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# On the Hopf algebra structure of the AdS/CFT S-matrix

Jan Plefka, Fabian Spill and Alessandro Torrielli

Humboldt–Universität zu Berlin, Institut für Physik
Newtonstraße 15, D-12489 Berlin

plefka,spill,torriell@physik.hu-berlin.de



## Abstract

We formulate the Hopf algebra underlying the $\mathfrak{su}(2|2)$ S-matrix relevant for the AdS/CFT correspondence, extending the one discovered by Janik in the $\mathfrak{su}(1|2)$ sector to the full algebra, by specifying the action of the coproduct and the antipode on the remaining generators. The nontriviality of the coproduct is determined by the lenght-changing effect, and results in an unusual central braiding. We fully determine the antiparticle representation by means of the derived antipode.

# 1 Introduction

Integrable structures play a very important role in the most recent developments concerning the AdS/CFT correspondence. After the discovery of one-loop integrability of the planar dilatation operator in the scalar sector of four-dimensional $\mathcal{N} = 4$ SYM theory [1], important progress has been made in uncovering integrable structures in both the gauge and the string theory side of the correspondence. On the gauge theory side a a huge effort has been spent in extending it to higher orders in perturbation theory as well as to the full set of field excitations [2]. Analogously, integrability has been established at the classical level of the string theory [3] and first steps to quantum integrability have been undertaken [5]. As a result of this work, the S-matrix of the problem has been fully fixed up to an overall undetermined scalar factor, which cannot be constrained only via symmetry arguments, and whose form should interpolate between the known expression obtained perturbatively on both sides of the correspondence.

The S-matrix of AdS/CFT was shown to possess a centrally extended $\mathfrak{su}(2|2) \otimes \mathfrak{su}(2|2)$ symmetry by Beisert [6]. It is completely fixed up to a scalar factor, and can be written in a manifestly $\mathfrak{su}(1|2)$ symmetric form, in terms of a combination of projectors into irreducible representations of the tensor product $\mathfrak{su}(1|2) \otimes \mathfrak{su}(1|2)$. The general structure of the phase factor can be written in terms of the local charges of the theory, with coefficients function of the ‘t Hooft coupling [7]. First orders in the expansion of these coefficients can be determined comparing with perturbation theory [8]. A remarkable progress in trying to determine the scalar factor was made by Janik, who advocated an underlying Hopf algebra structure from which he derived an equation analog to the crossing symmetry condition of relativistic S-matrices [9]. In order to derive this equation he made use of the antipode acting on $\mathfrak{su}(1|2)$ generators, where one can avoid lenght-changing effects. The resulting conditions, together with unitarity, put strong constraints on the form of the scalar factor. This equation was checked against the known perturbative expansion and agreement was found [10] (see also [11]).

Unfortunately these equations do not constrain the overall factor completely, and some new insights are needed. In this paper we take a step further and described the Hopf algebra structure underlying the full $\mathfrak{su}(2|2)$ symmetry algebra. This can be done by simply analyzing the condition of invariance of the $\mathfrak{su}(1|2)$ S-matrix under the remaining dynamic $\mathfrak{su}(2|2)$ generators. The effect of lenght-changing operators produces momentum dependent phase factors which can be interpreted as producing a nontrivial coproduct. We derive the antipode and verify the full Hopf algebra axioms. By using the derived antipode we are able to fully determine the antiparticle representation directly from the Hopf algebra structure.

The most important step after this would be to determine the Universal R-matrix of the problem. Hints at his existence come from the fact that we already have a representation of it. The requirement of the quasi- triangularity condition would lead to an expression of the Universal R-matrix which would solve the crossing relation directly at the algebraic level, withour making use of representation-dependent coefficients. This could be a very powerful and clean way of fixing the structure of the overall factor, and would teach a lot about the origin of this symmetry from the string theory perspective.

# 2 Deriving the Coproduct

Working in the $\mathfrak{su}(1|2)$ language, the S-matrix can be determined as a combination of projectors into the irreducible representations of the tensor product $\mathfrak{su}(1|2) \otimes \mathfrak{su}(1|2)$, weighted with representation-dependent coefficients. Invariance of the S-matrix under all $\mathfrak{su}(1|2)$ generators amounts in fact to impose a trivial coproduct condition. More specifically, let us indicate collectively with $\mathfrak{T}$ any of the $\mathfrak{su}(1|2)$ generators $\mathfrak{R}^1{}_1$, $\mathfrak{R}^2{}_2 = -\mathfrak{R}^1{}_1$, $\mathfrak{L}^\alpha{}_\beta$, $\mathfrak{Q}^1{}_1$, $\mathfrak{Q}^2{}_1$, $\mathfrak{S}^1{}_1$ and $\mathfrak{S}^1{}_2$, together with the central charge $\mathfrak{C}$. The ambient space we set is the universal enveloping algebra $U(\mathfrak{su}(2|2))$ containing generators of the Lie algebra and all their products. Together with the (undeformed) defining commutation relations, and the standard notion of unit, which define the multiplication structure, we will endow this space with a coalgebra structure by specifying the coproduct and the counit, making it a bialgebra. Finally, the antipode will determine the Hopf algebra structure on $U(\mathfrak{su}(2|2))$.

Invariance of The S-matrix under $\mathfrak{T}$ amounts to the following condition [6]

$$[\mathfrak{T}_1 + \mathfrak{T}_2, \mathcal{S}_{12}] = 0. \tag{1}$$

This can be rewritten in terms of an R-matrix

$$\mathcal{S} = \sigma \circ \mathcal{R}, \tag{2}$$

where $\sigma(a \otimes b) = b \otimes a$, as a coproduct relation

$$\Delta^{op}(\mathfrak{T})\, \mathcal{R} = \mathcal{R}\, \Delta(\mathfrak{T}), \tag{3}$$

with the coproduct $\Delta$ defined as

$$\Delta(\mathfrak{T}) = (\mathfrak{T} \otimes id + id \otimes \mathfrak{T}) \tag{4}$$

and $\Delta^{op} = \sigma \circ \Delta$. One recalls that $\mathcal{S} : V_1 \otimes V_2 \to V_2 \otimes V_1$, $V_i$ being modules for $\mathfrak{su}(2|2)$ representations. In order to get eq. (1) from eq.s (3) and (4) one has simply to project on some definite representations 1 and 2 the two factors of the tensor product in the abstract algebra. We recall that a bialgebra such that the opposite coproduct $\Delta^{op}$ is equal to the coproduct $\Delta$ is referred to as "co-commutative".

An R-matrix of the form

$$\mathcal{R}_{12} = \sum_i S_i P_i, \tag{5}$$

where $P_i$ are the projectors into the irreducible representations of $\mathfrak{su}(1|2) \otimes \mathfrak{su}(1|2)$, and $S_i$ are arbitrary coefficients, solves eq. (1). There are three such projectors, whose expression in terms of the quadratic $\mathfrak{su}(1|2)$ Casimir can be found in [9]. The coefficients $S_i$ are then fixed by requiring invariance under the remaining generators of $\mathfrak{su}(2|2)$ which are not in the $\mathfrak{su}(1|2)$ subalgebra[1], namely $\mathfrak{R}^2{}_1$, $\mathfrak{R}^1{}_2$, $\mathfrak{Q}^1{}_2$, $\mathfrak{Q}^2{}_2$, $\mathfrak{S}^2{}_1$ and $\mathfrak{S}^2{}_2$, together with the central charges $\mathfrak{P}$ and $\mathfrak{K}$. These generators are called "dynamic". All these generators change the length of the spin chain when acting on all magnons. In order to close their action on states of a same chain, one can use the basic relation (2.13) of [6] to move all length-changing operators to the right of all excitations, and exploit the limit of infinite chain. This produces the appearance of braiding factors. For example, invariance under $\mathfrak{Q}^1{}_2$ leads to eq. (34) of [9],

$$(e^{-ip_1}\, [\tilde{\mathfrak{Q}}^1{}_2]_2 \otimes id_1 + id_2 \otimes [\tilde{\mathfrak{Q}}^1{}_2]_1)\mathcal{S} = \mathcal{S}(e^{-ip_2}\, [\tilde{\mathfrak{Q}}^1{}_2]_1 \otimes id_2 + id_1 \otimes [\tilde{\mathfrak{Q}}^1{}_2]_2), \tag{6}$$

[1] In a quite different context, this is however similar to the standard Jimbo-equation procedure for determining the R-matrix for quantum affine algebras, see for instance[12].

where the subscript indicates the representations, and the tilded version of the operator means the same action of the untilded one but disregarding length-changing effects, which are taken into account by the braiding factors. In this case one defines (cfr. [9])

$$\tilde{\mathfrak{Q}}^1{}_2 = a\,|\psi^1 >< \chi| - b\,|\varphi >< \psi^2|. \tag{7}$$

One can rewrite eq. (6) in terms of a deformed coproduct

$$\Delta^{op}(\tilde{\mathfrak{Q}}^1{}_2)\,\mathcal{R} = \mathcal{R}\,\Delta(\tilde{\mathfrak{Q}}^1{}_2), \tag{8}$$

with the coproduct $\Delta$ defined as

$$\Delta(\tilde{\mathfrak{Q}}^1{}_2) = (\tilde{\mathfrak{Q}}^1{}_2 \otimes e^{-ip} + id \otimes \tilde{\mathfrak{Q}}^1{}_2). \tag{9}$$

We have lifted the coproduct relation to a representation-independent level, where we understand now $e^{-ip}$ as a central element of the (universal enveloping algebra of) the $\mathfrak{su}(2|2)$ algebra[2]. One can immediately realize that such coproduct and its opposite are different. The existence of an element of the tensor algebra $U(\mathfrak{su}(2|2)) \otimes U(\mathfrak{su}(2|2))$ such that (8) holds makes the bialgebra "quasi-cocommutative".

The presence of $e^{-ip}$ is connected to the fact that the dynamic generator $\mathfrak{Q}^1{}_2$ *add* one $\mathcal{Z}$ field to the chain. Let us spell out the coproducts one obtains in an analog way for all the other dynamic generators :

$$\begin{aligned}
\Delta(\tilde{\mathfrak{Q}}^2{}_2) &= (\tilde{\mathfrak{Q}}^2{}_2 \otimes e^{-ip} + id \otimes \tilde{\mathfrak{Q}}^2{}_2),\\
\Delta(\tilde{\mathfrak{S}}^2{}_1) &= (\tilde{\mathfrak{S}}^2{}_1 \otimes e^{ip} + id \otimes \tilde{\mathfrak{S}}^2{}_1),\\
\Delta(\tilde{\mathfrak{S}}^2{}_2) &= (\tilde{\mathfrak{S}}^2{}_2 \otimes e^{ip} + id \otimes \tilde{\mathfrak{S}}^2{}_2),\\
\Delta(\tilde{\mathfrak{R}}^1{}_2) &= (\tilde{\mathfrak{R}}^1{}_2 \otimes e^{-ip} + id \otimes \tilde{\mathfrak{R}}^1{}_2),\\
\Delta(\tilde{\mathfrak{R}}^2{}_1) &= (\tilde{\mathfrak{R}}^2{}_1 \otimes e^{ip} + id \otimes \tilde{\mathfrak{R}}^2{}_1).
\end{aligned} \tag{10}$$

One can notice the conjugate braiding $e^{ip}$ for the generators *subtracting* one $\mathcal{Z}$ field to the chain.

The central charges $\mathfrak{P}$ and $\mathfrak{K}$ also add length-changing operators to all states. Therefore, their coproduct should also be deformed in the following fashion[3]

$$\begin{aligned}
\Delta(\tilde{\mathfrak{P}}) &= (\tilde{\mathfrak{P}} \otimes e^{-ip} + id \otimes \tilde{\mathfrak{P}}),\\
\Delta(\tilde{\mathfrak{K}}) &= (\tilde{\mathfrak{K}} \otimes e^{ip} + id \otimes \tilde{\mathfrak{K}}).
\end{aligned} \tag{11}$$

# 3 The deformed Hopf Algebra structure

## 3.1 The Coproduct

We have seen in the previous section that in order to implement the length changing effects on the algebraic level we deform the universal enveloping algebra of our symmetry algebra. We do it

[2]One could make use of the natural choice $e^{-ip} = 1 + \frac{1}{\alpha}\mathfrak{P}$ imposed by physical requirement, where $\alpha$ is a parameter related to the coupling constant, and another similar relation for $e^{ip} = 1 + \frac{1}{\beta}\mathfrak{K}$ .

[3]Making use of these equations one can realize that $\Delta^{op} = \Delta$ on $\tilde{\mathfrak{P}}$ and $\tilde{\mathfrak{K}}$. Together with their centrality, this makes their coproduct relation with the R-matrix automatically satisfied for them.

in such a way that the ordinary commutation relations remain unchanged, so only the coalgebra structure gets modified. Since one has no length changing effects on the $\mathfrak{su}(1|2)$ sector, one define the coproduct to be the trivial one:

$$\Delta(J) = 1 \otimes J + J \otimes 1 \qquad \forall J \in su(1|2) \tag{12}$$

For $D \in \mathfrak{su}(2|2)/\mathfrak{su}(1|2)$ we need to introduce a braiding $B(D)$ to account for those length changing effects:

$$\Delta(D) = 1 \otimes D + D \otimes B(D) \tag{13}$$

In the following we check that this indeed gives a consistent Hopf Algebra structure, provided that we also modify the antipode $S$, and that there are consistent relations between the different braiding factors $B(D)$. First let us analyse the commutation relations of $\mathfrak{su}(2|2)$ [6]. We have

$$[\mathfrak{su}(1|2), \mathfrak{su}(1|2)] \subseteq \mathfrak{su}(1|2) \tag{14}$$
$$[\mathfrak{su}(1|2), \mathfrak{su}(2|2)] \subseteq \mathfrak{su}(2|2) \tag{15}$$
$$[\mathfrak{su}(2|2), \mathfrak{su}(2|2)] \subseteq \mathfrak{su}(1|2) \tag{16}$$
$$\tag{17}$$

Let $J \in \mathfrak{su}(1|2)$, $D \in \mathfrak{su}(2|2)/\mathfrak{su}(1|2)$, then $W := [J, D] \in \mathfrak{su}(2|2)/\mathfrak{su}(1|2)$[4]. The coproduct is required to be a homomorphism, that is, it has to respect the commutation relations:

$$\Delta W = \Delta([J, D]) = [\Delta J, \Delta D] \tag{18}$$

This equality holds iff

$$B(D) = B(W). \tag{19}$$

Similarly, if $D_1, D_2 \in \mathfrak{su}(2|2)/\mathfrak{su}(1|2)$, we again demand the equality

$$\Delta([D_1, D_2]) = [\Delta D_1, \Delta D_2] \tag{20}$$

Thus we conclude that

$$B(D_1) = B(D_2)^{-1} \tag{21}$$

whenever the commutator doesn't vanish, in particular, we expect the braiding to be an invertible element.

The commutation relations imply

$$B(\mathfrak{Q}_2^1) = B(\mathfrak{Q}_2^2) \tag{22}$$
$$B(\mathfrak{R}_1^2) = B(\mathfrak{S}_\alpha^2) \tag{23}$$
$$B(\mathfrak{Q}_2^\alpha) = B(\mathfrak{R}_2^1) \tag{24}$$
$$B(\mathfrak{R}_2^1) = B^{-1}(\mathfrak{S}_\alpha^2) \tag{25}$$
$$B(\mathfrak{Q}_2^\alpha) = B^{-1}(\mathfrak{S}_\beta^2) \tag{26}$$

Our result is that there can be just one braiding factor $B \equiv B(\mathfrak{Q}_2^1)$ and it's inverse, consistently with the analysis of the previous section.

[4] $[A, B]$ denotes the usual supercommutator: $[A, B] := AB - (-1)^{deg(A)deg(B)} BA$

Now let us derive the braiding of the central charges $\mathfrak{P}, \mathfrak{K}$ and $\mathfrak{C}$, which we can read off from the relations

$$\begin{aligned} \{\mathfrak{Q}_a^\alpha, \mathfrak{Q}_b^\beta\} &= \epsilon^{\alpha\beta}\epsilon_{ab}\mathfrak{P} && (27) \\ \{\mathfrak{S}_\alpha^a, \mathfrak{S}_\beta^b\} &= \epsilon_{\alpha\beta}\epsilon^{ab}\mathfrak{K} && (28) \\ \mathfrak{C} &= \frac{1}{2}\left([\mathfrak{Q}_1^1, \mathfrak{S}_1^1] - [\mathfrak{Q}_2^2, \mathfrak{S}_2^2]\right). && (29) \end{aligned}$$

We get

$$\begin{aligned} B(\mathfrak{P}) &= B(\mathfrak{Q}_2^\alpha) && (30) \\ B(\mathfrak{K}) &= B(\mathfrak{S}_\beta^2), && (31) \end{aligned}$$

while $\mathfrak{C}$ remains unbraided. This is again consistent with what we argued before.

The action of the coproduct on the braiding $B$ is determined from

$$(\Delta \otimes id)(\Delta(A)) = (id \otimes \Delta)(\Delta(A)), \tag{32}$$

which has to be satisfied by every coalgebra. We get

$$\Delta(B) = B \otimes B, \tag{33}$$

which means that our braiding is grouplike.

## 3.2 The Antipode and the Counit

The Antipode has to fullfill the equation

$$\mu(S \otimes id)\Delta(A) = \mu(id \otimes S)\Delta(A) = i \circ \epsilon(A) \tag{34}$$

Here $\epsilon$ denotes the Counit,which is given by $\epsilon(A) = 0 \quad \forall A \in \mathfrak{su}(2|2) \ltimes R^2$, $i$ is the unit and $\mu$ the multiplication. For $J \in \mathfrak{su}(1|2)$ the antipode is the trivial one:

$$S(J) = -J \tag{35}$$

If $D \in \mathfrak{su}(2|2)/\mathfrak{su}(1|2)$, we expect the braiding to appear in the antipode. Indeed,

$$\mu(S \otimes id)\Delta(D) = S(D)B(D) + D = 0 \tag{36}$$

gives

$$S(D) = -DB^{-1}(D) \tag{37}$$

Furthermore, the action of $S$ on the braiding itself is

$$S(B(D)) = B^{-1}(D). \tag{38}$$

¿From

$$(id \otimes \epsilon)\Delta(A) = A = (\epsilon \otimes id)\Delta(A) \tag{39}$$

we see that

$$\epsilon(B) = 1. \tag{40}$$

### 3.3 Charge Conjugation

For the representations we use the same convention as [6].The represented generators $\pi(A)$ act on the 4-dimensional graded vectorspace spanned by $|\phi>, |\chi>, |\psi^1>, |\psi^2>$, and the representations are labled by the numbers $a, b, c, d$ , with the constraint $ad - bc = 1$. For the $\mathfrak{su}(1|2)$ subalgebra we get

$$\begin{aligned}
\mathfrak{Q}_1^\alpha &= a\,|\psi^\alpha><\phi| + b\,\epsilon^{\alpha\beta}|\chi><psi^\beta| &(41)\\
\mathfrak{S}_\alpha^1 &= c\,\epsilon_{\alpha\beta}|\psi^\beta><\chi| + d\,|\phi><\psi^\alpha| &(42)\\
\mathfrak{R}_1^1 &= \frac{1}{2}(|\phi><\phi| - |\chi><\chi|) &(43)\\
\mathfrak{L}_\beta^\alpha &= |\psi^\alpha><\psi^\beta| - \frac{1}{2}\delta_\beta^\alpha|\psi^\gamma><\psi^\gamma| &(44)
\end{aligned}$$

[5]

For the generators in $\mathfrak{su}(2 \mid 2)/\mathfrak{su}(1 \mid 2)$ we get

$$\begin{aligned}
\mathfrak{Q}_2^\alpha &= a\,|\psi^\alpha><\chi| - b\,\epsilon^{\alpha\beta}|\phi><\psi^\beta| &(45)\\
\mathfrak{S}_\alpha^2 &= -c\,\epsilon_{\alpha\beta}|\psi^\beta><\phi| + d\,|\chi><\psi^\alpha| &(46)\\
\mathfrak{R}_2^1 &= |\phi><\chi| &(47)\\
\mathfrak{R}_1^2 &= |\chi><\phi|. &(48)
\end{aligned}$$

For the centre we have

$$\begin{aligned}
\mathfrak{C} &= \frac{1}{2}(ad + cb) &(49)\\
\mathfrak{P} &= ab &(50)\\
\mathfrak{K} &= cd. &(51)
\end{aligned}$$

### 3.4 Charge Conjugation

In this section we derive the charge conjugation $C$, which has to fullfil

$$\pi(S(A)) = C^{-1}\bar{\pi}(A)^{st}C. \tag{52}$$

We will adopt the notation of [9], and we will show how it is possible to get all the parameters $\bar{a}, \bar{b}, \bar{c}, \bar{d}$, which parametrise $\bar{\pi}$, directly from the knowledge of the full Hopf Algebra. We start with

$$\begin{aligned}
C &= |\chi><\phi|\frac{a_1 b_1}{\bar{a}\bar{b}} + |\phi><\chi| &(53)\\
&\quad -\frac{b_1}{\bar{a}}|\psi^2><\psi^1| + \frac{b_1}{\bar{a}}|\psi^1><\psi^2|. &(54)
\end{aligned}$$

[5] It is understood that $A \equiv \pi(A)$

Then the inverse is given by

$$C^{-1} = |\phi><\chi|\frac{\bar{a}\bar{b}}{a_1 b_1} + |\chi><\phi| \tag{55}$$
$$-\frac{\bar{a}}{b_1}|\psi 1><\psi^2| + \frac{\bar{a}}{b_1}|\psi^2><\psi^1|. \tag{56}$$

For $J \in \mathfrak{su}(1|2)$ the antipode is the trivial one and we get the equations

$$C\pi(J)C^{-1} = -\bar{\pi}(J)^{st}, \tag{57}$$

whilst for $D \in \mathfrak{su}(2|2)/\mathfrak{su}(1|2)$ we get

$$C\pi(D)C^{-1} = -B(D)\bar{\pi}(D)^{st}. \tag{58}$$

We therefore get the parameters of $\bar{\pi}$ in terms of the parameters of $\pi$:

$$C\mathfrak{S}^1_1 C^{-1} = -\frac{d_1 a_1}{\bar{b}}|\chi><\psi 2| + \frac{c_1 b_1}{\bar{a}}|\psi^1><\phi| \tag{59}$$
$$\equiv -\bar{\pi}^{st}(\mathfrak{S}^1_1) = \bar{c}|\chi>|\psi^2> - \bar{d}|\psi^1><\phi| \tag{60}$$

$$\Rightarrow \bar{d} = -\frac{c_1 b_1}{\bar{a}} \tag{61}$$
$$\bar{c} = -\frac{d_1 a_1}{\bar{b}} \tag{62}$$

$$C\mathfrak{Q}^1_2 C^{-1} = -\frac{a_1 b_1}{\bar{a}}|\psi^2><\phi| - \bar{a}|\chi><\psi^1| \tag{63}$$
$$\equiv -\pi(B(\mathfrak{Q}^1_2))\bar{\pi}^{st}(\mathfrak{Q}^1_2) = \pi(B(\mathfrak{Q}^1_2))(\bar{a}|\chi><\psi^1| + \bar{b}<\psi^2|<\phi|) \tag{64}$$

$$\Rightarrow \pi(B(\mathfrak{Q}^1_2)) = -\frac{a_1 b_1}{\bar{a}\bar{b}} \tag{65}$$

$$C\mathfrak{S}^2_2 C^{-1} = \frac{d_1 \bar{a}}{b_1}|\phi><\psi^1| - \frac{c_1 \bar{b}}{a_1}|\psi^2><\chi| \tag{66}$$
$$\equiv -\bar{\pi}^{st}(\mathfrak{S}^2_2)\pi(B(\mathfrak{S}^2_2)) = -\pi(B(\mathfrak{S}^2_2))(\bar{d}|\psi^2><\chi| - \bar{c}|\phi><\psi^1|) \tag{67}$$

$$\Rightarrow \pi(B(\mathfrak{S}^2_2)) = \frac{d_1 \bar{a}}{b_1 \bar{c}} \tag{68}$$
$$= \frac{c_1 \bar{b}}{\bar{d} a_1} \tag{69}$$

$$C\mathfrak{R}_1^2C^{-1} = \frac{\bar{a}\bar{b}}{a_1b_1}|\phi >< \chi| \equiv -\pi(B(\mathfrak{R}_1^2))|\phi >< \chi| \tag{70}$$

$$\Rightarrow \pi(B(\mathfrak{R}_1^2)) \quad = \quad -\frac{\bar{a}\bar{b}}{a_1b_1} \tag{71}$$

With the condition

$$a_1b_1 = \alpha(e^{-ip} - 1), \tag{72}$$

which comes from the requirement to have total zero momentum for physical states, and with $\pi(B(\mathfrak{R}_1^2)) = e^{ip}$, we get:

$$e^{ip} \quad = \quad \frac{\alpha}{\alpha + a_1b_1} = -\frac{\bar{a}\bar{b}}{a_1b_1} \tag{73}$$

$$\Rightarrow \bar{b} = -\frac{a_1b_1\alpha}{\bar{a}}\frac{1}{\alpha + a_1b_1} \tag{74}$$

This coincides with eq. (58) of Janik [9].

## 4 Conclusions

In this paper we have shown how it is possible to extend the Hopf algebra structure discovered by Janik for the subsector $\mathfrak{su}(1|2)$ to the full $\mathfrak{su}(2|2)$ algebra, by determing the action of the coproduct and of the antipode on the remaining generators. The result is a nontrivial action via a central element of the Hopf algebra. This is a bit unusual from the mathematical point of view, and it originates a slightly different structure from the one familiar from quantum groups. However, this algebra most probably resides at the core of the problem of determing additional symmetries and constraints for the S-matrix of the AdS/CFT correspondence. We have verified the Hopf algebra axioms for this structure, and we have used it to determine the antiparticle transformation directly from the algebra.

The next step will be to construct the correspondent Universal R-matrix. The condition of "quasi-traingularity"

$$\begin{aligned}(\Delta \otimes id)(\mathcal{R}) &= \mathcal{R}_{13}\mathcal{R}_{23}\\ (id \otimes \Delta)(\mathcal{R}) &= \mathcal{R}_{13}\mathcal{R}_{12}\end{aligned} \tag{75}$$

implies that $\mathcal{R}$ satisfies the Yang-Baxter equation, and, in the presence of an antipode, the crossing condition

$$\begin{aligned}(S \otimes id)(\mathcal{R}) &= \mathcal{R}^{-1}\\ (id \otimes S^{-1})(\mathcal{R}) &= \mathcal{R}^{-1}\end{aligned} \tag{76}$$

which in turn implies $(S \otimes S)\mathcal{R} = id$. The condition (75) is reminiscent of the "bootstrap" principle for a relativistic S-matrix, and is in fact its translation in Hopf-algebraic terms. The fact that the S-matric of AdS/CFT satisfies Yang-Baxter and crossing relations is a strong hint that it might also satisfy the quasi-traingularity condition.

Being expressed in a representation-independent way purely in terms of the abstract algebra generators, the Universal R-matrix may give a significant help determining the overall factor in a much cleaner way directly in terms of the generators of the universal enveloping algebra needed for fixing such Universal R-matrix at the algebraic level, and one is likely to gain also a better understanding of the origin of the phase factor, of its structure, and of the origin of the whole Hopf algebra structure from the point of view of the string theory sigma-model. We plan to investigate these issues in a future publication.

# 5 Note

While we were finishing the paper, we received notice of the publication of similar results by Gómez and Hernández in hep-th/0608029.

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# References

[1] J. A. Minahan and K. Zarembo, JHEP **0303** (2003) 013 [arXiv:hep-th/0212208].

[2] N. Beisert, C. Kristjansen and M. Staudacher, Nucl. Phys. B **664** (2003) 131 [arXiv:hep-th/0303060]. N. Beisert, Nucl. Phys. B **676** (2004) 3 [arXiv:hep-th/0307015]. N. Beisert and M. Staudacher, Nucl. Phys. B **670** (2003) 439 [arXiv:hep-th/0307042]. N. Beisert, Nucl. Phys. B **682** (2004) 487 [arXiv:hep-th/0310252]. D. Serban and M. Staudacher, JHEP **0406** (2004) 001 [arXiv:hep-th/0401057]. N. Beisert, V. Dippel and M. Staudacher, JHEP **0407** (2004) 075 [arXiv:hep-th/0405001]. N. Beisert, Phys. Rept. **405** (2005) 1 [arXiv:hep-th/0407277]. M. Staudacher, JHEP **0505** (2005) 054 [arXiv:hep-th/0412188]. N. Beisert and M. Staudacher, Nucl. Phys. B **727** (2005) 1 [arXiv:hep-th/0504190].

[3] I. Bena, J. Polchinski and R. Roiban, Phys. Rev. D **69** (2004) 046002 [arXiv:hep-th/0305116]. V. A. Kazakov, A. Marshakov, J. A. Minahan and K. Zarembo, JHEP **0405** (2004) 024 [arXiv:hep-th/0402207].

[4] C. G. . Callan, H. K. Lee, T. McLoughlin, J. H. Schwarz, I. J. Swanson and X. Wu, Nucl. Phys. B **673** (2003) 3 [arXiv:hep-th/0307032].

[5] C. G. . Callan, T. McLoughlin and I. J. Swanson, Nucl. Phys. B **694** (2004) 115 [arXiv:hep-th/0404007]. C. G. . Callan, T. McLoughlin and I. J. Swanson, Nucl. Phys. B **700** (2004) 271 [arXiv:hep-th/0405153]. G. Arutyunov, S. Frolov and M. Staudacher, JHEP **0410**

(2004) 016 [arXiv:hep-th/0406256]. S. Frolov, J. Plefka and M. Zamaklar, arXiv:hep-th/0603008. S. Frolov and A. A. Tseytlin, Nucl. Phys. B **668** (2003) 77 [arXiv:hep-th/0304255]. G. Arutyunov, S. Frolov, J. Russo and A. A. Tseytlin, Nucl. Phys. B **671** (2003) 3 [arXiv:hep-th/0307191]. G. Arutyunov, J. Russo and A. A. Tseytlin, Phys. Rev. D **69** (2004) 086009 [arXiv:hep-th/0311004]. A. A. Tseytlin, arXiv:hep-th/0311139. G. Arutyunov and S. Frolov, JHEP **0502** (2005) 059 [arXiv:hep-th/0411089]. L. F. Alday, G. Arutyunov and A. A. Tseytlin, JHEP **0507** (2005) 002 [arXiv:hep-th/0502240].

[6] N. Beisert, arXiv:hep-th/0511082.

[7] N. Beisert and T. Klose, J. Stat. Mech. **0607** (2006) P006 [arXiv:hep-th/0510124].

[8] G. Arutyunov, S. Frolov and M. Staudacher, JHEP **0410** (2004) 016 [arXiv:hep-th/0406256]. R. Hernandez and E. Lopez, JHEP **0607** (2006) 004 [arXiv:hep-th/0603204]. L. Freyhult and C. Kristjansen, Phys. Lett. B **638** (2006) 258 [arXiv:hep-th/0604069].

[9] R. A. Janik, Phys. Rev. D **73** (2006) 086006 [arXiv:hep-th/0603038].

[10] G. Arutyunov and S. Frolov, Phys. Lett. B **639** (2006) 378 [arXiv:hep-th/0604043].

[11] N. Beisert, arXiv:hep-th/0606214

[12] W. Delius, M. D. Gould and Y. Z. Zhang, Nucl. Phys. B **432** (1994) 377 [arXiv:hep-th/9405030]. A. J. Bracken, M. D. Gould, Y. Z. Zhang and G. W. Delius, J. Phys. A **27** (1994) 6551 [arXiv:hep-th/9405138].