Frequency Domain Analysis in Rotor/Stator Contact Problems

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Abstract

There are a variety of abnormal running conditions in rotating machinery which lead to rotor/stator dynamics interactions which, in turn, can cause a rich mixture of effects associated with rub-related phenomena. These effects manifest themselves in the occurrence of multiple solutions for steady-state vibration response scenarios, including amplitude jumps during rotor acceleration, and vibration responses at different/multiple frequencies of excitation forces such as unbalance. This paper describes a numerical algorithm based on the harmonic balance method to calculate the periodic response of a non-linear system under periodic excitation. The algorithm also calculates the stability of the periodic solutions found, marks turning and bifurcation points, and follows a solution branch over varying system parameters via arc-length continuation.

1 INTRODUCTION

The motivation for this study comes from rotor/stator contact induced vibration problems in turbo-machinery. They can include: rotors touching seals, rotor touching retainer bearings when main active magnetic bearings fail, inter-shaft contact in multiple spool engines, rotor blades contacting the stator, increased bearing clearance through wear or outright bearing failure. In many of these scenarios the rotor continues to rotate and so, depending on the problem, it is often the steady-state response to the out-of-balance excitation forces which is of concern, rather than a particular transient event.

Both the harmonic balance method (HBM) and continuation schemes are well-known numerical tools to study nonlinear dynamics problems. However, they seem to be used rarely in conjunction with each other in engineering applications, as continuation appears more frequently with time-domain methods, such as shooting or boundary value problem solvers, whereas the HBM is essentially a frequency-domain method. Rotor dynamics examples using time-domain methods with continuation are: Sundararajan and Noah (1997) for squeeze-film-damper and journal bearing analysis and Petrov (1996) for shroud/blade friction. Harmonic balance is not only convenient for purposes of linearisation of systems with small non-linearities (Géradin and Kill, 1988), but can can also be applied to large non-linearities (Choi and Noah, 1988). Kim et al. (1991) analysed the behaviour of rotors with bearing housing clearances using the HBM at discrete speeds but without continuation.

As with most numerical techniques, calculating the Jacobian (most probably by numerical finite difference estimation) is part of solving the equations set up by the harmonic balance method (Gasch and Knothe, 1989). It will be shown that the availability of the Jacobian means that the HBM lends itself nicely to studies of the stability of a solution without having to go back into the time domain (Floquet analysis). Such an algorithm for a non-linear system is based on the approach for *linear* time-variant systems. Naturally, the algorithm described here is also applicable to other common non-linear elements in structural dynamics.

2 HARMONIC BALANCE FORMULATION

Given the computing resources, the HBM is easily applicable to problems with a large number of degrees-of-freedom (DOFs). Typically, such a problem consists of finite element models of large parts of the structure or substructures where a linear representation is adequate, and some 'problematic' DOFs for special areas, e.g. where friction, impacts, or other interaction occurs. Usually, the linear DOFs outnumber the nonlinear ones by a large ratio. The example that will be used later on deals with the dynamics when rotor and stator come into contact. The rotor and stator are modelled as linear structures, and there will be some linear external forces like gravity and out-of-balance. The contact region supplies the nonlinear forcing at a few degrees of freedom on both rotor and stator.

For simplicity, the complete system is split into its linear part, represented by the usual mass, stiffness, damping matrices, with some linear external force vector, $\{f_\mathrm{u}\}$ (e.g. unbalance), and its nonlinear part, which is represented here as a single force vector $\{f_\mathrm{c}\}$ combining all nonlinear effects (contact between rotor and stator):

$$[M]\{\ddot{r}\} + [C]\{\dot{r}\} + [K]\{r\} = \{f_\mathrm{u}(t)\} + \{f_\mathrm{c}(r)\} \quad (1)$$

The harmonic balance method offers an alternative to time-domain methods for analysis of cases where a steady-state, periodic solution to the equation of motion is sought. The general idea is to represent each time history, $r(t)$, by its frequency content, $R(\omega)$, to obtain a set of equations by balancing the terms with the same frequency components and to start an iterative procedure to find the roots of these equations. An integer variable, ν, is introduced to accommodate possible sub-harmonics of an external excitation frequency, Ω (e.g. shaft rotation). The displacements, $r(t)$, and forces, $f(t)$, are represented as truncated Fourier series with N harmonics:

$$r(t) = \sum_{n=1}^{N} R_n \, \mathrm{e}^{\mathrm{i}\frac{n\Omega}{\nu}t} \qquad f_\mathrm{c}(t) = \sum_{n=1}^{N} F_{\mathrm{c}_n} \, \mathrm{e}^{\mathrm{i}\frac{n\Omega}{\nu}t} \qquad f_\mathrm{u}(t) = \sum_{n=1}^{N} F_{\mathrm{u}_n} \, \mathrm{e}^{\mathrm{i}\frac{n\Omega}{\nu}t} \quad (2)$$

Substituting these expressions, (2), into the rotor equation of motion, (1), and balancing the harmonic terms yields, for a harmonic n:

$$\left(-\left(\frac{n\Omega}{\nu}\right)^2 [M] + \mathrm{i}\frac{n\Omega}{\nu}[C] + [K] \right) \{R_n\} = \{F_{\mathrm{c}_n}\} + \{F_{\mathrm{u}_n}\}. \quad (3)$$

Bringing all N harmonics into one equation can be symbolised as

$$\left[\tilde{K}\right]\left\{R\right\} - \left\{F_{\mathrm{c}}\right\} - \left\{F_{\mathrm{u}}\right\} = 0 \tag{4}$$

and $\left\{R\right\}$ and $\left\{F\right\}$ are the vectors of Fourier coefficients of displacements and forces, respectively. As the Fourier coefficients, F_{c_n}, of the non-linear forces, f_{c}, are functions of the displacements (and thus their respective Fourier coefficients),

$$F_{\mathrm{c}_n} = F_{\mathrm{c}_n}(R_0(\omega_0), R_1(\omega_1), \ldots, R_N(\omega_N)), \tag{5}$$

equation (4) is non-linear and must be solved iteratively. This iteration process (Kim et al., 1991) can be sketched as:

$$R(\omega)^{(k)} \xrightarrow{\mathrm{FT}^{-1}} r(t)^{(k)} \rightarrow f_{\mathrm{c}}(t)^{(k+1)} \xrightarrow{\mathrm{FT}} F_{\mathrm{c}}(\omega)^{(k+1)} \rightarrow R(\omega)^{(k+1)}$$

Finite element models of rotor/stator structures can contain quite a large number of degrees-of-freedom. Setting up equation (4) then leads to a much bigger problem with $2N+1$ times more unknowns (real and imaginary components for N harmonics and a DC component).

Reduction The harmonic balance method offers an elegant means of reducing the problem order, so that only the non-linear DOFs need to be kept (Kim et al., 1991). Clearly being able to keep only the non-linear DOFs vastly increases the speed in cases of linear structures which have a few additional non-linear elements, as is typical for many classes of problems. The equation (3) is re-ordered for every harmonic, n (the subscripts $_n$ are omitted in this section for clarity):

$$\begin{bmatrix} \tilde{K}_{\mathrm{mm}} & \tilde{K}_{\mathrm{ms}} \\ \tilde{K}_{\mathrm{sm}} & \tilde{K}_{\mathrm{ss}} \end{bmatrix} \begin{pmatrix} R_{\mathrm{m}} \\ R_{\mathrm{s}} \end{pmatrix} = \begin{pmatrix} F_{\mathrm{c}} \\ 0 \end{pmatrix} + \begin{pmatrix} F_{\mathrm{u_m}} \\ F_{\mathrm{u_s}} \end{pmatrix} \tag{6}$$

where subscripts $_\mathrm{m}$ and $_\mathrm{s}$ stand for non-linear (master) and linear (slave) degrees-of-freedom, respectively. It follows that

$$\left[\hat{\tilde{K}}\right] = \left[\tilde{K}_{\mathrm{mm}}\right] - \left[\tilde{K}_{\mathrm{ms}}\right]\left[\tilde{K}_{\mathrm{ss}}\right]^{-1}\left[\tilde{K}_{\mathrm{sm}}\right] \tag{7}$$

$$\left\{\hat{F}_{\mathrm{u}}\right\} = \left\{F_{\mathrm{u_m}}\right\} - \left[\tilde{K}_{\mathrm{ms}}\right]\left[\tilde{K}_{\mathrm{ss}}\right]^{-1}\left\{F_{\mathrm{u_s}}\right\} \tag{8}$$

and

$$\left[\hat{\tilde{K}}\right]\left\{R_{\mathrm{m}}\right\} - \left\{F_{\mathrm{c}}\right\} - \left\{\hat{F}_{\mathrm{u}}\right\} = 0 \tag{9}$$

In contrast to the widely-used Guyan reduction, equation (9) is an exact reduction of the original problem as long as the prerequisites for applying the harmonic balance method are met and the number of harmonics included in the decomposition is sufficient.

Continuation Usually, the system behaviour is of interest over a range of values for at least one parameter (e.g. speed of shaft rotation), so that the solution has to be calculated at different parameter values consecutively. Any continuation scheme is just as applicable in the frequency domain as it is in the time-domain. The task of finding a periodic solution for equation (1) can be transformed into an equivalent root-finding problem in either time or frequency domain, for example by means of finite difference, shooting, HBM. An arc-length continuation scheme may then be employed to move along the arc-length α of a solution branch of the root-finding problem, facilitating the passing of turning points (overhung part of the solution branches). Continuation schemes are standard tools and the reader is referred to Seydel (1994); Nayfeh and Balachandran (1995) for a detailed treatment.

3 STABILITY

When HBM with arc-length continuation is employed, there is nothing in the algorithm per se that can warn the user that a particular solution branch followed has stepped over a turning or bifurcation point and the solution has switched stability, from stable to unstable, or vice versa. For example, there might only be a little change in the conditioning of the Jacobian of the system before and after such a change, nor is there a change in the convergence behaviour of the algorithm. This is a practical problem (not a theoretical one, as the Jacobian F_y is indeed singular exactly on a turning or bifurcation point) as change in conditioning of the Jacobian could occur far more rapidly than the step-length is able to resolve. However, at little cost, stability can be analysed with a modification of an algorithm employed for linear time-variant systems. The algorithm is called Hill's method and transforms a linear time-variant system into an eigenvalue problem of a linear time-invariant system (Gasch and Knothe, 1989). In order to use the same approach for non-linear systems, the stability analysis is carried out by investigating the effect of a perturbation around a periodic solution $r^\star(t)$. Let the perturbation be described as $p(t)$, where $p(t)$ consists of a decay term $\mathrm{e}^{\lambda t}$ and a periodic term $s(t)$ (Genta, 1999):

$$p(t) = \mathrm{e}^{\lambda t}\, s(t) \qquad s(t) = \sum_{n=-N}^{N} S_n\, \mathrm{e}^{\mathrm{i}\frac{n\Omega}{\nu} t} \tag{10}$$

Assigning $r(t) = r^\star(t) + p(t)$ and by substituting the Fourier representations of $r^\star(t), s(t)$, the harmonic components can be balanced in an analogue manner to equation (4):

$$\begin{aligned}\left[\tilde{K}\right]\{R^\star\} + \left(\lambda^2\left[\tilde{M}\right] + \lambda\left[\tilde{C}\right] + \left[\tilde{K}\right]\right)\{S\}\,\mathrm{e}^{\lambda t} \\ = \{F_\mathrm{u}\} + \left\{F_\mathrm{c}\left(R^\star + S\,\mathrm{e}^{\lambda t}\right)\right\}\end{aligned} \tag{11}$$

where $\left[\tilde{M}\right]$, $\left[\tilde{C}\right]$ are constructed in a similar manner to $\left[\tilde{K}\right]$ in equation (4) and $\left\{R^\star(\omega)\right\}$, $\left\{S(\omega)\right\}$ are the vectors of Fourier coefficients for $r^\star(t)$, $s(t)$, respectively.

In what follows, an attempt is being made to find a cost-effective linearisation for the term $\left\{F_\mathrm{c}\left(R^\star + S\,\mathrm{e}^{\lambda t}\right)\right\}$ so that equation (11) can be developed further. Consider a variant of equation (4):

$$\{F_\mathrm{c}\} = \left[\tilde{K}\right]\{R\} - \{F_\mathrm{u}\} - \{E(R)\} \tag{12}$$

where $\{E(R)\}$ is the error in the balancing terms. Developing this as a Taylor series around a known solution of equation (4), $\{R^\star\}$, one obtains

$$\{F_\mathrm{c}(R)\} = \left[\tilde{K}\right]\{R\} - \{F_\mathrm{u}\} - [E'(R^\star)]\left(\{R\} - \{R^\star\}\right) + \text{higher order terms} \tag{13}$$

with the abbreviation $[E'] = \left[\frac{\partial E}{\partial R}\right]$. Substituting equation (13) into (11) and neglecting terms of higher order, equation (11) simplifies to the following eigenvalue problem:

$$\left(\lambda^2\left[\tilde{M}\right] + \lambda\left[\tilde{C}\right] + [E'(R^\star)]\right)\{S\} = 0. \tag{14}$$

It is important to note that the term $[E'(R^\star)]$ is already available as a by-product of the quasi-newton solution technique, most probably as a numerical approximation, as it is the Jacobian of the objective function defined by equation (4).

Solving for the eigenvalues of equation (14), one obtains a set of λ_i with real and imaginary parts, where a negative real part indicates stability of the solution, as the perturbation $p(t)$ decays with time, and a positive real part indicates instability. So by solving this eigenvalue problem at the end of the overall iteration procedure, and simply checking if any λ_i possesses a real part > 0, one can easily determine whether a periodic solution $r^\star(t)$ is unstable. This also helps with finding possible bifurcation points. A change in stability of a solution branch is a sufficient indicator that a turning or bifurcation point has been passed, and the algorithm could be directed to determine the cross-over point within this interval of change more closely. Should this be of interest and the cross-over point found, the rank of F_y and $\begin{bmatrix} F_y & F_\Omega \end{bmatrix}$ at the cross-over determines whether the point in question is a turning or bifurcation point (Seydel, 1994). If indeed it is a bifurcation point a further solution branch may be followed.

4 NUMERICAL EXAMPLE

Although the algorithm described above is easily scalable to systems with a large number of degrees of freedom, a simple modified Jeffcott rotor (Figure 1) is used here for clarity. The equations of motion for a Jeffcott rotor interacting with a linear stator structure are:

$$m_\mathrm{r}\ddot{r}_\mathrm{r} + c_\mathrm{r}\dot{r}_\mathrm{r} + k_\mathrm{r}r_\mathrm{r} = -f_\mathrm{c} + \Omega^2 m_\mathrm{r}\epsilon_\mathrm{m}\,\mathrm{e}^{\mathrm{i}\Omega t} \tag{15}$$

$$m_\mathrm{s}\ddot{r}_\mathrm{s} + c_\mathrm{s}\dot{r}_\mathrm{s} + k_\mathrm{s}r_\mathrm{s} = f_\mathrm{c} \tag{16}$$

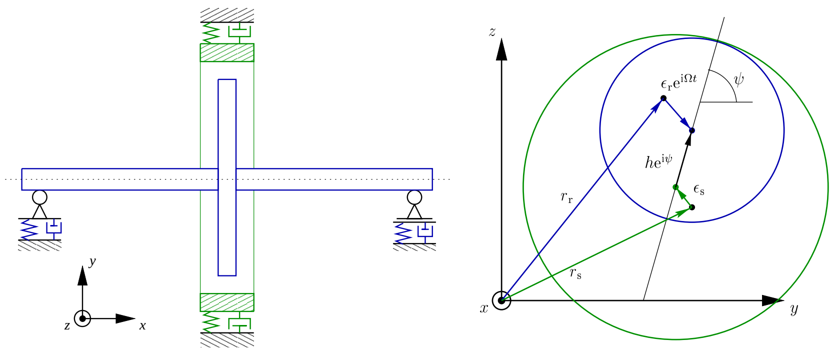

Figure 1: A Jeffcott rotor with stator

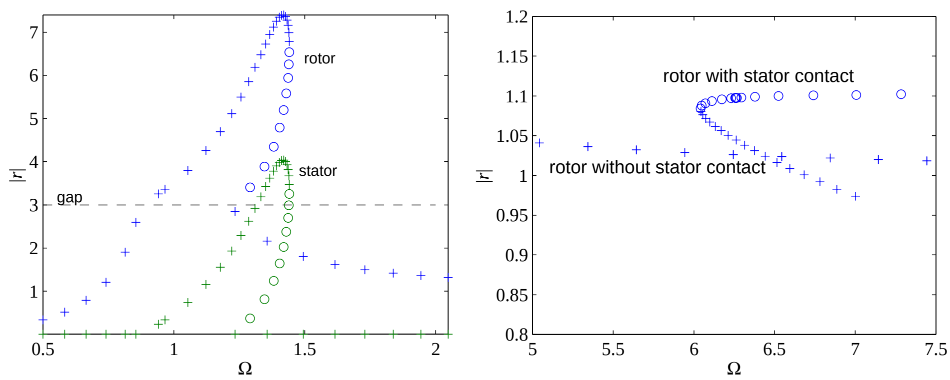

Figure 2: response magnitude at constant speed. solution: + stable, o unstable

where $r_{\mathrm{r}}, r_{\mathrm{s}}$ are the rotor and stator displacements in the complex plane and f_{c} is the contact force between rotor and stator, $f_{\mathrm{c}} = k_{\mathrm{c}}\delta$ where δ is the depth of the contact described below in (17) and k_{c} the local (in this case linear) contact stiffness (for more realistic simulations one would have to choose a nonlinear contact force, see Fumagalli and Schweitzer (1996)). For the purpose of numerical simulation, a small contact penetration δ of the rotor and stator rings is allowed. The contact stiffness k_{c} in this penetration region is being set to a value orders of magnitude higher than the rotor shaft or stator suspension stiffness, so that the penetration depth is orders of magnitude lower than rotor and stator deflections. The contact depth is defined as

$$\delta(t) = \begin{cases} r_{\mathrm{r}} + \epsilon_{\mathrm{r}}\,\mathrm{e}^{\mathrm{i}\Omega t} - r_{\mathrm{s}} - \epsilon_{\mathrm{s}} - h\,\mathrm{e}^{\mathrm{i}\psi} & \text{if } |r_{\mathrm{r}} + \epsilon_{\mathrm{r}}\,\mathrm{e}^{\mathrm{i}\Omega t} - r_{\mathrm{s}} - \epsilon_{\mathrm{s}}| > h, \\ 0 & \text{otherwise} \end{cases} \tag{17}$$

where $r_{\mathrm{r}}, r_{\mathrm{s}}$ are rotor and stator displacements, h is the gap size, ϵ_{r} a possible offset of the rotor disc and ϵ_{s} a stator offset. These entities are depicted in Figure 1.

For the special case of full annular rub with $\epsilon_{\mathrm{r}}, \epsilon_{\mathrm{s}} = 0$ and isotropic rotor supports, the equations of motion become quasi-static for pure forward or backward whirl. At a given speed, the steady-state conditions of rotor whirl are such that the radial deflection of the rotor is constant. The only frequency component in the unbalance response spectrum is thus the engine-order speed, Ω. This simple case is used here to illustrate the stability and continuation study. In Figure 2 the magnitudes of the rotor and stator responses $r_{\mathrm{r}}, r_{\mathrm{s}}$ are plotted versus the rotor speed of rotation Ω. One can see that at speeds $\Omega < 0.9$ the rotor unbalance response is too low to overcome the clearance ($h = 3$, dashed line) and rotor and stator are not in contact (stator response zero). At speeds $0.9 < \Omega < 1.4$ rotor and stator are in contact (non-zero stator response), albeit the overhung part of the curve represents an unstable solution. At speeds $\Omega > 1.4$, well past the natural frequency of the rotor, which has been normalised to $\omega_{\mathrm{r}} = 1$, the super-critically running rotor loses contact with the stator.

Figure 2b shows a second solution branch at $\Omega > 6$, which is not seen in the 1 DOF Duffing type oscillators that display only the overhung behaviour in Figure 2. It must be noted that by following the branch previously discussed, the one that lost contact with the

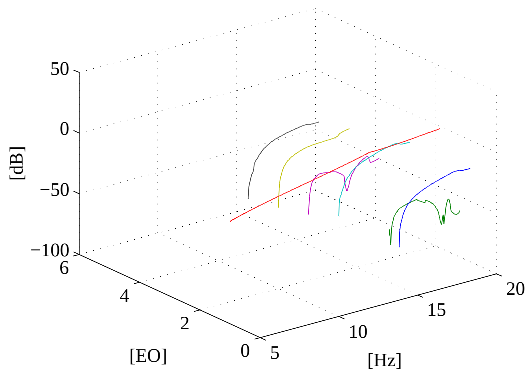

Figure 3: harmonics 1/3, 1/2, 1, 2, 3, 4, 5 EO versus speed of rotation [Hz]

stator and is coming into this picture from the left, there is no indication of the existence of the second branch. The branches of the rotor deflections intersect, but in the whole space of rotor and stator deflections these curves do not come near each other, so there is no warning in terms of changes in stability or conditioning of the Jacobian.

This second solution branch was found by brute force, using hundreds of random initial guesses at different speeds, in a quest to find out whether the system could vibrate in an 'inverted' modeshape, where the stator, which has twice the rotor mass in this example, is moving like a hola-hoop around the rotor. The success rate of finding the second branch from the random initial guesses as a whole were poor: many did not converge to any solution, most settled on the solution where rotor and stator were out of contact. The ones that did settle on the second solution branch seemed to settle just as easily in terms of convergence on the unstable part of the branch as on the stable one, albeit a physical system would object to that. Once a solution on that branch is found, it is easy to follow by continuation.

Making the system more general with $\epsilon_r, \epsilon_s \neq 0$ and non-isotropic rotor and stator mounts gives rise to more complicated motion. Under these circumstance the system has speed regimes where the rotor is rubbing along the stator only over parts of the orbit, causing sub- and super-harmonics to emerge. Even a full annular rub in these circumstances would consist of higher harmonics (but no sub-harmonics). Figure 3 shows the frequency content of the rotor motion at various speeds. The harmonics are expressed in engine order ratios and their magnitude is given in dB. At lower speeds (<15Hz) only the 1EO component is present, as rotor and stator are out of contact and without the nonlinear forces no other frequency components but 1EO (unbalance) is expected. As soon as rotor and stator come into contact at higher speeds do the other frequency components start to come into play.

5 CONCLUSIONS

The presented frequency-domain algorithm calculates the periodic solution, traces the solution along a varying parameter and determines the stability of a solution branch. For the investigation of stability previous studies have reverted back to the time-domain. A major motivation for working in the frequency domain is the computational speed advantages it has over time-domain methods.

The solution process of the HBM method itself is less expensive than time-domain methods, and the reduction to the non-linear degrees-of-freedom offers vast savings in large finite element models with only a few non-linear components. It was found that in the given numerical example the harmonic balance method was over 100 times faster than time-domain shooting and boundary-value-problem solving.

HBM also functioned properly in many instances where the time integration routines had difficulties. The source of these difficulties lies in the contact problem. As both rotor and stator have non-negligible mass (and thus dynamics in their own right), the penalty stiffness k_c determines the violence of the rotor/stator impacts and thus tests the robustness and influences the speed of convergence of any chosen method.

A more sophisticated contact model can alleviate this problem to a large extent. However, there are situations where the harmonic balance method does not find a solution. This is usually an indication that not enough or not the right harmonic components are included in the setup. This can probably be circumvented by choosing a hybrid approach of an initial setup in time-domain and continuation in frequency-domain. This approach will be investigated in future work.

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