $\widetilde{W}_{3i+1}$ by putting $\{ q_v \} = \{ k \}$ in the procedure given above.

This procedure stops when no more pinches can be removed.

We shall split the proof that the conjugacy problem for $K(M)$ reduces to the confluence problem for M into three sections by looking at the conjugacy problem for words U, V from K_2, K_3 and K_4 in turn. Our practice will be to assume that U, V are cyclically pinch-reduced w.r.t. $(K_4, K_3, k), (H_4, H_3, h)$ and (P_4, P_3, p) without further comment. By Collins lemma U, V can be conjugate only if they are k-, h- and p-circumparallel (not necessarily by the same cyclic permutation of course). We shall therefore assume that this is always the case.

Finally in this section we prove two useful lemmas and also show that the confluence problem for M reduces to the conjugacy problem for $K(M)$.

LEMMA 5.1 *Suppose that $p_1^{-1}p_2t(\alpha,\beta)p_2^{-1}p_1$ is equal to a p-free word. Then $p_1 \equiv p_2$.*

Proof. The proof relies on the fact that M is deterministic (cf. lemma 1, §4 of **[21]**). Suppose for example that $p_2 \equiv r$ and r corresponds to the quadruple (a, b, c, R) of M. If $p_1 \not\equiv p_2$ then $t(\alpha,\beta) \in A_K(p_2^{-1}) = \langle\, x^{m^2}, y, t(c,0)\,\rangle$ and so $\alpha = \alpha_1 m^2 + c$ for some $\alpha_1 \in \mathbf{Z}$. Thus

$$p_1^{-1}p_2t(\alpha,\beta)p_2^{-1}p_1 = p_1^{-1}t(\alpha_1 m + a, \beta m + b)p_1,$$

and so $t(\alpha_1 m + a, \beta m + b) \in A(p_1) = \langle\, x^m, y^m, t(a', b')\,\rangle$ where p_1 corresponds to a quadruple of M which begins with a', b'. However the last containment implies that $a = a'$, $b = b'$ so that p_1 corresponds to a quadruple which begins with a, b. Since there is only one such quadruple and it corresponds to p_2 it follows that $p_1 \equiv p_2$. □

LEMMA 5.2 *$S^{-1}t(\alpha,\beta)S = t(\alpha',\beta')$ for some S if and only if $(\alpha,\beta), (\alpha',\beta')$ conflow in M via paths Π, Π'. Moreover S determines Π, Π' and conversely.*

Proof. Suppose that $(\alpha,\beta), (\alpha',\beta')$ conflow via paths Π, Π' of M. Let S, S' be the p-words which corespond to the paths. Then $S^{-1}ht(\alpha,\beta)S = S'^{-1}ht(\alpha',\beta')S'$.

Conversely suppose that $S^{-1}t(\alpha,\beta)S = t(\alpha',\beta')$. Lemma 5.1 implies that $S \equiv S_1S_2^{-1}$ for some positive words S_1, S_2. Thus

$$S_1^{-1}t(\alpha,\beta)S_1 = S_2^{-1}t(\alpha',\beta')S_2. \tag{5.1}$$

Since S_1, S_2 are positive they correspond to paths Π, Π' of M. We use induction on $l(S_1) + l(S_2)$ to show that equation (5.1) implies that $(\alpha,\beta), (\alpha',\beta')$ conflow via Π, Π'.

If $l(S_1) + l(S_2) = 0$ the claim is trivial so suppose that $l(S_1) > 0$. Let $S_1 \equiv lS_0$ where l corresponds to the quadruple (a, b, c, L) of M (the case $S \equiv rS_0$ is similar). Since $S \equiv S_1S_2^{-1}$ is freely reduced it follows that both sides of equation (5.1) pinch reduce to p-free words. Thus $t(\alpha,\beta) \in A_K(l)$ and so there are integers α_1, β_1 such that $\alpha = m\alpha_1 + a$, $\beta = m\beta_1 + b$. Now equation (5.1) holds if and only if

$$S_0^{-1}t(\alpha_1, m^2\beta_1 + c)S_0 = S_2^{-1}t(\alpha',\beta')S_2$$

and the induction is complete. □

Lemma 5.3 *$(\alpha,\beta),(\alpha',\beta')$ conflow in M if and only if $kt(\alpha,\beta)\sim_{K(M)} kt(\alpha',\beta')$.*

Proof. By Collins' lemma $kt(\alpha,\beta)\sim_{K_4} kt(\alpha',\beta')$ if and only if there is a $D\in A(k)=\langle\, r_i, l_j, t^{-1}ht\,\rangle$ such that $D^{-1}t(\alpha,\beta)D=t(\alpha',\beta')$. Removing all pinches w.r.t. (K_3,K_2,h) we see that D may be assumed to be an $\{\,r_i,l_j\,\}$-word. The result now follows from lemma 5.2. □

6. *The conjugacy problem for K_2 in K_4*

Since the stable letters of K_3 and K_4 induce the identity isomorphism we need only consider conjugacy within K_2 itself. We change the presentation for K_2 by replacing each r_i with $r_i'=x^{a_i}y^{b_i}r_ix^{-c_i}$ and each l_j with $l_j'=x^{a_j}y^{b_j}l_jx^{-c_j}$. For simplicity we shall relabel r_i',l_j' to be just r_i,l_j and still put $\{\,p_v\,\}=\{\,r_i,l_j\,\}$. This change lasts throughout this section only and so there can be no confusion. We thus obtain the Britton tower

$$\begin{aligned}
&Q_0=K_0, Q_1=K_1,\\
&Q_2=\langle\, Q_1, r_i, l_j \mid\ x^mr_i=r_ix^{m^2}, y^mr_i=r_iy, tr_i=r_it,\\
&\qquad\qquad x^ml_j=l_jx, y^ml_j=l_jy^{m^2}, tl_j=l_jt;\\
&\qquad\qquad i\in I, j\in J\,\rangle.
\end{aligned}$$

This can be rewritten as

$$\begin{aligned}
&T_0=\langle\, y\mid \quad\rangle,\\
&T_1=\langle\, T_0, x\mid\ yx=xy\,\rangle,\\
&T_2=\langle\, T_1, r_i, l_j\mid\ x^mr_i=r_ix^{m^2}, y^mr_i=r_iy,\\
&\qquad\qquad x^ml_j=l_jx, y^ml_j=l_jy^{m^2}; i\in I, j\in J\,\rangle,\\
&T_3=\langle\, T_2, t\mid\ r_it=tr_i, l_jt=tl_j;\ i\in I, j\in J\,\rangle.
\end{aligned}$$

Note that all our reduction procedures are recursive for Q_2,T_3 since they are recursive for K_2. We consider $?(U\sim_{T_3}V)$ for words U,V of T_3 which are cyclically pinch-reduced w.r.t. (Q_2,Q_1,p) and (T_3,T_2,t) (there is clearly no problem in computing such a reduction). By Collins' lemma U,V must be p- and t-circumparallel if they are to be conjugate and so we assume that this is the case.

Lemma 6.1 *Let U,V be t-free but not p-free. Then $?(U\sim_{T_3}V)$ is solvable.*

Proof. Suppose $W^{-1}UW = V$. Since U, V are t-free we may assume that W is also t-free. Moreover we may take

$$U \equiv p_1^{\epsilon_1} x^{\alpha_1} y^{\beta_1} \cdots p_s^{\epsilon_s} x^{\alpha_s} y^{\beta_s},$$
$$V \equiv p_1^{\epsilon_1} x^{\alpha'_1} y^{\beta'_1} \cdots p_s^{\epsilon_s} x^{\alpha'_s} y^{\beta'_s},$$

and $W \in A_R(p_1^{\epsilon_1})$ so that $W = x^\alpha y^\beta$ for some α, β. Thus $W^{-1}UW = V$ if and only if

$$x^\alpha y^\beta = x^{-\alpha'_s} y^{-\beta'_s} p_s^{-\epsilon_s} \cdots x^{-\alpha'_1} y^{-\beta'_1} p_1^{-\epsilon_1} x^\alpha y^\beta p_1^{\epsilon_1} x^{\alpha_1} y^{\beta_1} \cdots p_s^{\epsilon_s} x^{\alpha_s} y^{\beta_s}.$$

From the relations of T_2 it is clear that we can compute linear expressions $d_i\alpha + e_i, f_i\beta + g_i$ for $1 \le i \le s+1$ such that the equation above holds if and only if $x^{d_i\alpha+e_i} y^{f_i\beta+g_i} \in A_R(p_i^{\epsilon_i})$ for $1 \le i \le s$ and $x^\alpha y^\beta = x^{d_{s+1}\alpha+e_{s+1}} y^{f_{s+1}\beta+g_{s+1}}$. Each of the first set of conditions can be replaced by at most two linear congruences one in α and one in β. The last condition can be replaced by the equations $\alpha = d_{s+1}\alpha + e_{s+1}$, $\beta = f_{s+1}\beta + g_{s+1}$. By elementary number theory we can decide whether or not α, β exist. □

LEMMA 6.2 *Let*

$$U \equiv x^{\alpha_0} y^{\beta_0} t^{\epsilon_1} x^{\alpha_1} y^{\beta_1} \cdots t^{\epsilon_s} x^{\alpha_s} y^{\beta_s},$$
$$V \equiv x^{\alpha'_0} y^{\beta'_0} t^{\epsilon_1} x^{\alpha'_1} y^{\beta'_1} \cdots t^{\epsilon_s} x^{\alpha'_s} y^{\beta'_s},$$

be pinch-reduced w.r.t. (T_3, T_2, t). *Then* $S^{-1}US = V$ *for some* S *if and only if* $\alpha'_i = m^{\sigma(S)}\alpha_i$, $\beta'_i = m^{-\sigma(S)}\beta_i$ *for* $0 \le i \le n$.

Proof. Suppose that $S^{-1}US = V$. It suffices to show that $x^{\alpha'_i} y^{\beta'_i} = S^{-1} x^{\alpha_i} y^{\beta_i} S$ for $0 \le i \le n$. For this note that

$$x^{\alpha'_0} y^{\beta'_0} t^{\epsilon_1} x^{\alpha'_1} y^{\beta'_1} \cdots t^{\epsilon_s} x^{\alpha'_s} y^{\beta'_s} = S^{-1} x^{\alpha_0} y^{\beta_0} S t^{\epsilon_1} S^{-1} x^{\alpha_1} y^{\beta_1} S \cdots t^{\epsilon_s} S^{-1} x^{\alpha_s} y^{\beta_s} S.$$

Clearly the right hand side is still pinch-reduced w.r.t. (T_3, T_2, t) and so we have that $x^{\alpha'_s} y^{\beta'_s} S^{-1} x^{-\alpha_s} y^{-\beta_s} S \in A(t^{-\epsilon}) = \langle r_i, l_j \rangle$. Thus $x^{\alpha'_s} y^{\beta'_s} = S^{-1} x^{\alpha_s} y^{\beta_s} S$. The result follows by induction on s.

The converse is trivial. □

LEMMA 6.3 *Let* U, V *be p-free. Then* $?(U \sim_{T_3} V)$ *is solvable.*

Proof. If U, V are t-free let $U \equiv x^\alpha y^\beta, V \equiv x^{\alpha'} y^{\beta'}$ and suppose $W^{-1}UW = V$. We may assume that W is t-free. Since $xy = yx$ we may also assume that W is $\{x, y\}$-free.

Thus W is a p-word. Put $\sigma = \sigma(W)$ so that $W^{-1}UW = x^{m^\sigma \alpha} y^{m^{-\sigma}\beta}$ and thus $\alpha' = m^\sigma \alpha$, $\beta' = m^{-\sigma}\beta$. The converse of this is trivial.

Suppose now that U, V are not t-free. By Collins' lemma we may take

$$U \equiv t^{\epsilon_1} x^{\alpha_1} y^{\beta_1} \cdots t^{\epsilon_s} x^{\alpha_s} y^{\beta_s},$$
$$V \equiv t^{\epsilon_1} x^{\alpha'_1} y^{\beta'_1} \cdots t^{\epsilon_s} x^{\alpha'_s} y^{\beta'_s},$$

and look for a conjugating word S. The result now follows from lemma 6.2. □

Lemma 6.4 *Suppose that U, V are not t-free and not p-free. Then $?(U \sim_{T_3} V)$ is solvable.*

Proof. We may put

$$U \equiv t^{\epsilon_1} X_1 \cdots t^{\epsilon_n} X_n,$$
$$V \equiv t^{\epsilon_1} Y_1 \cdots t^{\epsilon_n} Y_n,$$

where X_i, Y_i are t-free for $1 \le i \le n$ and consider the question of whether or not $S^{-1}US = V$ for some S. Using λ- and ρ-reduction w.r.t. (Q_2, Q_1, p) we may reduce $?(U \sim_{T_3} V)$ to the question of whether or not there is a $g \ge 0$ such that $S^{-g}U'S^g = V'$ where U', V' are p-free and $S \equiv \pi_p(U)$. We may put

$$U' \equiv x^{\alpha_0} y^{\beta_0} t^{\epsilon_1} x^{\alpha_1} y^{\beta_1} \cdots t^{\epsilon_n} x^{\alpha_n} y^{\beta_n},$$
$$V' \equiv x^{\alpha'_0} y^{\beta'_0} t^{\epsilon_1} x^{\alpha'_1} y^{\beta'_1} \cdots t^{\epsilon_n} x^{\alpha'_n} y^{\beta'_n},$$

and now lemma 6.2 shows that g exists if and only if $\alpha'_i = m^{g\sigma(S)}\alpha_i$, $\beta'_i = m^{-g\sigma(S)}\beta_i$ for $0 \le i \le n$. If $\sigma(S) = 0$ or each equation is zero then we must have $U' \equiv V'$ otherwise g is uniquely determined. □

This completes the proof that K_2 has solvable conjugacy problem.

7. *The conjugacy problem for K_3 in K_4*

We need only consider (normal) words U, V which are k-free but not h-free and decide (w.r.t. an oracle) when they are conjugate by a word in K_3. We may put

$$U \equiv h^{\epsilon_1} X_1 \cdots h^{\epsilon_s} X_s$$
$$V \equiv h^{\epsilon_1} Y_1 \cdots h^{\epsilon_s} Y_s$$

where X_i, Y_i are h-free for $1 \le i \le s$ and consider two cases depending on whether or not U, V are p-free. First we prove some preliminary lemmas.

Lemma 7.1 *Let* $W \equiv Z_0 h^{\epsilon_1} Z_1 \cdots h^{\epsilon_n} Z_n$ *be a* p, k*-free* $\mathcal{C}^*$*-normal word. Then* $W \in A_P(p^\epsilon)$ *if and only if* $Z_i \in A_K(p^\epsilon)$ *for* $0 \le i \le n$.

Proof. Clearly if each $Z_i \in A_K(p^\epsilon)$ then $W \in A_P(p^\epsilon)$. For the converse of this write W as a word T in the generators of $A_P(p^\epsilon)$. We may assume that T is k-free and pinch-reduced w.r.t. (K_3, K_2, h). Thus $T \equiv T_0 h^{\epsilon_1} T_1 \cdots h^{\epsilon_n} T_n$ where $T_i \in A_K(p^\epsilon)$ for $0 \le i \le n$. But now $W \equiv \mathcal{C}^*(T) \equiv \mathcal{C}^*(T_0) h^{\epsilon_1} \mathcal{C}^*(T_1) \cdots h^{\epsilon_n} \mathcal{C}^*(T_n)$ so that $Z_i \equiv \mathcal{C}^*(T_i) \in A_K(p^\epsilon)$ for $0 \le i \le n$. □

Let $X \equiv x^{\alpha_0}$, $Y \equiv y^{\beta_0}$, $T \equiv t^\delta(\alpha_1, \beta_1)$ and suppose that $X, Y, T \in A_K(p^\epsilon)$. Then there are integers $\alpha_0', \beta_0', \alpha_1', \beta_1'$ such that the images of X, Y, T under the isomorphism induced by p^ϵ are equal to the words $X' \equiv x^{\alpha_0'}$, $Y' \equiv y^{\beta_0'}$, $T' \equiv t^\delta(\alpha_1', \beta_1')$ respectively. By the *result of the direct application* of p^ϵ to any one of X, Y, T we shall mean the word X', Y', T' respectively. We extend this definition to any $\mathcal{C}^*$-normal word $W \in A_K(p^\epsilon)$ in the obvious way.

Lemma 7.2 *Let* $W \equiv x^\alpha y^\beta t^{\epsilon_1}(\alpha_1, \beta_1) \cdots t^{\epsilon_n}(\alpha_n, \beta_n)$ *be* $\mathcal{C}^*$*-normal. Then*

1. $W \in A_K(p^\epsilon)$ *if and only if* $x^\alpha y^\beta \in A_K(p^\epsilon)$ *and* $t(\alpha_i, \beta_i) \in A_K(p^\epsilon)$ *for* $1 \le i \le n$.

2. *If* $W \in A_K(p^\epsilon)$ *then* $\mathcal{C}^*(p^{-\epsilon} W p^\epsilon)$ *is the result of the direct application of* p^ϵ *to* W.

Proof. Part 1 follows easily from the fact that $K_1 = \langle t \mid \quad \rangle * (\langle x \mid \quad \rangle \times \langle y \mid \quad \rangle)$.

For part 2 suppose, for example, that $\epsilon = 1$ and p corresponds to the quadruple (a, b, c, R) of M. By part 1 we have $x^\alpha y^\beta \in A_K(p)$ and $t(\alpha_i, \beta_i) \in A_K(p)$ for $1 \le i \le n$. Hence there are integers $\alpha', \beta', \alpha_i', \beta_i'$ such that

$$\begin{aligned} \alpha &= m\alpha', & \beta &= m\beta', \\ \alpha_i &= m\alpha_i' + a, & \beta_i &= m\beta_i' + b, \quad \text{for } 1 \le i \le n, \end{aligned}$$

and so the result of the direct application of p^ϵ to W is the word

$$x^{m^2\alpha'} y^{\beta'} t^{\zeta_1}(m^2\alpha_1' + c, \beta_1') \cdots t^{\zeta_n}(m^2\alpha_n' + c, \beta_n').$$

Since no occurences of t cancel in W it follows that none cancel in the displayed word and the result follows. □

Lemma 7.3 *Let* $W \equiv Z_0 h^{\epsilon_1} Z_1 \cdots h^{\epsilon_n} Z_n$ *be* $\mathcal{C}^*$*-normal where* Z_i *is an* $\{x, y, t\}$*-word for* $0 \le i \le n$. *Suppose* $W \in A_P(p^\epsilon)$. *Then* $Z_i \in A_K(p^\epsilon)$ *and* $\mathcal{C}^*(p^{-\epsilon} W p^\epsilon) \equiv Z_0' h^{\epsilon_1} Z_1' \cdots h^{\epsilon_n} Z_n'$ *where* Z_i' *is the result of the direct application of* p^ϵ *to* Z_i *for* $0 \le i \le n$.

Proof. Follows from lemmas 7.1 and 7.2. □

We are now ready to deal with $?(U \sim_{K_3} V)$.

Case I. U, V are p-free.

LEMMA 7.4 *Let*

$$X_i \equiv x^{\alpha_i} y^{\beta_i} t^{\zeta_{i1}}(\alpha_{i1}, \beta_{i1}) \cdots t^{\zeta_{if_i}}(\alpha_{if_i}, \beta_{if_i}),$$
$$Y_i \equiv x^{\alpha'_i} y^{\beta'_i} t^{\eta_{i1}}(\alpha'_{i1}, \beta'_{i1}) \cdots t^{\eta_{ig_i}}(\alpha'_{ig_i}, \beta'_{ig_i}),$$

be $\mathcal{C}^*$-normal words for $1 \le i \le n$. Then there is an S such that $S^{-1}X_iS = Y_i$ for $1 \le i \le n$ if and only if

1. $f_i = g_i, \zeta_{ij} = \eta_{ij}$ *for* $1 \le i \le n$ *and* $1 \le j \le f_i$,

2. *there are paths* Π, Π' *of the modular machine* M *such that* $(\alpha_{ij}, \beta_{ij}), (\alpha'_{ij}, \beta'_{ij})$ *conflow via* Π, Π' *and* $m^{\sigma(\Pi)}\alpha_i = m^{\sigma(\Pi')}\alpha'_i$, $m^{-\sigma(\Pi)}\beta_i = m^{-\sigma(\Pi')}\beta'_i$ *for* $1 \le i \le n$ *and* $1 \le j \le f_i$.

Proof. Suppose that there is an S such that $S^{-1}X_iS = Y_i$ for $1 \le i \le n$. Part 1 now follows from lemma 7.2 and induction on $l(S)$.

For part 2 we apply lemma 7.2 to deduce that

$$S^{-1}t(\alpha_{ij}, \beta_{ij})S = t(\alpha'_{ij}, \beta'_{ij}),$$
$$S^{-1}x^{\alpha_i}y^{\beta_i}S = x^{\alpha'_i}y^{\beta'_i},$$

for $1 \le i \le n$ and $1 \le j \le f_i$. The first set of equations together with lemma 5.2 imply that $(\alpha_{ij}, \beta_{ij}), (\alpha'_{ij}, \beta'_{ij})$ conflow via paths Π, Π' of M for $1 \le i \le n$ and $1 \le j \le f_i$. The second set of equations yields $m^{\sigma(\Pi)}\alpha_i = m^{\sigma(\Pi')}\alpha'_i$, $m^{-\sigma(\Pi)}\beta_i = m^{-\sigma(\Pi')}\beta'_i$ for $1 \le i \le n$.

The converse follows by induction on $l(\Pi) + l(\Pi')$. □

LEMMA 7.5 *The problem of deciding whether or not there is an S such that $S^{-1}US = V$ is reducible to the confluence problem for M.*

Proof. Follows from lemmas 7.3, 7.4 and theorem 2.2. □

Case II. U is not p-free.

Lemma 7.6 *Given any S there are constants μ, ν which are recursively computable from S such that if*

$$W \equiv x^{\alpha_0} y^{\beta_0} t^{\epsilon_1}(\alpha_1, \beta_1) \cdots t^{\epsilon_n}(\alpha_n, \beta_n)$$

is any $\mathcal{C}^*$-normal word with

$$S^{-1}WS = x^{\alpha'_0} y^{\beta'_0} t^{\epsilon_1}(\alpha'_1, \beta'_1) \cdots t^{\epsilon_n}(\alpha'_n, \beta'_n)$$

then

$$\alpha'_0 = m^{\sigma}\alpha_0, \qquad \beta'_0 = m^{-\sigma}\beta_0,$$
$$\alpha'_i = m^{\sigma}\alpha_i + \mu, \qquad \beta'_i = m^{-\sigma}\beta_i + \nu, \quad \text{for } 1 \le i \le n,$$

where $\sigma = \sigma(S)$.

Proof. By straightforward induction on $l(S)$. □

Using λ- and ρ-reduction w.r.t. (P_4, P_3, p) we may reduce $?(U \sim_{K_3} V)$ to the question of whether or not there is a $g \ge 0$ such that $S^{-g}U'S^g = V'$ where U', V' are p-free and $S \equiv \pi_p(U)$.

Lemma 7.7 *Let*

$$X_i \equiv x^{\alpha_i} y^{\beta_i} t^{\zeta_{i1}}(\alpha_{i1}, \beta_{i1}) \cdots t^{\zeta_{if_i}}(\alpha_{if_i}, \beta_{if_i}),$$
$$Y_i \equiv x^{\alpha'_i} y^{\beta'_i} t^{\zeta_{i1}}(\alpha'_{i1}, \beta'_{i1}) \cdots t^{\zeta_{if_i}}(\alpha'_{if_i}, \beta'_{if_i}).$$

be $\mathcal{C}^*$-normal words for $1 \le i \le n$. Suppose also that S is given. Then the problem of deciding whether or not there is a $g > 0$ such that $S^{-g}X_iS^g = Y_i$ for $1 \le i \le n$ is solvable.

Proof. We may assume that $X_j \ne Y_j$ for some j. Suppose that $S^{-g}X_iS^g = Y_i$ for $1 \le i \le n$. By lemma 7.6 (and induction on g) we can recursively compute constants μ, ν, σ from S such that

$$\alpha'_i = m^{g\sigma}\alpha_i, \qquad \beta'_i = m^{-g\sigma}\beta_i,$$
$$\alpha'_{ij} = m^{g\sigma}\alpha_{ij} + \mu \textstyle\sum_{k=0}^{g-1} m^{k\sigma}, \qquad \beta'_{ij} = m^{-g\sigma}\beta_{ij} + \nu \textstyle\sum_{k=0}^{g-1} m^{-k\sigma},$$

for $1 \le i \le n$ and $1 \le j \le f_i$.

If $\sigma = 0$ we have

$$\alpha'_i = \alpha_i, \qquad \beta'_i = \beta_i,$$
$$\alpha'_{ij} = \alpha_{ij} + g\mu, \qquad \beta'_{ij} = \beta_{ij} + g\nu,$$

for all i, j. If $\mu = \nu = 0$ then $X_i = Y_i$ for all i, contrary to assumption. Thus g is uniquely determined.

If $\sigma \neq 0$ then we may assume that $\alpha_i' = \alpha_i, \beta_i' = \beta_i$ for all i, j for otherwise g is again uniquely determined. Now we have

$$\alpha_{ij}' = m^{g\sigma}\alpha_{ij}^* + \mu^*, \qquad \beta_{ij}' = m^{-g\sigma}\beta_{ij}^* + \nu^*,$$

where $\mu^* = -\mu/(m^\sigma - 1)$, $\alpha_{ij}^* = \alpha_{ij} - \mu^*$, $\nu^* = -\nu/(m^{-\sigma} - 1)$, $\beta_{ij}^* = \beta_{ij} - \nu^*$ for all i, j. (Recall that, by the definition of a modular machine, $m > 1$.) If $\alpha_{ij}^* = \beta_{ij}^* = 0$ for all i, j then $X_i = Y_i$ for all i, contrary to assumption. It follows that $\alpha_{ij}^* \neq 0$ or $\beta_{ij}^* \neq 0$ for some i, j and once again g is uniquely determined. □

LEMMA 7.8 *With U', V' as above the problem of deciding whether or not there is a $g > 0$ such that $S^{-g}U'S^g = V'$ is solvable.*

Proof. We may assume that U', V' are $\mathcal{C}^*$-normal and $U' \not\equiv V'$ (note that $\mathcal{C}^*$ is recursive for K_3). If in addition we assume that U', V' are λ-reduced w.r.t. (K_3, K_2, h) then U', V' both begin with h^{ϵ_1} (note that λ-reduction w.r.t. (K_3, K_2, h) is recursive). Let

$$U' \equiv h^{\epsilon_1}X_1' \cdots h^{\epsilon_s}X_s',$$
$$V' \equiv h^{\epsilon_1}Y_1' \cdots h^{\epsilon_s}Y_s'.$$

By lemma 7.3 (and induction on $l(S^g)$) it follows that $S^{-g}U'S^g = V'$ if and only if $Y_i' \equiv \mathcal{C}^*(S^{-g}X_i'S^g)$ for $1 \leq i \leq s$. The result now follows by part 1 of lemma 7.4 and by lemma 7.7 □

We have thus shown that the conjugacy problem for K_3 is reducible to the confluence problem for M. The opposite reduction follows from lemma 5.2 and the observation that $ht(\alpha, \beta) \sim_{K_3} ht(\alpha', \beta')$ if and only if $S^{-1}t(\alpha, \beta)S = t(\alpha', \beta')$ for some S. We have thus established the following result.

THEOREM 7.1 *The word problem for K_3 is solvable and its conjugacy problem is equivalent to the confluence problem for M.*

This theorem together with theorem 2.3 (and the remarks following it) yields the main result of [**4**] and [**13**].

THEOREM 7.2 (Bokut', Collins) *Let* **a** *be any r.e. Turing degree. Then there is a finitely presented group whose word problem is solvable and whose conjugacy problem has degree* **a**.

8. *The conjugacy problem for K_4*

In this section we consider words which are not k-free. The main part of our treatment is based on that of Collins [**15**], §2 with his Γ replaced by our t.

Let K_h be the group obtained from K_4 by killing h. Then $K_h \cong H_3$ via the isomorphism which sends k to h and fixes the other generators. It now follows from §7 that if U, V are h-free then $?(U \sim_{K_4} V)$ is reducible to the confluence problem for M. Thus we need only look at words which are not h-free and not k-free.

We shall use D as a variable for $\{\, r_i, l_j, t^{-1}ht \,\}$-words throughout. We may assume that U, V are both ρ-reduced w.r.t.(K_4, K_3, k) and put

$$U \equiv X_0 k^{s_0} \cdots X_n k^{s_n},$$
$$V \equiv Y_0 k^{q_0} \cdots Y_d k^{q_d},$$

where $s_i, q_j \neq 0$, the words X_i, Y_j are k-free for $0 \leq i \leq n$, $0 \leq j \leq d$ and X_i, Y_j are non-empty for $i, j > 0$.

Lemma 8.1 *If $D_1 U D_2 = V$ for some D_1, D_2 then $d = n$ and $q_i = s_i$ for $0 \leq i \leq n$.*

Proof. Easy induction on $l_k(U)$. □

In view of this lemma we shall henceforth assume that $d = n$ and $q_i = s_i$ for $0 \leq i \leq n$. We consider the cases

1. $n > 0$,
2. $n = 0$ and $X_0 \notin A_K(k)$,
3. $U = D_1 k^s, V = D_2 k^s$ for some D_1, D_2.

Before dealing with these cases we note that $K(M)$ is defined by the following Britton tower:

$$
\begin{aligned}
T_0 &= \langle\, y \mid \quad \rangle, \\
T_1 &= \langle\, T_0, x \mid \; yx = xy \,\rangle, \\
T_2 &= \langle\, T_1, r_i, l_j \mid \; x^m r_i = r_i x^{m^2}, y^m r_i = r_i y, \\
&\qquad\qquad x^m l_j = l_j x, y^m l_j = l_j y^{m^2}; i \in I, j \in J \,\rangle, \\
T_3 &= \langle\, T_2, h \mid \; r_i h = h r_i, l_j h = h l_j;\ i \in I, j \in J \,\rangle, \\
T_4 &= \langle\, T_3, t \mid \; x^{c_i} r_i^{-1} x^{-a_i} y^{-b_i} t = t x^{c_i} r_i^{-1} x^{-a_i} y^{-b_i}, \\
&\qquad\qquad x^{c_j} l_j^{-1} x^{-a_j} y^{-b_j} t = t x^{c_j} l_j^{-1} x^{-a_j} y^{-b_j}; i \in I, j \in J \,\rangle, \\
T_5 &= \langle\, T_4, k \mid \; r_i k = k r_i, l_j k = k l_j, t^{-1} h t k = k t^{-1} h t k;\ i \in I, j \in J \,\rangle.
\end{aligned}
$$

Clearly we may assume that each D is pinch-reduced w.r.t. (T_4, T_3, t).

Case I. $n > 0$.

First we define two reduction procedures which will allow us to change U, V in such a way that we need look only at a simple kind of conjugating element.

α-reduction. Let X be k-free. If $X \equiv X' h^\epsilon R$ where R is h-free and $R = S_1 t S_2$ for some S_1, S_2 then $X' S_1 t$ is a *primitive α-reduction* of X. Clearly a sequence of primitive α-reductions must eventually halt in a word $\alpha(X)$. Moreover there is a D such that $X = \alpha(X) D$.

β-reduction. Dual to α-reduction. This time we have $X = D\beta(X)$.

(The corresponding procedures of **[15]** are called ρ- and λ-reduction.) We observe that $\alpha(X)$ and D are $H_0(M)$-computable. First of all note that $X = X' S_1 t t^{-1} h^\epsilon t S_2$ and so we need only show that primitive α-reduction is $H_0(M)$-recursive. We consider the equation $R = S_1 t S_2$. We may assume that X is pinch-reduced w.r.t (K_2, K_1, p) (since such a reduction is $H_0(M)$-computable). Now consider a word $S_1 t S_2$ where, as always, S_1, S_2 are freely reduced. The only possible pinch in this word is of form ptp^{-1} and this is a pinch if and only if p corresponds to a quadruple $(a, b, 0, P)$ of M (here $P \equiv R$ if $p \in \{\, r_i \,\}$ otherwise $P \equiv L$). Thus if there is a pinch we have $S_1 \equiv S_3 p, S_2 \equiv p^{-1} S_4$ and $S_1 t S_2 = S_3 t(a, b) S_4$ and by lemma 5.1 the right hand side is pinch-reduced. Thus the only candidates for S_1, S_2 in the equation $R = S_1 t S_2$ are words S_1', S_2' or $S_1' p, p^{-1} S_2'$ where $S_1' S_2' \equiv \pi_p(R)$.

Similar remarks apply to β-reduction.

We proceed to reduce U. We first find $\alpha(X_0)$ and D_0 such that $X_0 = \alpha(X_0)D_0$. We can commute D_0 past k^{s_0} and then find $\alpha(D_0X_1)$ and D_1 such that $D_0X_1 = \alpha(D_0X_1)D_1$. Iterating this procedure we obtain

$$U' \equiv X'_0 k^{s_0} \cdots X'_n k^{s_n} D_n$$

where X'_i is α-reduced for $0 \leq i \leq n$. Next we β-reduce X'_n to find D'_1 and X''_n such that $X'_n = D'_1 X''_n$. At this point we consider two subcases.

Subcase I. If $X''_n = S_1 t^{-1} S t S_2$ and $St \neq tS$ for some S_1, S, S_2 then we define $D^*_n = S_2 D_n$, $X^*_n = t^{-1}St$ and

$$U^* \equiv D^*_n X'_0 k^{s_0} X'_1 k^{s_1} \cdots X'_{n-1} D'_1 S_1 k^{s_{n-1}} X^*_n k^{s_n}.$$

Subcase II. Otherwise we define $X^*_n \equiv X''_n$ and

$$U^* \equiv D_n X'_0 k^{s_0} X'_1 k^{s_1} \cdots X'_{n-1} D'_1 k^{s_{n-1}} X^*_n k^{s_n}.$$

V^* is defined similarly.

It is clear that $U^* = D^{-1}UD$ for some D. Furthermore U^* is $H_0(M)$-computable from U. In order to see this we have to show that the question of whether or not there are S_1, S, S_2 such that $X''_n = S_1 t^{-1} S t S_2$ and $St \neq tS$ is $H_0(M)$-decidable. However this can be seen by an argument similar to the one used to show that α-reduction is $H_0(M)$-recursive.

LEMMA 8.2 *Suppose that $D^{-1}U^*D = V^*$ where $D \neq S$ for any S. Then Subcase I holds and $D = t^{-1}h^f tS$ for some $f \neq 0$ and S.*

Proof. Let $D \equiv S_1 t^{-1} h^\epsilon t D''$. Using k as a stable letter we obtain $X^*_n D = D' Y^*_n$ for some D'. We claim that X^*_n, Y^*_n are h-free. From the way in which U^*, V^* are defined X^*_n, Y^*_n are α- and β-reduced. This means that $X^*_n D$, $D Y^{*-1}_n$, $Y^{*-1}_n D'^{-1}$, $D'^{-1} X^*_n$ are all pinch-reduced w.r.t. (K_3, K_2, h). But $X^*_n D Y^{*-1}_n D'^{-1} = 1 = D'^{-1} X^*_n D Y^{*-1}_n$ and D is not h-free. This is contradictory unless both Y^*_n and X^*_n are h-free.

The equation $X^*_n D = D' Y^*_n$ now yields $X^*_n S_1 t^{-1} = S'_1 t^{-1} S_2$ for some S'_1, S_2 and hence $X^*_n = S'_1 t^{-1} S_2 t S_1^{-1}$. Also $S_2 t \neq t S_2$ since otherwise $X^*_n = S'_1 S_2 S_1^{-1}$ which contradicts the fact that U is ρ-reduced w.r.t. (K_4, K_3, k). This completes the proof that Subcase I holds.

For the rest of the lemma we have $X_n^* \equiv t^{-1}S^*t$ where $S^*t \neq tS^*$. Also $t^{-1}S^*tS_1t^{-1} = S_1't^{-1}S_2$. Using t as a stable letter we see that $S_1t = tS_1$ and so $D = t^{-1}h^\epsilon tS_1D''$. Now X_n^* commutes with $t^{-1}h^\epsilon t$ and if we define

$$U_1 \equiv t^{-1}h^{-\epsilon}tD_n^*X_0'k^{s_0}\cdots X_{n-1}'D_1'S_1t^{-1}h^\epsilon tk^{s_{n-1}}X_n^*k^{s_n}$$

then

1. $t^{-1}h^{-\epsilon}tU^*t^{-1}h^\epsilon t = U_1$ and
2. U_1 yields after a single h-pinch w.r.t. (H_4, H_3, h) a word U_1^* which ends in a k-symbol and is ρ-reduced w.r.t. (K_4, K_3, k).

The lemma now follows by iterating the above argument (lemma 8.1 ensures that we are still in Case I). □

Case II. $n = 0$ and $X_0 \notin A_K(k)$.

This case is similar to Case I. We have $U \equiv Xk^s, V \equiv Yk^s$ where neither X nor Y is a D. First we transform U into $X'D_1k^s$ where $X' = \alpha(X)$ and then into $D_1'X''k^sD_1$ where $X'' = \beta(X')$. If there are S_1, S, S_2 such that $X'' = S_1t^{-1}StS_2$ and $St \neq tS$ then we put $D_2 \equiv S_2D_1D_1'S_1$, $X^* \equiv t^{-1}St$ and $U^* \equiv D_2X^*k^s$. Otherwise $D_2 \equiv D_1D_1'$, $X^* \equiv X''$ and $U^* \equiv D_2X^*k^s$. We define V^* similarly.

Clearly U^*, V^* are $H_0(M)$-computable from U, V respectively. Moreover U, V are conjugate by some D if and only if U^*, V^* are also conjugate by some D.

Lemma 8.3 *If X^* is not h-free and V^* is a right conjugate of U^* by D w.r.t. (H_4, H_3, h) then D is an S.*

Proof. This follows from the fact that X^* is α-reduced. □

Lemma 8.4 *Suppose that X^* is h-free and V^* is a right conjugate of U^* by D w.r.t. (H_4, H_3, h) where $D \neq S$ for any S. Then $D \equiv t^{-1}h^ftS$ for some $f \neq 0$ and S.*

Proof. Using the fact that V^* is a right conjugate of U^* by D we can show that $X^* \equiv t^{-1}St$ where $St \neq tS$. Let $D_2 \equiv D_2't^{-1}h^{-\delta}tS_0$, $D \equiv S_1t^{-1}h^\epsilon tD'$ so that $\delta = \epsilon$ and $tS_0t^{-1}Stk^sS_1t^{-1} \in A_H(h) = \langle r_i, l_j, tkt^{-1} \rangle$. By writing this word in the generators of $A_H(h)$ and removing k-pinches we find that $tS_0t^{-1}StS_1t^{-1} = S_2$ for some S_2. Thus S_0 and S_1 both commute with t and so $D_2 = D_2'S_0t^{-1}h^{-\delta}t, D = t^{-1}h^\epsilon tS_1D'$. Since X^* commutes with $t^{-1}ht$ the lemma follows by induction on $l_h(D)$. □

Before dealing with Case III we provide a further reduction for Cases I and II. Lemmas 8.2, 8.3 and 8.4 show that in order to decide whether or not there is a D such that $D^{-1}U^*D = V^*$ we need only look at elements D of form $t^{-1}h^f tS$. The next lemma puts a bound on $|f|$ and thus reduces the problem to looking for a D of form S.

LEMMA 8.5 *Let $l_h(U) = g$ and $|f| > g$. If $h^{-f}tUt^{-1}h^f$ has $|f|$ h-pinches w.r.t. (H_4, H_3, h) then $h^{-f}tUt^{-1}h^f = h^{-g}tUt^{-1}h^g$ when $f > 0$ and $h^{-f}tUt^{-1}h^f = h^g tUt^{-1}h^{-g}$ when $f < 0$*

Proof. This follows from the fact that h induces the identity isomorphism in (H_4, H_3, h). □

Case III. $U = D_1 k^s, V = D_2 k^s$ for some D_1, D_2.

It is now convenient to extend the range of each D so that it can be any product of the symbols $r_i^{\pm 1}, l_j^{\pm 1}, t^{-1}h^n t$ for $n \in \mathbf{Z} - \{0\}$. We put

$$D_1 \equiv S_{10}t^{-1}h^{q_1}tS_{11} \cdots t^{-1}h^{q_d}tS_{1d}.$$

Since D_1 is pinch-reduced w.r.t. (T_4, T_3, t) it follows that $S_{11}, \ldots, S_{1,d-1}$ do not commute with t. Furthermore we may assume that D_1, D_2 are cyclically pinch-reduced w.r.t. (K_3, K_2, h). We transform D_1 according to two cases:

1. $d = 1$. We define $D_1^* \equiv S_{11}S_{10}t^{-1}h^{q_1}t$.

2. $d > 1$. If $S_{1d}S_{10}$ does not commute with t then we define $D_1^* \equiv S_{1d}S_{10}t^{-1}h^{q_1}t$ $S_{11} \cdots t^{-1}h^{q_d}t$ otherwise $D_1^* \equiv S_{1d}S_{10}S_{11}t^{-1}h^{q_2}tS_{12} \cdots t^{-1}h^{q_d+q_1}t$.

Of course D_1^* is cyclically pinch-reduced w.r.t. (K_3, K_2, h). Let $U^* \equiv D_1^*k^s$ and define $V^* = D_2^*k^s$ similarly. Clearly U, V are conjugate by some D if and only if U^*, V^* are also conjugate by some D. If $d > 1$ then D_1^* is cyclically pinch-reduced w.r.t. (T_4, T_3, t) unfortunately this need not be so if $d = 1$.

LEMMA 8.6 *Suppose that $d = 1$ and D_1^* is not cyclically pinch-reduced w.r.t. (T_4, T_3, t). If there is a D such that $D^{-1}U^*D = V^*$ then there is an S such that $S^{-1}U^*S = V^*$*

Proof. We have $D_1^* \equiv S_1t^{-1}h^q t$ where $S_1t = tS_1$ (this implies that $S_1 \in \{r_i, l_j\}$ but we do not need this). We shall show that $D = t^{-1}h^f tS$ from which it will follow that $S^{-1}U^*S = V^*$. If V^* is a right conjugate of U^* by D w.r.t. (H_4, H_3, h) then we write

$D \equiv S_2 t^{-1} h^f t D'$ and deduce that $tS_2 = S_2 t$ as in lemma 8.2. The lemma then follows by induction.

If V^* is a left conjugate of U^* by D w.r.t. (H_4, H_3, h) then we obtain $tS_2^{-1} S_1 t^{-1} = S_3$ for some S_3. Since $S_1 t = tS_1$ we have $tS_2^{-1} t^{-1} S_1 = S_4$ whence it follows that $S_2 t = tS_2$ and again induction completes the proof. □

LEMMA 8.7 *Suppose that D_1^*, D_2^* are both cyclically pinch-reduced w.r.t. (T_4, T_3, t). Then there is a D such that $D^{-1} U^* D = V^*$ if and only if there exist $\widetilde{D}_1, \widetilde{D}_2$ and S such that*

1. *$\widetilde{D}_1, \widetilde{D}_2$ are cyclic permutations of D_1^*, D_2^* respectively,*
2. *$S^{-1} \widetilde{D}_1 k^s S = \widetilde{D}_2 k^s$.*

Proof. If there exist $\widetilde{D}_1, \widetilde{D}_2$ and S which satisfy the stated conditions then certainly there is a D such that $D^{-1} U^* D = V^*$.

Conversely suppose that $D^{-1} U^* D = V^*$. Then $D^{-1} D_1^* D = D_2^*$ and this holds in K_3. Since D_1^*, D_2^* are cyclically pinch-reduced w.r.t. (K_3, K_2, h) Collins' lemma implies that there exist X, Y, S such that $S^{-1} X S = Y$ where X, Y are of form

$$X \equiv tS_{1i} t^{-1} h^{q_{i+1}} t S_{1,i+1} \cdots S_{1,i-1} t^{-1} h^{q_i},$$
$$Y \equiv tS_{2j} t^{-1} h^{q'_{j+1}} t S_{2,j+1} \cdots S_{2,j-1} t^{-1} h^{q'_j}.$$

We show that $St = tS$ by using t as a stable letter. First note that X, Y are pinch-reduced w.r.t. (T_4, T_3, t) since D_1^*, D_2^* are both cyclically pinch-reduced w.r.t. (T_4, T_3, t). From the equality $S^{-1} X S = Y$ we deduce that $t^{-1} S^{-1} t$ is a pinch so that $St = tS$. The lemma follows if we define

$$\widetilde{D}_1 \equiv S_{1i} t^{-1} h^{q_{i+1}} t S_{1,i+1} \cdots S_{1,i-1} t^{-1} h^{q_i} t,$$
$$\widetilde{D}_2 \equiv S_{2j} t^{-1} h^{q'_{j+1}} t S_{2,j+1} \cdots S_{2,j-1} t^{-1} h^{q'_j} t.$$

□

Lemmas 8.6 and 8.7 show that given U^*, V^* we can compute a finite set of words $\{ (U_i^*, V_i^*) \}$ such that $U^* \sim_{K_4} V^*$ if and only if for some i there is an S such that $S^{-1} U_i^* S = V_i^*$.

We have thus seen that in each case we can reduce $?(U \sim_{K_4} V)$ to the question of whether or not there is an S such that $S^{-1} U' S = V'$ where U', V' are $H_0(M)$-computable from U, V respectively. It remains to show that the latter problem is reducible to the confluence problem for M.

First of all we assume that U', V' are both p-free. Put

$$U' \equiv Z_0 k^{\epsilon_1} Z_1 \cdots k^{\epsilon_n} Z_n,$$
$$Z_i \equiv Z_{i0} h^{\delta_{i1}} Z_{i1} \cdots h^{\delta_{iq_i}} Z_{iq_i},$$

where each Z_{ij} is p, h, k-free.

LEMMA 8.8 *$U' \in A_P(p^\epsilon)$ if and only if $Z_{ij} \in A_K(p^\epsilon)$ for all i, j. Furthermore $\mathcal{C}^*(p^{-\epsilon}U'p^\epsilon) \equiv Z'_0 k^{\epsilon_1} Z'_1 \cdots k^{\epsilon_n} Z'_n$ where each Z'_i is the word obtained from Z_i by the direct application of p^ϵ to each Z_{ij}.*

Proof. If each $Z_{ij} \in A_K(p^\epsilon)$ then clearly $U' \in A_P(p^\epsilon)$. For the converse of this, lemma 7.1 shows that it suffices to show that $Z_i \in A_P(p^\epsilon)$ for $0 \leq i \leq n$. Write U' as a word T in the generators of $A_P(p^\epsilon)$. We may assume that T is pinch-reduced w.r.t. (K_4, K_3, k). Thus $T \equiv T_0 k^{\epsilon_1} T_1 \cdots k^{\epsilon_n} T_n$ where $T_i \in A_P(p^\epsilon)$ for $0 \leq i \leq n$. Put $T'_i \equiv \mathcal{C}^*(T_i)$ and $T' \equiv T'_0 k^{\epsilon_1} T'_1 \cdots k^{\epsilon_n} T'_n$. Of course $T', T'_i \in A_P(p^\epsilon)$ for $0 \leq i \leq n$. If T' is in $\mathcal{C}^*$-normal form then we are done. Otherwise the only possible forbidden subwords are of the type $h^\delta t(\alpha, \beta) k^{\epsilon_i}$ where $(\alpha, \beta) \in H_0(M)$. Let the leftmost one occur as $T''_i h^\delta t(\alpha, \beta) k^{\epsilon_i}$ where $T'_i \equiv T''_i h^\delta t(\alpha, \beta)$. Lemma 7.1 implies that $t(\alpha, \beta) \in A_K(p^\epsilon)$. Moreover the forbidden subword is transformed to $t(\alpha, \beta) k^{\epsilon_i} t^{-1}(\alpha, \beta) h^\delta t(\alpha, \beta)$. If we note that $T''_i t(\alpha, \beta), t^{-1}(\alpha, \beta) h^\delta t(\alpha, \beta) T_{i+1} \in A_P(p^\epsilon)$ then the proof is complete by induction on the number of steps required to put T' into $\mathcal{C}^*$-normal form.

For the second part we have $p^{-\epsilon}U'p^\epsilon = Z'_0 k^{\epsilon_1} Z'_1 \cdots k^{\epsilon_n} Z'_n$. By lemma 7.3 each Z'_i is $\mathcal{C}^*$-normal. It therefore suffices to show that no $Z'_i k^{\epsilon_{i+1}}$ has a forbidden subword for $0 \leq i < n$. If there is such a subword then it must be of form $h^\delta t(\alpha', \beta') k^\epsilon$ where $(\alpha', \beta') \in H_0(M)$. However this means that U' has the subword $h^\delta t(\alpha, \beta) k^\epsilon$ where $p^{-\epsilon} t(\alpha, \beta) p^\epsilon = t(\alpha', \beta')$. Moreover $(\alpha, \beta) \in H_0(M)$ so that U' is not $\mathcal{C}^*$-normal, contrary to assumption. □

LEMMA 8.9 *The problem of deciding whether or not there is an S such that $S^{-1}U'S = V'$ is reducible to the confluence problem for M.*

Proof. Follows from lemmas 8.8, 7.4 and theorem 2.2. □

If U', V' are not p-free then by using λ- and ρ-reduction w.r.t. (P_4, P_3, p) we may reduce $?(U' \sim_{K_4} V')$ to the question of whether or not there is a $g \geq 0$ such that $S^{-g}U''S^g = V''$ where U'', V'' are p-free and $S \equiv \pi_p(U')$. We may also assume that U'', V'' are $\mathcal{C}^*$-normal.

Lemma 8.10 *The question of whether or not there is a $g > 0$ such that $S^{-g}U''S^g = V''$ is solvable.*

Proof. This follows from lemma 8.8, part 1 of 7.4 and lemma 7.7. □

This completes the proof of theorem 1.3 and of Collins' theorem.

References

1. S. Aanderaa and D. E. Cohen, 'Modular machines, the word problem for finitely presented groups and Collins' theorem', *Word problems II, the Oxford book* (ed. S. I. Adian, W. W. Boone, G. Higman, North-Holland, Amsderdam, 1980), pp. 1–16.

2. S. Aanderaa and D. E. Cohen, 'Modular machines and the Higman–Clapham–Valiev embedding theorem', *Word problems II, the Oxford book* (ed. S. I. Adian, W. W. Boone, G. Higman, North-Holland, Amsderdam, 1980), pp. 17–28.

3. L. A. Bokut', 'On a property of the Boone groups' (in Russian), *Algebra i Logika Sem.*, 5 (1966) No. 5, 5–23; 6 (1967) No.1, 15–24.

4. L. A. Bokut', 'Degrees of unsolvability of the conjugacy problem for finitely presented groups' (in Russian), *Algebra i Logika Sem.*, 7 (1968), no. 5, 4–70; no. 6, 4–52.

5. L. A. Bokut', 'Malcev's problem and groups with a normal form', *Word problems II, the Oxford book* (ed. S. I. Adian, W. W. Boone, G. Higman, North-Holland, Amsderdam, 1980), pp. 29–53.

6. W. W. Boone, 'Certain simple unsolvable problems of group theory', *Indag. Math.*,16 (1954), 231–237, 492–497; 17 (1955), 252–256, 571–577; 19 (1957), 22–27, 227–232.

7. W. W. Boone, 'The word problem', *Ann. of Math.*, 70 (1959), 207–265.

8. W. W. Boone, 'Word problems and recursively enumerable degrees of unsolvability. A sequel on finitely presented groups', *Ann. of Math.*, 84 (1966), 49–84.

9. J. L. Britton, 'The word problem', *Ann. of Math.*, 77 (1963), 16–32.

10. C. R. J. Clapham, 'Finitely presented groups with word problems of arbitrary degrees of unsolvability', *Proc. London Math. Soc.* (3), 14 (1964), 633–676.

11. C. R. J. Clapham, 'An embedding theorem for finitely generated groups', *Proc. London Math. Soc.* (3), 17 (1967), 419–430.

12. D. E. Cohen, 'Degree problems for modular machines', *J. Symb. Logic*, 45 (1980), 510–528.

13. D. J. Collins, 'Recursively enumerable degrees and the conjugacy problem', *Acta. Math.*, 122 (1969), 115–160.

14. D. J. Collins, 'Truth-table degrees and the Boone groups', *Annals of Math.*, 94 (1971), 392–396.

15. D. J. Collins, 'Representation of Turing reducibility by word and conjugacy problems in finitely presented groups', *Acta. Math.*, 128 (1972), 73–90.

16. A. A. Fridman, 'On the relation between the word problem and the conjugacy problem in finitely defined groups' (in Russian), *Trudy Moskov. Mat. Obšč.*, 9 (1960), 329–365.

17. A. A. Fridman, 'Degrees of unsolvability of the word problem in finitely defined groups' (in Russian), *Dokl. Akad. Nauk SSSR*, 147 (1962), 805–808; English translation in *Soviet Math. Dokl.*, 3 (1962), 1733–1737

18. A. A. Fridman, *Degrees of unsolvability of the word problem for finitely presented groups* (in Russian, Izdatel'stvo Nauka, Moscow, 1967).

19. R. M. Friedberg, 'Two recursively enumerable sets of incomparable degrees of unsovability (solution of Post's problem, 1944), *Proc. Nat. Acad. Sci. U.S.A.*, 43 (1957), 236–238.

20. G. Higman, 'Subgroups of finitely presented groups', *Proc. Roy. Soc. London Ser. A*, 262 (1961), 455–475.

21. K. A. Kalorkoti, 'Decision problems in group theory', *Proc. London Math. Soc.* (3), 44 (1982), 312–332.

22. C. F. Miller III, 'Conjugacy and word problems in finitely generated groups', *J. Symb. Logic*, 32 (1967), 440–441.

23. C. F. Miller III, 'Characterisation of recursively enumerable degrees by conjugacy and word problems for finitely generated groups', *J. Symb. Logic*, 32 (1967), 441.

24. C. F. Miller III, *On group-theoretic decision problems and their classification* (Annals of Mathematics Studies, no. 68, Princeton University Press, Princeton N. J., 1971).

25. A. A. Mucnik, 'Negative answer to the problem of reducibility of the theory of algorithms' (in Russian), *Dokl. Akad. Nauk SSSR*, 108 (1956), 194–197.

26. P. S. Novikov, 'Unsolvability of the conjugacy problem in the theory of groups' (in Russian), *Izv. Akad. Nauk SSSR, Ser. Math.*, 18 (1954), 485–524.

27. P. S. Novikov, 'On the algorithmic unsolvability of the word problem in group theory' (in Russian), *Trudy Mat. Inst. Steklov*, 44 (1955).

28. M. K. Valiev, 'On polynomial reducibility of the word problem under embedding of recursively presented groups in finitely generated groups', Lecture Notes in Computer Science 32, (ed. J. Bečvář, Springer, Berlin, 1975), pp. 432–438.

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