Non-linear output frequency response functions for multi-input non-linear Volterra systems

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(Received 26 October 2006; in final form 20 December 2006)

The concept of non-linear output frequency response functions (NOFRFs) is extended to the non-linear systems that can be described by a multi-input Volterra series model. A new algorithm is also developed to determine the output frequency range of non-linear systems from the frequency range of the inputs. These results allow the concept of NOFRFs to be applied to a wide range of engineering systems. The phenomenon of the energy transfer in a two degree of freedom non-linear system is studied using the new concepts to demonstrate the significance of the new results.

1. Introduction

Linear systems, which have been widely studied by practitioners in many different fields, have provided a basis for the development of the majority of control system synthesis, mechanical system analysis and design, and signal processing methods. However, there are certain types of qualitative behaviour encountered in engineering, which cannot be produced by linear models (Pearson 1994), for example, the generation of harmonics and inter-modulation behaviour. In cases where these effects are dominant or significant non-linear behaviours exist, non-linear models are required to describe the system, and non-linear system analysis methods have to be applied to investigate the system dynamics.

The Volterra series approach (Worden et al. 1997) is a powerful tool for the analysis of non-linear systems, which extends the familiar concept of the convolution integral for linear systems to a series of multi-dimensional convolution integrals. The Fourier transforms of the Volterra kernels are known as the kernel transforms, higher-order frequency response functions (HFRFs) (Lang and Billings 1996), or generalized frequency response functions (GFRFs), and these provide a convenient tool for analyzing non-linear systems in the frequency domain. If a differential equation or discrete-time model is available for a system, the GFRFs can be determined using the algorithm in Billings and Tsang (1989), Peyton Jones and Billings (1989), Billings and Peyton Jones (1990). The GFRFs can be regarded as the extension of the classical frequency response function (FRF) of linear systems to the non-linear case. However, the GFRFs are much more complicated than the FRF. GFRFs are multidimensional functions (Zhang and Billings 1993, 1994), which can be difficult to measure, display and interpret in practice. Recently, the novel concept of non-linear output frequency response functions (NOFRFs) was proposed by Lang and Billings (2005). The concept can be considered to be an alternative extension of the FRF to the non-linear case. NOFRFs are one dimensional functions of frequency, which allow the analysis of non-linear systems to be implemented in a manner similar to the analysis of linear systems and which provides great insight into the mechanisms which dominate many non-linear behaviours. The NOFRF concept was recently used to investigate the sub-resonance phenomena for a class of non-linear systems (Peng et al. 2006). The results revealed the existence of resonances at frequencies different from the frequencies at the input excitation in this class of oscillators.

The objective of this paper is to extend the concept of NOFRFs to multi-input non-linear Volterra systems so
that the concept of NOFRFs can be applied to MIMO systems which allow the basic idea to be used in more complicated systems. The phenomenon of energy transfer in a 2DOF non-linear system is investigated using the extended concept to demonstrate the effectiveness and significance of the results obtained in the present study.

2. The concept of non-linear output frequency response functions

NOFRFs were recently proposed and used to investigate the behaviour of structures with polynomial-type non-linearities. The definition of NOFRFs is based on the Volterra series theory of non-linear systems.

Consider the class of non-linear systems which are stable at zero equilibrium and which can be described in the neighbourhood of the equilibrium by the Volterra series

\[ y(t) = \sum_{n=1}^{N} \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} h_n(\tau_1, \ldots, \tau_n) \prod_{i=1}^{n} u(t - \tau_i) d\tau_i, \quad (1) \]

where \( y(t) \) and \( x(t) \) are the output and input of the system, \( h(\tau_1, \ldots, \tau_n) \) is the \( n \)th order Volterra kernel, and \( N \) denotes the maximum order of the system non-linearity. Lang and Billings (1996) have derived an expression for the output frequency response of this class of non-linear systems to a general input. The result is

\[
\begin{align*}
Y(j\omega) &= \sum_{n=1}^{N} Y_n(j\omega) \quad \text{for } \forall \omega \\
Y_n(j\omega) &= \frac{1}{\sqrt{n}} \left( \frac{2\pi}{n} \right)^{n-1} \frac{1}{\omega_1 \cdots \omega_n} \int_{-\infty}^{\infty} h_n(j\omega_1, \ldots, j\omega_n) \\
&\quad \times \prod_{i=1}^{n} X(j\omega_i) d\sigma_{\omega_1} \\
&\quad \times \prod_{i=1}^{n} X(j\omega_i) d\sigma_{\omega_n}. \quad (2)
\end{align*}
\]

This expression is just a different form description for non-linear system output frequency responses. Compared with other descriptions, equation (2) provides a more physically meaningful insight into the composition of non-linear output frequency responses and reveals how non-linear mechanisms operate on the input spectra to produce the system output frequency response. In (2), \( Y(j\omega) \) and \( X(j\omega) \) are the spectra of the system output and input respectively, \( Y_n(j\omega) \) represents the \( n \)th order output frequency response of the system,

\[
H_n(j\omega_1, \ldots, j\omega_n) = \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} h_n(\tau_1, \ldots, \tau_n) \\
\times e^{-j\omega_1 \tau_1 - \cdots - j\omega_n \tau_n} d\tau_1 \cdots d\tau_n \quad (3)
\]

is the definition of the generalized frequency response function (GFRF), and

\[
\int_{\omega_1 + \cdots + \omega_n = \omega} H_n(j\omega_1, \ldots, j\omega_n) \prod_{i=1}^{n} X(j\omega_i) d\sigma_{\omega_1} \cdots d\sigma_{\omega_n}
\]

denotes the integration of \( H_n(j\omega_1, \ldots, j\omega_n) \prod_{i=1}^{n} X(j\omega_i) \) over the \( n \)-dimensional hyper-plane \( \omega_1 + \cdots + \omega_n = \omega \). Equation (2) is a natural extension of the well-known linear relationship \( Y(j\omega) = H_1(j\omega)X(j\omega) \) to the non-linear case.

For linear systems, the possible output frequencies are the same as the frequencies in the input. For non-linear systems described by equation (1), however, the relationship between the input and output frequencies is more complicated. Given the frequency range of the input, the output frequencies of system (1) can be determined using an explicit expression derived by Lang and Billings (1996).

Based on the above results for output frequency responses of non-linear systems, a new concept known as the non-linear output frequency response function (NOFRF) was recently introduced by Lang and Billings (2005). The concept was defined as

\[
G_n(j\omega) = \frac{\int_{\omega_1 + \cdots + \omega_n = \omega} H_n(j\omega_1, \ldots, j\omega_n) \prod_{i=1}^{n} X(j\omega_i) d\sigma_{\omega_1} \cdots d\sigma_{\omega_n}}{\int_{\omega_1 + \cdots + \omega_n = \omega} \prod_{i=1}^{n} X(j\omega_i) d\sigma_{\omega_1} \cdots d\sigma_{\omega_n}} \quad (4)
\]

under the condition that

\[
U_n(j\omega) = \int_{\omega_1 + \cdots + \omega_n = \omega} \prod_{i=1}^{n} X(j\omega_i) d\sigma_{\omega_1} \cdots d\sigma_{\omega_n} \neq 0. \quad (5)
\]

Note that \( G_n(j\omega) \) is valid over the frequency range of \( U_n(j\omega) \), which can be determined using the algorithm in Lang and Billings (1996).

By introducing the NOFRFs \( G_n(j\omega), n = 1, \ldots, N \), equation (4) can be written as

\[
Y(j\omega) = \sum_{n=1}^{N} Y_n(j\omega) = \sum_{n=1}^{N} G_n(j\omega) U_n(j\omega) \quad (6)
\]

which is similar to the description of the output frequency response of linear systems. For a linear system, the relationship between \( Y(j\omega) \) and \( X(j\omega) \) can be illustrated as in figure 1. Similarly, the non-linear system input and output relationship of equation (1) can be illustrated in figure 2.

The NOFRFs reflect a combined contribution of the system and the input to the frequency domain output behaviour. It can be seen from equation (4) that \( G_n(j\omega) \) depends not only on \( H_n(n = 1, \ldots, N) \) but also on the input \( X(j\omega) \). For a non-linear system, the dynamical
properties are determined by the GFRFs $H_n$ ($n = 1, \ldots, N$). However, from equation (3) it can be seen that a GFRF is multidimensional (Zhang and Billings 1993, 1994), and may become difficult to measure, display and interpret in practice. Feijoo et al. (2004, 2005) demonstrated that the Volterra series can be described by a series of associated linear equations (ALEs) whose corresponding associated frequency response functions (AFRFs) are easier to analyze and interpret than the GFRFs. According to equation (4), the NOFRF $G_n(j\omega)$ is a weighted sum of $H_n(j\omega_1, \ldots, j\omega_n)$ over $\omega_1 + \cdots + \omega_n = \omega$ with the weights depending on the input. Therefore $G_n(j\omega)$ can be used as an alternative representation of the structural dynamical characteristics described by $H_n$. The most important property of the NOFRF $G_n(j\omega)$ is that it is one dimensional, and thus allows the analysis of non-linear systems to be implemented in a convenient manner similar to the analysis of linear systems. Moreover, there is an effective algorithm Lang and Billings (2005) available which allows the evaluation of the NOFRFs to be implemented directly using system input output data.

3. NOFRFS for multi-input non-linear Volterra systems

3.1 Multi-input non-linear Volterra systems

Multi-input multi-output non-linear Volterra systems can be expressed so that each output can be modeled as a multi-input Volterra series. The extension of the single input single output Volterra series representation (1) to this more general case is as follows:

$$y_i(t) = \sum_{n=1}^{N} y_i^{(n)}(t),$$  \hfill (7)

where

$$y_i^{(n)}(t) = \sum_{N_1+\cdots+N_m=n} \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} h^{(n)}_{i,P_1=1,P_2=1} \prod_{i=1}^{N_1} X_i(j\omega_i) \prod_{i=1}^{N_2} X_2(j\omega_i) \cdots \prod_{i=1}^{N_m} X_m(j\omega_i) \, d\sigma_{i\omega_1},$$  \hfill (11)

and $h^{(n)}_{i,P_1=1,P_2=1}$ represents the $n$th order kernel associated with the $i$th output and $N_1$th input $x_1(t)$, $N_2$th input $x_2(t)$, $\ldots$, $N_m$th input $x_m(t)$.

Equation (8) can be rewritten as

$$y_i^{(n)}(t) = \sum_{N_1+\cdots+N_m=n} \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} h^{(n)}_{i,P_1=1,P_2=1} \prod_{i=1}^{N_1} X_i(t - \tau_i) \prod_{i=1}^{N_2} X_2(t - \tau_{i_2}) \cdots \prod_{i=1}^{N_m} X_m(t - \tau_1) \, d\tau_1 \cdots d\tau_n,$$  \hfill (9)

where

$$x_{N_1+\cdots+N_m+1}(\tau_1, \ldots, \tau_n) = x_1(t - \tau_1) \cdots x_1(t - \tau_{N_1}) \times x_2(t - \tau_{N_1+1}) \cdots x_2(t - \tau_{N_1+N_2}) \times \cdots x_m(t - \tau_{N_1+\cdots+N_m+1}).$$  \hfill (10)

In the single input case, the Volterra series has only one kernel for each order of nonlinearity. For example, $h(\tau_1, \ldots, \tau_n)$ is the second order kernel. It can be seen, however, that in the multi-input case more kernels are involved for each order of non-linearity. For example, for a two input system there are three 2nd order kernels for the $i$th output which are $h^{(2)}_{i,P_1=2,P_2=0}(\tau_1, \tau_2)$, $h^{(2)}_{i,P_1=0,P_2=2}(\tau_1, \tau_2)$ and $h^{(2)}_{i,P_1=1,P_2=1}(\tau_1, \tau_2)$.

The frequency domain description of (7) and (8) can be expressed as

$$Y_i(j\omega) = \sum_{n=1}^{N} Y_i^{(n)}(j\omega).$$  \hfill (11)

$$Y_i^{(n)}(j\omega) = \left(\frac{1}{2\pi}\right)^{n-1} \sum_{N_1+\cdots+N_m=n} \int_{j\omega_1+\cdots+j\omega_n=\omega} H^{(n)}_{i,P_1=1,P_2=1} \prod_{i=1}^{N_1} X_i(j\omega_i) \prod_{i=1}^{N_2} X_2(j\omega_i) \cdots \prod_{i=1}^{N_m} X_m(j\omega_i) \, d\sigma_{i\omega_1}.$$  \hfill (12)

This is an extension of equation (12) for the single-input case to the multi-input case. Define $N_0 = 0$, then
equation (12) can be written as

\[
Y_i^{(n)}(\omega) = \left(\frac{1}{2\pi}\right)^n \frac{1}{\sqrt{n}} \sum_{N_1+\ldots+N_m=n} \int_{-\infty}^{\infty} H_{(i,P_1=N_1,\ldots,P_m=N_m)}^{(n)}(j\omega_1,\ldots,j\omega_n) \times (j\omega_1,\ldots,j\omega_n) \prod_{j=1}^{m} N_j \prod_{j=1}^{m} X_j(j\omega) d\sigma_{\omega_1} \ldots d\sigma_{\omega_n}.
\]

(13)

For a given set of \(N_1, N_2, \ldots, N_m\), define

\[
Y_i^{(n)}(\omega) = \sum_{N_1+\ldots+N_m=n} Y_i^{(n)}(\omega),
\]

(14)

then equation (13) can be written in a compact form

\[
Y_i^{(n)}(\omega) = \sum_{N_1+\ldots+N_m=n} Y_i^{(n)}(\omega).
\]

(15)

3.2 Definition of the NOFRFs for multi-input non-linear Volterra systems

Define

\[
U^{(n)}_{(i,P_1=N_1,\ldots,P_m=N_m)}(\omega) = \left(\frac{1}{2\pi}\right)^n \frac{1}{\sqrt{n}} \int_{-\infty}^{\infty} H_{(i,P_1=N_1,\ldots,P_m=N_m)}^{(n)}(j\omega_1,\ldots,j\omega_n) \times (j\omega_1,\ldots,j\omega_n) \prod_{j=1}^{m} N_j \prod_{j=1}^{m} X_j(j\omega) d\sigma_{\omega_1} \ldots d\sigma_{\omega_n}
\]

(16)

then (14) can be rewritten as

\[
Y_i^{(n)}(\omega) = \frac{\int_{0}^{\omega} \cdots \int_{0}^{\omega} H^{(n)}_{(i,P_1=N_1,\ldots,P_m=N_m)}(j\omega_1,\ldots,j\omega_n) \times (j\omega_1,\ldots,j\omega_n) \prod_{j=1}^{m} N_j \prod_{j=1}^{m} X_j(j\omega) d\sigma_{\omega_1} \cdots d\sigma_{\omega_n}}{\int_{0}^{\omega} \cdots \int_{0}^{\omega} \prod_{j=1}^{m} N_j \prod_{j=1}^{m} X_j(j\omega) d\sigma_{\omega_1} \cdots d\sigma_{\omega_n}}
\]

(17)

where

\[
G^{(n)}_{(i,P_1=N_1,\ldots,P_m=N_m)}(\omega) = \frac{\int_{0}^{\omega} \cdots \int_{0}^{\omega} H^{(n)}_{(i,P_1=N_1,\ldots,P_m=N_m)}(j\omega_1,\ldots,j\omega_n) \times (j\omega_1,\ldots,j\omega_n) \prod_{j=1}^{m} N_j \prod_{j=1}^{m} X_j(j\omega) d\sigma_{\omega_1} \cdots d\sigma_{\omega_n}}{\int_{0}^{\omega} \cdots \int_{0}^{\omega} \prod_{j=1}^{m} N_j \prod_{j=1}^{m} X_j(j\omega) d\sigma_{\omega_1} \cdots d\sigma_{\omega_n}}
\]

(18)

will be referred to as the non-linear output frequency response function for multi-input non-linear Volterra systems, and is a natural extension of equation (14) to more general case. Substituting (18) into (15) yields

\[
Y_i^{(n)}(\omega) = \sum_{N_1+\ldots+N_m=n} G^{(n)}_{(i,P_1=N_1,\ldots,P_m=N_m)}(\omega) \times U^{(n)}_{(i,P_1=N_1,\ldots,P_m=N_m)}(\omega)
\]

(19)

It is easy to verify that \(G^{(n)}_{(i,P_1=N_1,\ldots,P_m=N_m)}(\omega)\) has the following important properties.

(i) \(G^{(n)}_{(i,P_1=N_1,\ldots,P_m=N_m)}(\omega)\) allows \(Y_i^{(n)}(\omega)\) to be described in a manner similar to the description for the output frequency response of linear systems.

(ii) \(G^{(n)}_{(i,P_1=N_1,\ldots,P_m=N_m)}(\omega)\) is valid over a frequency range where \(U^{(n)}_{(i,P_1=N_1,\ldots,P_m=N_m)}(\omega) \neq 0\).

(iii) \(G^{(n)}_{(i,P_1=N_1,\ldots,P_m=N_m)}(\omega)\) is insensitive to the change of the input spectra by a constant gain, that is

\[
G^{(n)}_{(i,P_1=N_1,\ldots,P_m=N_m)}(\omega) |_{X_i(j\omega) = a X_i(j\omega)} = G^{(n)}_{(i,P_1=N_1,\ldots,P_m=N_m)}(\omega) |_{X_i(j\omega) = a X_i(j\omega)}
\]

(20)

3.3 Determination of the output frequency range of multi-input non-linear Volterra systems

For a non-linear system that can be modelled as a single-input Volterra series, given the frequency range of the input, Lang and Billings (1996) derived an explicit expression for the output frequency range. In the following, a method will be derived to determine the output frequency range of multi-input non-linear Volterra systems.

Obviously, the frequency range of \(U^{(n)}_{(i,P_1=N_1,\ldots,P_m=N_m)}(\omega)\) is given as the range of

\[
\omega = \sum_{k=1}^{m} \omega^{(k)}
\]

(21)
where $\omega^{(k)}$ is associated to the $k$th input $x_k(t)$ of order $N_k$ and

$$\omega^{(k)} = \omega_{N_0} + \cdots + N_{k-1} + \cdots + \omega_{N_0} + \cdots + N_k. \quad (22)$$

Define the frequency range of the $k$th input $x_k(t)$ ($k = 1, \ldots, m$) as

$$[-b_k, -a_k] \cup [a_k, b_k].$$

Now, assume $L_k$ of $N_k$ components are located in $[-b_k, -a_k]$, and the remainder are located $[a_k, b_k]$. In this case, the frequency range is

$$[(N_k - L_k)a_k - L_kb_k , (N_k - L_k)b_k - L_ka_k]. \quad (23)$$

that is

$$[N_ka_k - L_k(b_k + a_k) , N_kb_k - L_k(a_k + b_k)]. \quad (24)$$

Therefore the range of $\omega^{(k)}$ is

$$\bigcup_{L_k=0}^{N_k} [(N_ka_k - L_k(b_k + a_k) , N_kb_k - L_k(a_k + b_k)]. \quad (25)$$

that is

$$\omega^{(k)} \in \bigcup_{L_k=0}^{N_k} [(N_ka_k - L_k(b_k + a_k) , N_kb_k - L_k(a_k + b_k)]. \quad (26)$$

This can easily be extended to case of $\omega^{(1)} + \cdots + \omega^{(m)}$

$$\omega = (\omega^{(1)} + \cdots + \omega^{(m)}) \in \bigcup_{L_1=0}^{N_1} \cdots \bigcup_{L_m=0}^{N_m} \left[ \sum_{i=1}^{m} N_i a_i - \sum_{i=1}^{m} L_i(a_i + b_i) \right]. \quad (27)$$

Therefore, the frequency range of $U_{(i,p_1=n_1,\ldots,p_m=n_m)}(j\omega)$ can be expressed

$$f_{(i,p_1=n_1,\ldots,p_m=n_m)}(j\omega) = \bigcup_{L_1=0}^{N_1} \cdots \bigcup_{L_m=0}^{N_m} \left[ \sum_{i=1}^{m} N_i a_i - \sum_{i=1}^{m} L_i(a_i + b_i) \right. \times \left. \sum_{i=1}^{m} N_i b_i - \sum_{i=1}^{m} L_i(a_i + b_i) \right]. \quad (28)$$

From equation (17), it can be shown that $Y_{(i,p_1=n_1,\ldots,p_m=n_m)}(j\omega)$ and $U_{(i,p_1=n_1,\ldots,p_m=n_m)}(j\omega)$ have the same frequency range. Furthermore, according to equation (15), it can easily be shown that the frequency range of $Y_{(i,p_1=n_1,\ldots,p_m=n_m)}(j\omega)$ is

$$f_{(i,p_1=n_1,\ldots,p_m=n_m)}(j\omega) = \bigcup_{N_1 + \cdots + N_m = n} f_{(i,p_1=n_1,\ldots,p_m=n_m)}(j\omega). \quad (29)$$

Finally, according to equation (11), the frequency range of $Y_i(j\omega)$ is given by

$$f_y = \bigcup_{n=1}^{N} f_{(i,p_1=n_1,\ldots,p_m=n_m)}(j\omega). \quad (30)$$

Therefore, given the frequency ranges of the inputs $x_k(t)$ ($k = 1, \ldots, m$) as

$$[-b_k, -a_k] \cup [a_k, b_k],$$

the output frequency range can be determined by equations (28)–(30). The validity of this method will be verified by numerical studies in §3.

### 3.4 Evaluation of the NOFRFs for multi-input systems

For single-input non-linear systems, Lang and Billings (2005) derived an effective algorithm for the estimation of the NOFRFs, which can be implemented directly using system input output data. To estimate the NOFRFs up to $N$th order, the algorithm generally requires experimental or simulation results for the system under $N$ different input signal excitations, which have the same waveforms but different intensities. This algorithm can be extended to estimate the NOFRFs in the multi-input case. As a multi-input system of non-linearity up to $N$th order involves more than $N$ NOFRFs, more than $N$ experiments or simulations under different signal excitations are needed to estimate the NOFRFs.

Combining equations (11) and (19) yields

$$y_i(j\omega) = \sum_{n=1}^{N} \sum_{P_1=n_1,\ldots,P_m=n_m} Y_{(i,P_1=n_1,\ldots,P_m=n_m)}(j\omega). \quad (31)$$

Equation (31) can be further written in the polynomial form

$$y_i(j\omega) = \sum_{k_1=1}^{m} G_{(i,k_1)}(j\omega) [U_{k_1}(j\omega)] + \sum_{k_1=1}^{m} \sum_{k_2=k_1}^{m} G_{(i,k_1,k_2)}(j\omega) \times [U_{k_1}(j\omega)U_{k_2}(j\omega)] + \cdots + \sum_{k_1=1}^{m} \cdots \sum_{k_N=k_{N-1}}^{m} G_{(i,N)}(j\omega) [U_{k_1}(j\omega) \cdots U_{k_N}(j\omega)], \quad (32)$$

where $G_{(i,k_1,\ldots,k_N)}(j\omega)$ represents a specific

$$G_{(i,k_1,\ldots,k_N)}(j\omega) = \frac{1}{n_1} \frac{1}{n_2} \cdots \frac{1}{n_m} (j\omega). \quad (33)$$
with \( N_1 + \cdots + N_m = n \), and \( n = 1, \ldots, N \), and

\[
U_{(P)=N_1, \ldots, P_m=N_m}(j\omega) = \left[ \frac{U_1(j\omega) \cdots U_1(j\omega)U_2(j\omega) \cdots U_2(j\omega)}{N_1} \right. \\
\times \left. \cdots \frac{U_m(j\omega) \cdots U_m(j\omega)}{N_m} \right]
\]

(34)

The number of terms contained in equation (32) can easily be calculated using the method given in Guo and Billings (2005), as

\[
C(N, m) = m + (m + 1)m/2! + \cdots + (m + N - 1) \cdots (m + 1)m/n!.
\]

(35)

It can be seen that there are \( (m + n - 1) \cdots (m + 1)m/n! \) terms for the \( n \)th order NOFRFs.

Sorting all \( G_{k_1 \cdots k_N}(j\omega) \), \( k_i = k_{i-1}, \ldots, m, i = 1, \ldots, n \), \( k_0 = 1 \) as a series, and labelling them as

\[
G_{(i,k)}(j\omega), \quad k = 1: C_{(n,m)},
\]

where

\[
C_{(n,m)} = (m + n - 1) \cdots (m + 1)m/n!.
\]

(37)

Denoting the corresponding \([U_{k_1}(j\omega) \cdots U_{k_N}(j\omega)]\) as

\[
U_{(k)}(j\omega), \quad k = 1: C_{(n,m)},
\]

then equation (31) can be rewritten as

\[
y_i(j\omega) = \left[ U_{(1)}^{(1)} \cdots U_{(C_{(1,m)})}^{(1)} \cdots U_{(1)}^{(N)} \cdots U_{(C_{(N,m)})}^{(N)} \right] [G_i],
\]

(39)

where

\[
Y_i(j\omega) = \left[ \begin{array}{cccc} 
\alpha_1 U_{(1)}^{(1)} & \cdots & \alpha_1 U_{(C_{(1,m)})}^{(1)} & \cdots & \alpha_1 U_{(1)}^{(N)} & \cdots & \alpha_1 U_{(C_{(1,m)})}^{(N)} \\
\vdots & & \vdots & & \vdots & & \vdots \\
\alpha_{C(N,m)} U_{(1)}^{(1)} & \cdots & \alpha_{C(N,m)} U_{(C_{(1,m)})}^{(1)} & \cdots & \alpha_{C(N,m)} U_{(1)}^{(N)} & \cdots & \alpha_{C(N,m)} U_{(C_{(1,m)})}^{(N)} \\
\end{array} \right] \left[ \begin{array}{c} 
G_i \\
\end{array} \right],
\]

(40)

\[
Y_i(j\omega) = \left[ \begin{array}{c} 
Y_i(j\omega) \\
\end{array} \right].
\]

(45)

Moreover, defining

\[
AU_{(1, \ldots, C(N,m))}(j\omega) = \left[ \begin{array}{cccc} 
\alpha_1 U_{(1)}^{(1)} & \cdots & \alpha_1 U_{(C_{(1,m)})}^{(1)} & \cdots & \alpha_1 U_{(1)}^{(N)} & \cdots & \alpha_1 U_{(C_{(1,m)})}^{(N)} \\
\vdots & & \vdots & & \vdots & & \vdots \\
\alpha_{C(N,m)} U_{(1)}^{(1)} & \cdots & \alpha_{C(N,m)} U_{(C_{(1,m)})}^{(1)} & \cdots & \alpha_{C(N,m)} U_{(1)}^{(N)} & \cdots & \alpha_{C(N,m)} U_{(C_{(1,m)})}^{(N)} \\
\end{array} \right]
\]

(46)
From equation (47),
\[ Y(j\omega) = AU^{1,\ldots,C(N,m)}(j\omega)[G]\]
(47)

From equation (47), \( [G^r] = [G^{n(i)}_{(i,1)} \ldots G^{n(i)}_{(i,C_r,m)}]^T \) can then be determined using an Least Square based approach to yield
\[
[G^r] = \left[ (AU^{1,\ldots,C(N,m)}(j\omega))^T (AU^{1,\ldots,C(N,m)}(j\omega)) \right]^{-1}
\times (AU^{1,\ldots,C(N,m)}(j\omega))^T Y(j\omega).
\]
(48)

From equation (35), it is known that the number of NOFRF terms will increase with the number of system inputs. For a single input non-linear system \((m = 1)\) with up to 4th order non-linearity \((N=4)\) has only 4 NOFRF terms; however, a nonlinear system of \(N=4\) and \(m=2\) will have 14 NOFRF terms. This implies that 14 different signal excitations are needed to generate the data of the output spectra to estimate these NOFRFs.

4. Energy transfer phenomena of a multi-input non-linear system

In this section, the concept of NOFRFs for multi-input non-linear systems is applied to investigate the energy transfer phenomena in a 2-DOF non-linear system (Worden et al. 1997). The differential equation of the considered non-linear system is given by
\[
m_1\dddot y_1(t) + (c_{11} + c_{12})\dot y_1(t) - c_{12}\ddot y_2(t) + (k_{11} + k_{12})y_1(t)
- k_{12}\dot y_2(t) + c_2(\dot y_1(t) - \dot y_2(t))^2 + c_3(\ddot y_1(t) - \ddot y_2(t))^3
+ k_3\dot y_2(t) + k_3\dddot y_2(t) = u_1(t)
\]
\[
m_2\dddot y_2(t) + (c_{12} + c_{22})\dot y_2(t) - c_{12}\ddot y_1(t) + (k_{12} + k_{22})y_2(t)
- k_{12}\dot y_1(t) - c_2(\dot y_1(t) - \dot y_2(t))^2
- c_3(\ddot y_1(t) - \ddot y_2(t))^3 = u_2(t),
\]
(49)

where \(y_1(t)\), \(y_2(t)\) are the two outputs of the system, \(m_1, m_2, c_{11}, c_{12}, c_{22}, c_2, c_3, k_{11}, k_{12}, k_{22}, k_2, k_3\) are the system parameters: mass, damping and stiffness respectively. The non-linear system can be illustrated as a mechanical oscillator shown in figure 3.

In the following study, the values of all the parameters used are
\(m_1 = m_2 = 1\) kg, \(c_{11} = c_{12} = c_{22} = 20\) N/m/s, \(c_2 = 1 \times 500\) N(m/s)^2, \(c_3 = 1 \times 10^4\) N(m/s)^3, \(k_{12} = k_{22} = 1 \times 10^4\) N/m, \(k_2 = 1 \times 10^7\) N/m^2, \(k_3 = 5 \times 10^9\) N/m^3, and the two input excitations are
\[
u_1(t) = \frac{3}{2\pi} \sin(2 \times 35 \times \pi \times t) - \sin(2 \times 10 \times \pi \times t)
- 10 \sec \leq t \leq 10 \sec
\]
(50)
\[
u_2(t) = \frac{3}{2\pi} \sin(2 \times 100 \times \pi \times t) - \sin(2 \times 85 \times \pi \times t)
- 10 \sec \leq t \leq 10 \sec.
\]
(51)

The frequency ranges of the first input and the second input are \([-10 -35] \cup [10 35]\) Hz and \([-85 -100] \cup [85 100]\) Hz respectively. These spectra are shown in figure 4. According to equation (27), it can be shown that the frequency range of \(U^{(2)}_{(p=2,p=0)}(j\omega)\) is \([2 \times 10 2 \times 35] \cup [10 -35 35 -10] \cup [-2 \times 35 -2 \times 10] = [-70 70]\) Hz. Similarly, it can be deduced that

Figure 3. A 2-DOF non-linear system.

Figure 4. The spectra of the two inputs for the system in equation (49).
the frequency range of \( U_{(P_1=0, P_2=2)}^{(2)}(j\omega) \) is \([-200 -170] \cup [-15 15] \cup [170 200]\) Hz. According to equation (29), it can be shown that the frequency range of \( U_{(P_1=1, P_2=1)}^{(2)}(j\omega) \) is \([-135 -95] \cup [-90 -50] \cup [50 90] \cup [95 135]\) Hz. These results are verified by the spectra of \( u_2^1(t) \) and \( u_1(t)u_2(t) \) shown in figure 5. Furthermore, using equation (31), the possible frequency range of the output can be calculated to be \([-200 -170] \cup [-135 135] \cup [170 200]\) Hz.

The forced response of the system is obtained through integrating equation (49) using a fourth-order Runge-Kutta method, and the results over \((3 \leq t \leq 3)\) are shown in figure 6. The initial conditions are \( y_1(0) = y_2(0) = 0 \) and \( \dot{y}_1(0) = \dot{y}_2(0) = 0 \). Figure 7 shows the spectra of the outputs, which clearly indicate the two outputs have the same frequency range over \([0 135] \cup [170 200]\) Hz, and this frequency range is the same as that determined using the analysis result by equation (30). From figure 7, it can be seen that considerable input energy is transferred by the system from the input frequency band \([10 35] \cup [85 100]\) Hz to the other frequency ranges \([0 10] \cup (35 70]\) Hz.

The NOFRFs of system (49) under the excitation (50) and (51) have been evaluated up to second order over the frequency range \([0 135] \cup [170 200]\) Hz. According to equation (36), to evaluate the NOFRFs of a 2-DOF non-linear system up to the second order, generally, five different signal excitations are needed. However, from the frequency ranges of \( U_{(P_1=1)}^{(1)}(j\omega) \), \( U_{(P_2=1)}^{(1)}(j\omega) \), \( U_{(P_1=2, P_2=0)}^{(2)}(j\omega) \), \( U_{(P_1=0, P_2=2)}^{(2)}(j\omega) \) and \( U_{(P_1=1, P_2=1)}^{(2)}(j\omega) \), the output frequency responses in equation (32) can be simplified as

\[
y_2(j\omega) = C_{(i, P_1=2, P_2=0)}^{(2)}(j\omega)U_{(P_1=2, P_2=0)}^{(2)}(j\omega) + G_{(i, P_1=0, P_2=2)}^{(2)}(j\omega)U_{(P_1=0, P_2=2)}^{(2)}(j\omega)
\]
\[
\omega \in [0 10]\) Hz \quad \text{(a)}
\]
\[
y_1(j\omega) = C_{(i, P_1=1)}^{(1)}(j\omega)U_{(P_1=1)}^{(1)}(j\omega) + G_{(i, P_1=2, P_2=0)}^{(2)}(j\omega)U_{(P_1=2, P_2=0)}^{(2)}(j\omega)
\]
\[
\times U_{(P_1=0, P_2=2)}^{(2)}(j\omega) + G_{(i, P_1=0, P_2=2)}^{(2)}(j\omega)U_{(P_1=0, P_2=2)}^{(2)}(j\omega)
\]
\[
\omega \in [10 15]\) Hz \quad \text{(b)}
\]
\[
y_2(j\omega) = C_{(i, P_1=1)}^{(1)}(j\omega)U_{(P_1=1)}^{(1)}(j\omega) + G_{(i, P_1=2, P_2=0)}^{(2)}(j\omega)U_{(P_1=2, P_2=0)}^{(2)}(j\omega)
\]
\[
\times U_{(P_1=1, P_2=1)}^{(2)}(j\omega) \quad \omega \in (15 35]\) Hz \quad \text{(c)}
\]

**Figure 5.** The spectra of \( u_2^1(t) \), \( u_2^2(t) \) and \( u_1(t)u_2(t) \) for the system in equation (49).
Functions for multi-input non-linear Volterra systems

\[ y_i(j\omega) = G^{(2)}_{i, p_1=2, p_2=0}(j\omega)U^{(2)}_{(p_1=2, p_2=0)}(j\omega) \]
\[ \omega \in (35, 50) \text{ Hz} \]  
\[ y_i(j\omega) = G^{(2)}_{i, p_1=2, p_2=0}(j\omega)U^{(2)}_{(p_1=2, p_2=0)}(j\omega) \]
\[ + G^{(2)}_{i, p_1=1, p_2=1}(j\omega)U^{(2)}_{(p_1=1, p_2=1)}(j\omega) \]
\[ \omega \in [50, 70] \text{ Hz} \]  
\[ y_i(j\omega) = G^{(2)}_{i, p_1=1, p_2=1}(j\omega)U^{(2)}_{(p_1=1, p_2=1)}(j\omega) \]
\[ \omega \in (70, 85) \cup (100, 135) \text{ Hz} \]  
\[ y_i(j\omega) = G^{(1)}_{i, p_1=1}(j\omega)U^{(1)}_{(p_1=1)}(j\omega) + G^{(2)}_{i, p_1=1, p_2=1}(j\omega)U^{(2)}_{(p_1=1, p_2=1)}(j\omega) \]
\[ \omega \in [85, 90] \cup [95, 100] \text{ Hz} \]

\[ y_i(j\omega) = G^{(1)}_{i, p_2=1}(j\omega)U^{(1)}_{(p_2=1)}(j\omega) \]
\[ \omega \in (90, 95) \text{ Hz} \]  
\[ y_i(j\omega) = G^{(2)}_{i, p_1=0, p_2=2}(j\omega)U^{(2)}_{(p_1=0, p_2=2)}(j\omega) \]
\[ \omega \in [170, 200] \text{ Hz} \]  
\[ \text{for } i = 1, 2. \]  

Equation (52) indicates that, to estimate the NOFRFs up to the second order, three different excitations are enough. Equations (52(a)–(i)) also clearly show how the energy transfer happens in the non-linear system (49) when subjected to the inputs (50) and (51). For example, from equation (52(a)), it is clearly that it is the 2nd order NOFRFs \( G^{(2)}_{1, p_1=2, p_2=0}(j\omega) \), \( G^{(2)}_{1, p_1=0, p_2=2}(j\omega) \) which transfer the energy from the frequency bands of the

![Figure 6. The output response of the system in equation (49).](image1)

![Figure 7. The output spectra of the system in equation (49).](image2)
first input ([10 35] Hz) and the second input (85 100] Hz) respectively to the frequency band [0 10] Hz in the output. The evaluated NOFRFs are shown in figures 8 and 9.

As the spectra in figure 7 show, most of the output energy is located in the frequency range [0 70] Hz. From figures 8 and 9, it can be seen that, in the frequency range of [0 70] Hz, the first input dependent NOFRFs, such as \( G^{(1)}_{(i, P_1=1)}(j\omega) \) and \( G^{(2)}_{(i, P_1=2, P_2=0)}(j\omega) \), are much bigger than the other NOFRFs. This implies that the output energy in this frequency range is mainly contributed by the first input. For example, according to equation (52(b)), the first output response at 14 Hz is contributed by the three terms \( G^{(1)}_{(1, P_1=1)}(j\omega) \times U^{(1)}_{(P_1=1)}(j\omega) \), \( G^{(2)}_{(1, P_1=2, P_2=0)}(j\omega) \times U^{(2)}_{(P_1=2, P_2=0)}(j\omega) \), and \( G^{(2)}_{(1, P_1=0, P_2=2)}(j\omega) \times U^{(2)}_{(P_1=0, P_2=2)}(j\omega) \). The contributions from these terms to the output are given in table 1.

From table 1, it can be seen that the contribution of \( G^{(2)}_{(1, P_1=0, P_2=2)}(j\omega) \) \( U^{(2)}_{(P_1=0, P_2=2)}(j\omega) \) is so small that it can be ignored. The output response at other frequencies can be analysed in a similar way.

Comparing equations (14) and (18), the main difference between the NOFRFs of the single-input and the multi-input non-linear systems is that the multi-input NOFRFs have more cross-NOFRF terms, for instance, \( G^{(2)}_{(i, P_1=1, P_2=1)}(j\omega) \), \( (i=1, 2) \) for the second order NOFRF. From equations (52(e–g)), it can be seen that, in this study, \( G^{(2)}_{(i, P_1=1, P_2=1)}(j\omega) \), \( (i=1, 2) \) only influence the components at [50 90] \( \cup [95 135] \) Hz. Equation (52(f)) also indicates that the output responses at (70 85) \( \cup [100 135] \) Hz are only determined by \( G^{(2)}_{(i, P_1=1, P_2=1)}(j\omega) \), \( (i=1, 2) \) and have a very small amplitude. At other frequency ranges, \( G^{(2)}_{(i, P_1=1, P_2=1)}(j\omega) \), \( (i=1, 2) \) will influence the output response together with other NOFRF terms, for example with \( G^{(2)}_{(i, P_1=2, P_2=0)}(j\omega) \), \( (i=1, 2) \) at [50 70] Hz. For the first output response at 55 Hz, the contributions by \( G^{(2)}_{(1, P_1=2, P_2=0)}(j\omega) \) and \( G^{(2)}_{(1, P_1=1, P_2=1)}(j\omega) \) are given in below table 2.
The results in table 2 show that, compared with \( G_{(2)}^{(1)}(P_1=2, P_2=0)(f_\omega) \), the contribution of \( G_{(1)}^{(2)}(P_1=1, P_2=1)(f_\omega) \) to the output response at 55 Hz is very small and can be ignored. Similarly, it can be found that the contribution of the cross-NOFRF to the output responses at other frequencies is also very small. To a certain degree, this implies that the influence of the cross-NOFRFs on the output responses can be ignored in this specific case.

The results shown in figures 8 and 9 indicate that the maximum gains in the NOFRFs of \( G_{(2)}^{(1)}(P_1=2, P_2=0)(f_\omega) \) appear near 16 Hz and 28 Hz, \((i=1,2)\). This means that, at these frequencies, the energy transfer through these NOFRFs becomes more efficient, and the frequency
components at these frequencies will become significant in the output spectra. This can be confirmed by the output spectra shown in figure 7 where some significant components can be found at these frequencies.

The above qualitative analysis gives a clear interpretation regarding why and how the generation of new frequencies happens in a multi-input non-linear system, and extends the procedure for the same analysis for single-input non-linear systems to the more general multi-input non-linear system case.

5. Conclusions and remarks

The Volterra series is a powerful tool that can be used to describe a wide class of non-linear systems which are stable at zero equilibrium. Generally, this class of non-linear systems can describe the non-linear phenomena including secondary resonance, the generation of superharmonics, and inter-modulations, et al., but excluding the chaos and bifurcation. In our previous study Lang and Billings (2005), based on the Volterra series, a new concept called NOFRF was proposed to study the energy transfer properties of non-linear systems in the frequency domain. In the present study, the concept of NOFRFs has been extended from the single-input non-linear system case to the multi-input non-linear system case. Given the frequency range of the inputs, a new method was also developed to determine the output frequency range. The phenomenon of energy transfer (Nayfeh and Mook 1979) in a 2DOF non-linear system subjected to two input excitations was investigated using the concept of NOFRFs for multi-input systems.

Multi-input systems are important in many engineering systems and structures. For example, multi-degree of freedom mechanical structures are a typical example of this category of systems. Therefore, the extension of the NOFRF concept to the more general multi-input case of non-linear systems is important for the potential applications of the NOFRF concept to a wide range of engineering areas.

Acknowledgements

The authors gratefully acknowledge the support of the Engineering and Physical Science Research Council, UK, for this work.

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