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Solving $3+1$ QCD on the Transverse Lattice Using $1+1$ Conformal Field Theory

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Abstract

A new transverse lattice model of $3+1$ Yang-Mills theory is constructed by introducing Wess-Zumino terms into the 2-D unitary non-linear sigma model action for link fields on a 2-D lattice. The Wess-Zumino terms permit one to solve the basic non-linear sigma model dynamics of each link, for discrete values of the bare QCD coupling constant, by applying the representation theory of non-Abelian current (Kac-Moody) algebras. This construction eliminates the need to approximate the non-linear sigma model dynamics of each link with a linear sigma model theory, as in previous transverse lattice formulations. The non-perturbative behavior of the non-linear sigma model is preserved by this construction. While the new model is in principle solvable by a combination of conformal field theory, discrete light-cone, and lattice gauge theory techniques, it is more realistically suited for study with a Tamm-Dancoff truncation of excited states. In this context, it may serve as a useful framework for the study of non-perturbative phenomena in QCD via analytic techniques.

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1. Introduction

The transverse lattice approach to $3+1$ Yang-Mills theory (QCD) originally developed by Bardeen and Pearson[1] over ten years ago, incorporates conceptual and computational advantages that are found separately in other formulations. Like the 4-D Euclidean lattice formulation, the physical degrees of freedom are link variables of a discrete lattice which are interpreted as phase factors $e^{i\int A}$. The transverse lattice models incorporate the non-perturbative dynamics of QCD and are well suited for studying the bound state spectrum[2]. However in the transverse lattice construction, the lattice is only two-dimensional. Local 2-D continuum gauge fields are also present to gauge the symmetries at each site. The local gauge invariance is then used to eliminate, via gauge fixing, the 2-D gauge fields in favor of a non-local Coulomb interaction for the link fields. This is accomplished in a light-cone gauge $A_- = 0$ and with light-cone quantization so that A_+ can be eliminated by using its equation of constraint. All physical states have positive light-cone energy P^+. This eliminates two degrees of freedom and simplifies the classification of the bound states (see ref. [3] for further discussion of the advantages of the light-cone approach).

The basic action for each link on the transverse lattice is the 2-D unitary $SU(N)$ principal chiral non-linear sigma model. Although this sigma model is exactly solvable via a Bethe-ansatz technique[4], this solution cannot be easily applied in the transverse lattice context. The Bardeen Pearson model is a linear sigma model approximation of the non-linear model in which the unitarity constraint of the link fields is relaxed. The $N \times N$ matrices of the linear sigma model are constrained by introducing potential terms into the theory which are designed to drive the system into the non-linear phase[5]. In the numerical work of Bardeen, Pearson, and Rabinovici[2], glueballs are constructed from local two-link and four-link bound states which are smeared over the 2-D lattice. This truncation of the Hilbert space is a non-perturbative light-front Tamm-Dancoff approximation to the QCD bound state problem[6]. The numerical results based on this approach were inconclusive. The links were weakly coupled via the Coulomb interation, and the spectrum was qualitatively similar to what would be obtained from a strong coupling expansion in ordinary lattice gauge theory[7].

There are a number of changes one could make to their original analysis that might improve the situation. This paper will focus on directly solving the non-linear

sigma model dynamics instead of using the linear sigma model approximation. By introducing Wess-Zumino[8] terms into the sigma model action, we will describe the non-linear sigma model dynamics in the basis of operators given by the well-studied and exactly solvable Wess-Zumino-Witten[9] (WZW) model. The WZW currents will be the linear variables which describe exactly the dynamics of the non-linear sigma model. The Wess-Zumino terms in the action will become irrelevant operators in the continuum limit.

The non-linear aspects of the principle chiral sigma model are retained in the WZW model. The unitary link fields (and products of link fields) appear as the primary fields of the WZW model, and play a crucial role in defining the highest weight states of the Hilbert space of the WZW model. The highest weight states correspond to zero modes of Wilson loops on the transverse lattice. This zero mode structure is lacking in the linear sigma model treatment of the transverse lattice theory. (The structure presumably corresponds to the space of soliton excitations of the linear sigma model fields.)

The advantage of exactly solvable non-linear sigma model dynamics must be weighed against the two potential disadvantages of this approach. First, this WZW model approach will only work for the discrete values of bare sigma model coupling constants which correspond to the non-trivial WZW fixed points. This will in turn place a constraint on the QCD coupling constants that this model can obtain in the continuum limit. These particular values are not special points in the context of $3+1$ QCD, but rather these are points where we can apply our limited knowledge of the 2-D non-linear sigma model to simplify the local dynamics of the link fields. Second, the continuum limit may be difficult to obtain because the irrelevant terms added to simplify the local link dynamics may be large for finite lattice spacings. This issue can only be resolved by explicit numerical simulation.

Preliminary work on the use of the gauged WZW model to describe the dynamics of lattice model links was discussed in ref. [10]. A transverse lattice model with one lattice dimension and two continuum dimensions was studied, and it was found that assigning the same Wess-Zumino term, with the same coupling constant, to each link leads to an order a term in the continuum limit, where a is the lattice spacing. This order a term generates the $2+1$ pure Chern-Simons action in the continuum limit, and its dynamics was discussed in some detail. The states of the Chern-Simons model correspond to zero modes of Wilson Loops; similar states will generate the vacuum sectors in the $3+1$ QCD model. An important lesson from this work is that one cannot simply assign the same Wess-Zumino term to each

link in the $3+1$ QCD case, because the leading terms in the continuum limit must go as order a^2 in this case, and not order a as in the $2+1$ Chern-Simons theory. This means that the Wess-Zumino terms must be staggered from site to site, with coupling constants $\pm k$. The study of how to correctly stagger the Wess-Zumino terms is the major topic of this paper.

Section 2 is review of the basic transverse lattice construction of QCD based on the (unitary) non-linear sigma model. The degrees of freedom on the lattice are introduced, and the "naive" continuum limit is taken by performing a Bloch wave expansion of the link fields. In section 3, we begin the analysis of adding Wess-Zumino terms to the action. It is found that the structure of the staggered Wess-Zumino terms which generate the Coulomb potential that in turn correctly drives the system to the desired continuum limit violates local gauge invariance by generating non-Abelian anomaly terms. In section 4 this difficulty is resolved by defining a new model which has a different structure of local gauge invariance, but has the correct continuum limit. In the new model, pairs of nearest neighbor links are associated with a single local 2-D gauge symmetry, and the anomalies from the Wess-Zumino terms for each pair of sites cancel. The local 2-D gauge symmetry is reduced by a factor of two from the transverse lattice construction with no Wess-Zumino terms. The remaining local gauge symmetry for each pair of links in the *bilocal* transverse lattice model is then properly gauge fixed in light-cone gauge. In section 5, the current algebraic solution of the WZW model is reviewed and the quantum theory of the new model is discussed. Particular emphasis is placed on the highest weight states of the current algebras, which generate the space of Wilson loop zero modes on the lattice. Aspects of future bound-state calculations and other applications of this construction are discussed in section 6.

Fig. 1. The degrees of freedom associated with each site $\vec{x}_\perp$ are the 2-D gauge fields $A_\pm(\vec{x}_\perp)$, and the link fields $U_x(\vec{x}_\perp)$ and $U_y(\vec{x}_\perp)$, defined on the links as in fig. 1.

2. The Basic Transverse Lattice Construction

In this section, the non-linear sigma model-based formulation of the transverse lattice construction of QCD is reviewed, and the process of taking the naive continuum limit is studied.

Consider the matrix-valued chiral fields $U_\alpha(\vec{x}_\perp; x^+, x^-)$, where $\alpha = 1, 2$, which belong to the fundamental representation of $SU(N)$. These fields lie on the links $[\vec{x}_\perp, \vec{x}_\perp + \vec{\alpha}]$ of a discrete square lattice of points $\vec{x}_\perp = a(n_x, n_y)$, with lattice spacing a and basis vectors $\vec{\alpha} = (a, 0)$ or $(0, a)$. The link fields are continuous functions of the light-cone coordinates $x^\pm = (x^0 \pm x^1)/\sqrt{2}$, so that the two-dimensional lattice describes a partially discretized $3+1$ dimensional Minkowski space field theory[1](see fig. 1).

The links fields are defined to transform on the left and right under independent local 2-D gauge transformations associated with the sites that the links connect,

$$\delta_G U_\alpha = \Lambda_{\vec{x}_\perp}(x^\mu) U_\alpha - U_\alpha \Lambda_{\vec{x}_\perp + \vec{\alpha}}(x^\mu) \ . \tag{2.1}$$

To construct a gauge invariant action, introduce $SU(N)_{\vec{x}_\perp}$ gauge fields $A_\pm(\vec{x}_\perp) = iA^a_\pm(\vec{x}_\perp)T^a$, where the group generators T^a satisfy $[T^a, T^b] = if^{abc}T^c$ and $\mathrm{Tr}\, T^a T^b = \frac{1}{2}\delta^{ab}$. The infinitesimal transformation law for the gauge fields is

$$\delta_G A_\pm(\vec{x}_\perp) = \partial_\pm \Lambda_{\vec{x}_\perp} + [\Lambda_{\vec{x}_\perp}, A_\pm(\vec{x}_\perp)] \ , \tag{2.2}$$

and the covariant derivative is

$$D_\mu U_\alpha(\vec{x}_\perp) = \partial_\mu U_\alpha - A_\mu(\vec{x}_\perp) U_\alpha + U_\alpha A_\mu(\vec{x}_\perp + \vec{\alpha}) \ . \tag{2.3}$$

The transverse lattice action is given by[1]

$$\begin{aligned} I_{\mathrm{TL}} = \sum_{\vec{x}_\perp} \mathrm{Tr} \int d^2x \Bigg\{ & \frac{a^2}{2g_1^2} F^{\mu\nu} F_{\mu\nu} + \frac{1}{g^2} \sum_\alpha D_\mu U_\alpha D^\mu U_\alpha^\dagger \\ & + \frac{1}{g_2^2 a^2} \sum_{\alpha \neq \beta} \Big[U_\alpha(\vec{x}_\perp) U_\beta(\vec{x}_\perp + \vec{\alpha}) U_\alpha^\dagger(\vec{x}_\perp + \vec{\beta}) U_\beta^\dagger(\vec{x}_\perp) - 1 \Big] \Bigg\}. \end{aligned} \tag{2.4}$$

[1] The indexes $\alpha, \beta, \ldots$ denote transverse coordinates x, y, and $\mu, \nu, \ldots$ denote longitudinal coordinates $x^\pm$.

As the lattice spacing a is taken to zero, the interaction terms will select smooth configurations as the dominant contributions to the quantum path integral; both the interactions mediated by the local 2-D gauge fields and the plaquette interactions will generate large potentials, unless the link configurations are smooth. For the plaquette term this is obvious; for the gauge interactions, this is clear only after studying the Coulomb potential obtained by gauge fixing in light-cone gauge $A_- = 0$, in the context of light-cone quantization[2]. We will discuss this process further in the next sections for the new transverse lattice model. Inserting the Bloch-wave expansion

$$U_\alpha = \exp\left[-aA_\alpha(\vec{x}_\perp + \tfrac{1}{2}\vec{\alpha})\right] \ , \tag{2.5}$$

and keeping only the lowest order contributions, one obtains from the gauged sigma model kinetic term

$$I_K = \frac{2a^2}{g^2} \sum_{\vec{x}_\perp,\alpha} \int d^2x F_{\mu\alpha}F^{\mu\alpha} + \mathcal{O}(a^4) \ , \tag{2.6}$$

and from the the plaquette term

$$I_P = \frac{a^2}{g_2^2} \sum_{\vec{x}_\perp,\alpha,\beta} \int d^2x (F_{\alpha\beta})^2 + \mathcal{O}(a^4) \ . \tag{2.7}$$

In deriving eqn. (2.6), the fields $A_\pm(\vec{x}_\perp)$ were also assumed to be slowly varying on the lattice. Combining these three terms and tuning the coupling constants to $g_1 = g_2 = g$ yields the continuum 4-D QCD action. For the quantum theory on the lattice, the Lorentz covariant critical point for each lattice spacing a is determined by examining specific properties of the states, such as the mass spectrum $3+1$ Lorentz multiplets and the covariant dispersion relations[2].

This is the non-linear sigma model (NLSM)-based transverse lattice model of QCD. In an ideal world, there would be an exact solution to the primary chiral sigma model, which could be used as a kernel to solve the entire model perturbatively in the interactions. The idea would be to use the states that are diagonalize with respect to the NLSM Hamiltonian as a basis for construction of the singlet bound states of the full theory. This is the approach pioneered by Schwinger in his solution to $1+1$ massless QED[11]. There, the full Hamiltonian was diagonalized in the basis of free fermions. While there has been some progress in understanding the quantum NLSM[4], the progress is still insufficient to generate a Schwinger-type solution to the problem at hand. A "Fock space" of operators with simple commutation relations is required.

3. Transverse Lattice with Wess-Zumino Terms

Now consider the NLSM with Wess-Zumino term[2][9],

$$I_{WZW} = \frac{1}{\lambda^2}\int d^2x \mathrm{Tr}\ \partial_\mu U \partial^\mu U^\dagger + k\Gamma\ , \tag{3.1}$$

where U is a unitary matrix. The non-local Wess-Zumino term Γ is well-defined only up to $\Gamma \to \Gamma + 2\pi$, and therefore the coupling constant k is an integer. The model is exactly solvable for the restricted critical values of the NLSM coupling constant

$$\lambda^2 = \frac{4\pi}{|k|}\ . \tag{3.2}$$

For these values there exist a complete basis of conserved vector and axial-vector currents from which a Fock space representation of the quantum theory can be constructed. For positive k, the conserved currents are

$$J_-(x^-) = (\partial_- U)U^\dagger\ , \quad J_+(x^+) = U^\dagger(\partial_+ U)\ , \tag{3.3}$$

and for negative k,

$$\tilde{J}_+(x^+) = (\partial_+ U)U^\dagger\ , \quad \tilde{J}_-(x^-) = U^\dagger(\partial_- U)\ . \tag{3.4}$$

An elegant current algebraic solution for the quantum theory was given by Dashen and Frishman[13] for $k = 1$, and was later generalized for arbitrary k by Knizhnik and Zamolodchikov[14]. This solution will be discussed further in section 5.

To apply the WZW NLSM technology to the transverse lattice formulation of QCD, we associate a WZW field U with each link $U_\alpha(\vec{x}_\perp)$[3]. For each link $U_\alpha(\vec{x}_\perp)$, the currents $J_{\pm,\alpha}$ and $\tilde{J}_{\pm,\alpha}$ are defined via eqns. (3.3) and (3.4). The gauge variation of the J currents is given by

$$\begin{aligned}\delta_G J_{+,\alpha} =& -\partial_+\Lambda_{\vec{x}_\perp+\vec{\alpha}} + [\Lambda_{\vec{x}_\perp+\vec{\alpha}}, J_{+,\alpha}] + U_\alpha^\dagger \partial_+ \Lambda_{\vec{x}_\perp} U_\alpha\ , \\ \delta_G J_{-,\alpha} =& \partial_-\Lambda_{\vec{x}_\perp} + [\Lambda_{\vec{x}_\perp}, J_{-,\alpha}] - U_\alpha \partial_- \Lambda_{\vec{x}_\perp+\vec{\alpha}} U_\alpha^\dagger\ ,\end{aligned} \tag{3.5}$$

where the currents $J_\pm$ are specified at $\vec{x}_\perp$. The variations of the $\tilde{J}$ currents are similar.

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[2] Normalization of the kinetic term differs from ref. [9] because of a different definition of the trace. See ref. [17] for a thorough discussion of such normalization issues.

[3] The analysis here generalizes the $2+1$ dimensional transverse lattice construction given in ref. [10].

The action is invariant under the global symmetry $U \to AUB$, where A, B are constant unitary matrices. The Wess-Zumino term must be gauged so it can be added to a transverse lattice action. (The kinetic term of the WZW model is easily gauged by introducing the covariant derivative as in the previous section.) The gauge variation of the Wess-Zumino term is local[9] and can be written in terms of the currents,

$$\begin{aligned}\delta_G \Gamma(U_\alpha(\vec{x}_\perp)) = \frac{1}{2\pi}\int d^2x \mathrm{Tr}\Big[&\Lambda_{\vec{x}_\perp}(\partial_+ J_{-,\alpha} - \partial_- \tilde{J}_{+,\alpha}) \\ &- \Lambda_{\vec{x}_\perp+\vec{\alpha}}(\partial_- J_{+,\alpha} - \partial_+ \tilde{J}_{-,\alpha})\Big] \ .\end{aligned} \tag{3.6}$$

It is straightforward to construct the "gauged" Wess-Zumino term for each link,

$$\begin{aligned}\tilde{\Gamma}(U_\alpha(\vec{x}_\perp)) = \Gamma + \frac{1}{2\pi}\int d^2x \mathrm{Tr}\ \Big\{\Big[&A_+(\vec{x}_\perp)J_{-,\alpha} - A_-(\vec{x}_\perp+\vec{\alpha})J_{+,\alpha} \\ &+ A_+(\vec{x}_\perp)U_\alpha A_-(\vec{x}_\perp+\vec{\alpha})U_\alpha^\dagger\Big] - \Big[A_-(\vec{x}_\perp)\tilde{J}_{+,\alpha} \\ &- A_+(\vec{x}_\perp+\vec{\alpha})\tilde{J}_{-,\alpha} + A_-(\vec{x}_\perp)U_\alpha A_+(\vec{x}_\perp+\vec{\alpha})U_\alpha^\dagger\Big]\Big\} \ .\end{aligned} \tag{3.7}$$

This term actually is not gauge invariant, but instead transforms as

$$\delta_G \tilde{\Gamma}(U_\alpha(\vec{x}_\perp)) = \frac{1}{2\pi}\int d^2x \mathrm{Tr}[\Lambda_{\vec{x}_\perp}\epsilon^{\mu\nu}\partial_\mu A_\nu(\vec{x}_\perp) - \Lambda_{\vec{x}_\perp+\vec{\alpha}}\epsilon^{\mu\nu}\partial_\mu A_\nu(\vec{x}_\perp+\vec{\alpha})] \ , \tag{3.8}$$

where $\epsilon^{+-} = 1$. This lack of gauge invariance has the same form as the non-Abelian anomaly in two dimensions, and is realized at the classical level. Note that the vector subgroup of $SU(N)_{\mathrm{left}} \otimes SU(N)_{\mathrm{right}}$ for each link is anomaly free[15]. Also, the global gauge symmetries for each site ($x^\pm$ independent gauge transformations) are unbroken.

Since more than one link is coupled to each site, the anomaly can cancel between the links. This is the mechanism that was introduced in ref. [10] to cancel the anomalies at each site of a $2+1$ dimensional transverse lattice model. For the case at hand, each link on the 2-D lattice is assigned a Wess-Zumino coupling constant $k_{\vec{x}_\perp,\alpha}$. The full action, including the Wess-Zumino term, is given by

$$\tilde{I}_{TL} = I_{\mathrm{TL}} + \sum_{\vec{x}_\perp,\alpha} k_{\vec{x}_\perp,\alpha}\tilde{\Gamma}(U_\alpha(\vec{x}_\perp)) \ . \tag{3.9}$$

and it is anomaly free for each site $\vec{x}_\perp$ if

$$\sum_\alpha \ k_{\vec{x}_\perp,\alpha} - k_{\vec{x}_\perp-\vec{\alpha},\alpha} = 0 \ . \tag{3.10}$$

Fig. 2. Up to an overall change of sign, the figures 2(a)-(c) denote the anomaly-free vertices allowed for the action $\tilde{I}_{\rm TL}$. The signs correspond to Wess-Zumino coupling $\pm k$, where k is a positive integer.

In the remainder of this paper, we will assign either $+k$ or $-k$ Wess-Zumino coupling to each link, where k is positive, and the NLSM coupling constant will be fixed to the critical point $g^2 = 4\pi/k$.

There are three pairs of anomaly free vertices (i.e., six total) that can be constructed for each site. The members of each pair are related by an overall flip of signs, and representatives of each pair are given in fig. 2. Each link is labeled by $\pm$ signs which denote the Wess-Zumino coupling.

The simplest configuration to consider is a lattice of all "+" links, so that all vertices are all of the type 2(a). Unfortunately, this does not work because the continuum limit of the gauged Wess-Zumino terms are order a,

$$\sum_{\vec{x}_\perp,\alpha} \tilde{\Gamma}(U_\alpha(\vec{x}_\perp)) = \sum_{\vec{x}_\perp,\alpha} \mathrm{Tr}\left\{\frac{a}{2\pi}[2F_{-+}A_\alpha + A_+\partial_\alpha A_- - A_-\partial_\alpha A_+] + \mathcal{O}(a^3)\right\} \quad (3.11)$$

This expression, for a one-dimensional lattice, is the pure Chern-Simons term in $2+1$ dimensions. It was discussed in some detail in ref. [10]. The leading $\mathcal{O}(a)$ part comes from the terms in the action that were added to gauge the Wess-Zumino term Γ. The Wess-Zumino term itself contributes only to order a^3, as can be seen by expanding the variation $\delta\Gamma$ with respect to the link fields U given in ref. [9].

A possible resolution is to stagger the gauged Wess-Zumino terms from site to site with alternating signs. Clearly, there are a number of ways to stagger these terms. The "correct" ways will be those which lead to the right continuum limit. In particular, we argued in section 2 that the Coulomb and plaquette interactions between links would drive the system to a smooth continuum limit. With the necessity of staggering, this is no longer as obvious for the Coulomb interactions, since the gauge couplings to each link are not the same on a staggered lattice. Recall that the Coulomb interactions are mediated by the longitudinal gauge fields $A_\pm$. These fields can be eliminated in the light-cone gauge $A_- = 0$, at the expense of generating a non-local Coulomb potential. We need to study the form of this potential on a staggered lattice.

In the gauge $A_- = 0$, the part of the path integral which depends upon the gauge field A_+ is

$$Z_{\rm GF} = \prod_{\vec{x}_\perp}[\det \partial_-]_{\vec{x}_\perp} \int [dA_+(\vec{x}_\perp)] e^{-i\int d^2x \left\{a^2/2g_1^2\ (\partial_- A_+(\vec{x}_\perp))^2 + A_+(\vec{x}_\perp)\mathcal{J}_-(\vec{x}_\perp)\right\}}\ , \quad (3.12)$$

where $\det \partial_-$ is the Fadeev-Popov determinant for each site, and the currents $\mathcal{J}_-(\vec{x}_\perp)$ are given by reading off the couplings in eqn. (2.4). The form of the current simplifies dramatically at the WZW critical points $\lambda^2 = 4\pi/k$. For these cases,

$$\mathcal{J}_-(\vec{x}_\perp) = \frac{k}{2\pi} \sum_{\alpha^\pm} \{ J_{-,\alpha^+}(\vec{x}_\perp) - \tilde{J}_{-,\alpha^-}(\vec{x}_\perp - \vec{\alpha}) \} \ , \tag{3.13}$$

where the sum over α^+ (α^-) is over links with $+k$ ($-k$) Wess-Zumino coupling. In the context of the WZW model, the currents $\mathcal{J}_-$ depend only upon x^-. The Coulomb interaction is obtained by completing the square in A_+. After completing the square, the integral over A_+ cancels the Fadeev-Popov determinant and the path integral (3.12) becomes

$$Z_{\rm GF} = \prod_{\vec{x}_\perp} e^{i \int d^2x g_1^2/2a^2 \left(\frac{1}{\partial_-} \mathcal{J}_-(\vec{x}_\perp) \right)^2} \ , \tag{3.14}$$

where

$$\frac{1}{\partial_-} \mathcal{J}_-(\vec{x}_\perp; x^-) = \tfrac{1}{2} \partial_- \int dy^- |y^- - x^-| \mathcal{J}_-(\vec{x}_\perp; y^-) + f_{\vec{x}_\perp}(x^+) \ . \tag{3.15}$$

The importance of keeping the integration constant $f_{\vec{x}_\perp}(x^+)$ in the context of the massless Thirring model was recently discussed in ref. [16]. In our context, it is easy to calculate by gauge fixing with the condition $A_+ = 0$, thereby introducing the currents

$$\mathcal{J}_+(\vec{x}_\perp) = \frac{k}{2\pi} \sum_{\alpha^\pm} \{ \tilde{J}_{-,\alpha^-}(\vec{x}_\perp) - J_{-,\alpha^+}(\vec{x}_\perp - \vec{\alpha}) \} \ . \tag{3.16}$$

These currents in the WZW model depend only on x^+. After completing the square in this case, the path integral (3.12) is

$$Z_{\rm GF} = \prod_{\vec{x}_\perp} e^{i \int d^2x g_1^2/2a^2 \left(\frac{1}{\partial_+} \mathcal{J}_+(\vec{x}_\perp) \right)^2} \ , \tag{3.17}$$

where

$$\frac{1}{\partial_+} \mathcal{J}_+(\vec{x}_\perp; x^-) = \tfrac{1}{2} \partial_+ \int dy^+ |y^+ - x^+| \mathcal{J}_+(\vec{x}_\perp; y^+) + \bar{f}_{\vec{x}_\perp}(x^-) \ . \tag{3.18}$$

Equating the two results for the same gauge-fixed path integral yields

$$f_{\vec{x}_\perp}(x^+) = \tfrac{1}{2} \partial_+ \int dy^+ |y^+ - x^+| \mathcal{J}_+(\vec{x}_\perp; y^+) \ , \tag{3.19}$$

Fig. 3. The two vertex configurations for which all four links couple symmetrically to each other. In figure 2(a) all four links contribute to the $\mathcal{J}_+$ current at the vertex, and in figure 2(b) all four contribute to $\mathcal{J}_-$.

and the path integral $Z_{\rm GF} = e^{iI_{\rm C}}$, where

$$I_{\rm C} = \frac{g_1^2}{4a^2} \sum_{\vec{x}_\perp} \int d^2x {\rm Tr} \left\{ \int dy^- \mathcal{J}_-(x^-)|x^- - y^-|\mathcal{J}_-(y^-) + \int dy^+ \mathcal{J}_+(x^+)|x^+ - y^+|\mathcal{J}_+(y^+) \right\} . \tag{3.20}$$

As in the Schwinger model, the Coulomb potential does not mix left- and right-mover currents. In the current algebra solution to the quantum theory discussed in the next section, the Coulomb terms are treated as potential terms. The currents $\mathcal{J}_-$ and $\mathcal{J}_+$ then remain functions of x^- or x^+ in the WZW model with Coulomb interactions. This is the same situation found in the massless Schwinger model (see the analysis of ref. [12]).

The Coulomb interactions are proportional to $1/a^2$. As $a \to 0$, configurations which minimize the full action should dominate the path integral. The question is whether these configurations correspond to the smooth continuum limit that we desire. Consider the link $U_{\alpha^+}(\vec{x}_\perp)$. According to eqns. (3.13) and (3.16), it interacts at $\vec{x}_\perp$ by contributing to the $\mathcal{J}_-(\vec{x}_\perp)$ current and interacts at $\vec{x}_\perp + \vec{\alpha}$ by contributing to $\mathcal{J}_+(\vec{x}_\perp + \vec{\alpha})$. Similarly, $U_{\alpha^-}(\vec{x}_\perp)$ interacts at $\vec{x}_\perp$ by contributing to the $\mathcal{J}_+(\vec{x}_\perp)$ current and interacts at $\vec{x}_\perp + \vec{\alpha}$ by contributing to $\mathcal{J}_-(\vec{x}_\perp + \vec{\alpha})$. For each of the anomaly-free vertices in figure 2, the links interact pairwise with each other. None of the vertices have all four links contributing to $\mathcal{J}_+$ or $\mathcal{J}_-$. Rather, two links contribute to $\mathcal{J}_+$ and two links contribute to $\mathcal{J}_-$. Therefore minimizing the Coulomb interaction as $a \to 0$ does not necessarily drive the system to the smooth continuum limit that is required to reproduce continuum QCD. Further evidence that the vertices of figure 2 do not generate the correct continuum limit was obtained by studying the vacuum structure, following the analysis discussed in section 5 for the correct result.

4. Bilocal Gauge Invariance, and a New Transverse Lattice Action

The two vertices that have all four links contributing to either $\mathcal{J}_+$ or $\mathcal{J}_-$ are given in figure 3. While these vertices have the correct behavior under the Coulomb interactions, they are both anomalous with respect to the local gauge invariance at

each site. Recovering QCD in the continuum limit is our paramount consideration, so we will consider breaking some of the local gauge symmetry. Specifically, we will gauge only the anomaly free local symmetries. The anomalous local symmetries will be become global symmetries. The unbroken local symmetries will then have to be gauge fixed, and the coupling of the links via Coulomb interactions will need to be re-examined.

The solution to the problems of the previous section will make use of the fact that the two vertices of figure 3 break gauge invariance in opposite ways. To be specific, label each site $\vec{x}_\perp = a(n_x, n_y)$ with the Z_2 quantum number

$$P_{\rm L}(\vec{x}_\perp) = (-1)^{n_x+n_y} \tag{4.1}$$

which will be referred to as *lattice parity* . Even (odd) sites have lattice parity $+1$ (-1). As in the previous section, we consider the transverse lattice action with Wess-Zumino terms, eqn (3.9). However, now we consider configurations which violate the gauge invariance constraint eqn. (3.10). The assignment of the Wess-Zumino coupling constant is given by

$$\begin{aligned} U_\alpha(\vec{x}_\perp) &= U_{\alpha^+}(\vec{x}_\perp) \ , \vec{x}_\perp \text{ even} \ , \\ U_\alpha(\vec{x}_\perp) &= U_{\alpha^-}(\vec{x}_\perp) \ , \vec{x}_\perp \text{ odd} \ , \end{aligned} \tag{4.2}$$

where the notation $\alpha^\pm$ corresponds to assigning Wess-Zumino coupling $\pm k$. The vertex at odd (even) sites is the type shown in fig. 3(a) (fig. 3(b)). The anomaly at each site is

$$\delta_G \tilde{I}_{\rm TL}(\vec{x}_\perp) = P_{\rm L}(\vec{x}_\perp) \frac{2k}{\pi} \int d^2x {\rm Tr}\ \Lambda(\vec{x}_\perp) \epsilon^{\mu\nu} \partial_\mu A_\nu(\vec{x}_\perp) \ . \tag{4.3}$$

Define nearest neighbor pairs $(\vec{x}_\perp^+, \vec{x}_\perp^-)$, which by the above construction have opposite anomalies, as

$$(\vec{x}_\perp^+, \vec{x}_\perp^-) = (\vec{x}_\perp, \vec{x}_\perp + (-1)^{n_x}\hat{x}), \quad \forall \vec{x}_\perp \text{ s.t. } P_{\rm L}(\vec{x}_\perp) = 1 \ , \tag{4.4}$$

where $\hat{x} = (a, 0)$. Every site on the square lattice belongs to one pair. By construction, $\vec{x}_\perp^+$ $(\vec{x}_\perp^-)$ is an even (odd) site.

For each of these nearest neighbor pairs, the anomaly breaks one of the local gauge symmetries, and preserves the other local symmetry and the two global symmetries. Gauging the local symmetries for this transverse lattice model no longer requires one independent vector potential for each site. Rather, the gauge fields at

Fig. 4. For the bilocal model, each gauge field $A_\mu(\vec{x}_\perp^+)$ interacts with seven links. Figure 4 shows the case where $\vec{x}_\perp^+$ is to the right of $\vec{x}_\perp^-$.

$\vec{x}_\perp^-$ sites can be parametrized as

$$A_\mu(\vec{x}_\perp^-) = G_{\vec{x}_\perp^-} A_\mu(\vec{x}_\perp^+) G^\dagger_{\vec{x}_\perp^-} \ , \tag{4.5}$$

where $G_{\vec{x}_\perp^-}$ is a constant ($x^\pm$ independent) unitary matrix which transforms as

$$\delta_G G_{\vec{x}_\perp^-} = \tilde{\Lambda}_{\vec{x}_\perp^-} G_{\vec{x}_\perp^-} \ , \tag{4.6}$$

under gauge transformations. The field $G_{\vec{x}_\perp^-}$ corresponds to the unbroken global symmetry at the $\vec{x}_\perp^-$ sites. So instead of having a 2-D gauge field A_μ for each site, we now have the set $(A_\mu, G_{\vec{x}_\perp^-})$ for each pair of sites. The links and vector fields transform as given by eqns. (2.1) and (2.2) as long as the the infinitesimal variations at the x^- sites satisfy the constraint

$$\Lambda_{\vec{x}_\perp^-}(x^\pm) = \tilde{\Lambda}_{\vec{x}_\perp^-} + G_{\vec{x}_\perp^-} \Lambda_{\vec{x}_\perp^+}(x^\pm) G^\dagger_{\vec{x}_\perp^-} \ . \tag{4.7}$$

With this construction, the gauge variations from the x^- sites cancel the anomaly from the x^+ sites. The path integral measure is redefined as

$$\prod_{\vec{x}_\perp} [dA_\mu] \to \prod_{\vec{x}_\perp^+} [dA_\mu(\vec{x}_\perp^+)] \prod_{\vec{x}_\perp^-} [dG_{\vec{x}_\perp^-}] \ , \tag{4.8}$$

where $[dG]$ is the left invariant Haar measure for G. The full action is given by eqn. (3.9), with the Wess-Zumino coupling constants given by eqn. (4.2), and the field identification (4.5).

It is important that the local gauge symmetry at each site remain unbroken. Otherwise, each pair of links $(U_{\alpha^+}, U_{\alpha^-})$ would transform the same way under the remaining local gauge invariance, effectively doubling the number of link degrees of freedom in the gauge theory, and the theory would not have QCD as the "naive" continuum limit. Each of the remaining local symmetries are associated with two nearest neighbor sites, paired together as prescribed by equation (4.4). The above construction will be referred to as a *bilocal* transverse lattice model. Each gauge field $A_\mu(\vec{x}_\perp^+)$ is coupled to seven links instead of four (see fig. 4). This difference is obviously significant at the lattice level. But again, the point is that there are many models which have the same continuum behavior which differ at scales of the lattice spacing. The bilocal model has the advantage of being more easily treatable at the lattice level than the basic transverse lattice model without Wess-Zumino terms.

The Coulomb dynamics of the bilocal model is studied by gauge fixing in the $A_- = 0$ gauge as in the previous section. The part of the path integral which depends upon the gauge field A_+ is

$$Z_{\rm GF} = \prod_{\vec{x}^+_\perp} [\det \partial_-]_{\vec{x}^+_\perp} \int [dA_+(\vec{x}^+_\perp)] e^{-i \int d^2x {\rm Tr} \, \left\{ a^2/g_1^2 (\partial_- A_+(\vec{x}^+_\perp))^2 + A_+(\vec{x}^+_\perp) \mathcal{J}_-(\vec{x}^+_\perp) \right\}} \, . \tag{4.9}$$

There is an additional factor of two in front of the $(\partial_- A_+)^2$ term, relative to eqn. (3.12), from the contribution to the kinetic energy term from the $\vec{x}^-_\perp$ sites. Note that the $G_{\vec{x}^-_\perp}$s cancel out of this expression. The current $\mathcal{J}_-(\vec{x}^+_\perp)$ is given by eqn. (3.13), and all four links connected to the site $\vec{x}^+_\perp$ contribute to it. Completing the square yields eqns. (3.14) and (3.15), up to the additional factor of two, and where $\mathcal{J}_-(\vec{x}^-_\perp) = 0$. The functions $f_{\vec{x}^+_\perp}(x^+)$ remain to be determined. Following the previous analysis, we gauge fix in the $A_+ = 0$ gauge and find

$$\begin{aligned} Z_{\rm GF} = & \prod_{\vec{x}^+_\perp} [\det \partial_+]_{\vec{x}^+_\perp} \int [dA_-(\vec{x}^+_\perp)] [dG_{\vec{x}^-_\perp}] \\ & \times e^{-i \int d^2x {\rm Tr} \, \left\{ a^2/g_1^2 (\partial_+ A_-(\vec{x}^+_\perp))^2 + A_-(\vec{x}^+_\perp) G^\dagger_{\vec{x}^-_\perp} \mathcal{J}_+(\vec{x}^-_\perp) G_{\vec{x}^-_\perp} \right\}} \, . \end{aligned} \tag{4.10}$$

After completing the square and comparing to the $A_- = 0$ case, one finds

$$f_{\vec{x}^+_\perp}(x^+) = \tfrac{1}{2} \partial_+ \int dy^+ |y^+ - x^+| \mathcal{J}_+(\vec{x}^-_\perp; y^+) \, . \tag{4.11}$$

The nonlocal Coulomb effective action for the bilocal theory is

$$\begin{aligned} I_{\rm C} = \frac{g_1^2}{8a^2} \int d^2x {\rm Tr} \, \Big\{ & \sum_{\vec{x}^+_\perp} \int dy^- \mathcal{J}_-(x^-) |x^- - y^-| \mathcal{J}_-(y^-) \\ & + \sum_{\vec{x}^-_\perp} \int dy^+ \mathcal{J}_+(x^+) |x^+ - y^+| \mathcal{J}_+(y^+) \Big\} \, . \end{aligned} \tag{4.12}$$

While in the gauge invariant bilocal model action the $G_{\vec{x}_\perp}$ dependence is required to preserve the global gauge invariance at x^- sites, the $G_{\vec{x}^-_\perp}$ dependence cancels in the gauge fixed action because it is bilinear in the currents. (The path integral over the $G_{\vec{x}^-_\perp}$ is finite since the group $SU(N)$ is compact.) These Coulomb terms have precisely the properties that we desired to obtain the correct continuum limit. All links connected to a given site $\vec{x}_\perp$ interact with each other via (4.12). While the pairing of sites (4.4) broke a discrete lattice symmetry by differentiating between x and y directions, this symmetry is restored in the Coulomb effective action. As the

lattice spacing a goes to zero, the Coulomb dynamics drives the system to a smooth continuum limit.

The naive continuum limit is studied, as in the previous sections, by inserting a Bloch-wave expansion into the gauge invariant action. Recall that the A_α dependence in (2.5) was determined by requiring that it transform as a gauge field,

$$\delta_G A_\alpha(\vec{x}_\perp + \tfrac{1}{2}\vec{\alpha}) = \frac{\Lambda_{\vec{x}_\perp+\vec{\alpha}} - \Lambda_{\vec{x}_\perp}}{a} + \left[\Lambda_{\vec{x}_\perp} A_\alpha - A_\alpha \Lambda_{\vec{x}_\perp+\vec{\alpha}}\right] + \mathcal{O}(a) \ . \tag{4.13}$$

For the bilocal model it is still possible to meaningfully expand the Λ's as

$$\Lambda_{\vec{x}_\perp+\vec{\alpha}} - \Lambda_{\vec{x}_\perp} = a\partial_\alpha \Lambda_{\vec{x}_\perp} + \mathcal{O}(a^2) \ , \tag{4.14}$$

since the constraint (4.7) allows for arbitrary $\tilde{\Lambda}_{\vec{x}_\perp^-}$ transformations at x^- sites, i.e. the global gauge invariance at each site is retained. A_α transforms as a gauge field as $a \to 0$ and the Bloch-wave expansion (2.5) is valid for the bilocal model. In the continuum limit, the undesired order a terms (3.11) cancel between even and odd sites as $a \to 0$. This is due to the staggering of the Wess-Zumino terms which is built into the model. The cancellation of the gauged Wess-Zumino terms between pairs of adjacent even and odd links is to order a^3, because parity prevents the gauged Wess-Zumino terms from contributing to order a^2. They are therefore irrelevant operators in the continuum.

In fact, the order a terms are explicitly cancelled locally in the gauge fixed lattice action, and there is no need to invoke a cancellation between sites, as in the gauge invariant analysis. To see this, expand the currents $J_\pm$ or $\tilde{J}_\pm$ for each link order by order in lattice spacing a. The currents $\mathcal{J}_+$ and $\mathcal{J}_-$ given by eqns. (3.13) and (3.16) are order a^2, and therefore the Coulomb interaction (4.12) for each site is order a^2. The kinetic and plaquette terms for the link fields are also order a^2 by the analysis in section 2. (The bare ungauged Wess-Zumino terms are order a^3 for each link field as discussed in section 3.)

The bilocal transverse lattice model satisfies the primary constraint that its continuum limit be QCD, at the expense of introducing a somewhat complicated structure of gauge invariance on the lattice.

5. Quantization of the Bilocal Transverse Lattice Model

In this section, the quantum theory of the new bilocal transverse lattice model constructed. The "discrete light-cone" approach of ref. [3] is followed when the

Hilbert space is defined. The goal of this section is to set up the theory for future computational study of the bound state problem.

The approach taken is to solve the WZW model for each link and treat the Coulomb and plaquette terms as additional potential terms. The current algebra solution of the WZW model was first given by Dashen and Frishman[13] for level $k = 1$, and later generalized to arbitrary level by Knizhnik and Zamolodchikov[14]. The current algebra is specified by Sugawara-type algebras for the left and right-moving currents. For even (odd) links with Wess-Zumino coupling k $(-k)$, the currents are by $J_\pm$ $(\tilde{J}_\pm)$. For right movers of an even link, the current commutation relations are

$$[J^a_-(x^-), J^b_-(y^-)] = if^{abc}J^c_-(x^-) - \frac{ik}{2\pi}\delta^{ab}\delta'(x^- - y^-) \ , \tag{5.1}$$

where J^a_- is[17]

$$\sum_a T^a J^a_-(\vec{x}_\perp) = \frac{-ik}{\sqrt{2}\pi} J_-(\vec{x}_\perp) \ . \tag{5.2}$$

The same type of expressions hold for currents $\tilde{J}^a_-$, J^a_+ and $\tilde{J}^a_+$ ($x^- \to x^+$ for the $+$ currents). Left- and right-mover currents commute, as do currents defined for different links. These algebras are the equal time commutation relations translated into light-cone coordinates. One can specify initial conditions on the light fronts $x^+ = 0$ and $x^- = 0$ for the massless currents J_- and J_+ in the WZW model. However, because of the complicated form of the plaquette interaction term, initial conditions will be specified on an equal time surface.

To make the connection between the commutators above and Kac-Moody algebras, infared cutoffs for the light-cone coordinates are introduced by hand. With $x^\pm$ defined on the interval $[-L, L]$, and fields taken to be periodic, a mode expansion for the currents takes the form

$$\begin{aligned} J^a_+ =& \frac{1}{2L}\sum_n e^{-i\pi n x^+/L} J^a_n \ , \quad \tilde{J}^a_+ = \frac{1}{2L}\sum_n e^{-i\pi n x^+/L} K^a_n \ , \\ J^a_- =& \frac{1}{2L}\sum_n e^{-i\pi n x^-/L} \bar{J}^a_n \ , \quad \tilde{J}^a_- = \frac{1}{2L}\sum_n e^{-i\pi n x^-/L} \bar{K}^a_n \ , \end{aligned} \tag{5.3}$$

where the site dependence of the currents has been suppressed. The currents J^a_n, $\bar{J}^a_n$ are defined for even links, and K^a_n,$\bar{K}^a_n$ are defined for odd links. The delta function in the current algebra (5.1) is easily defined for period boundary conditions, and the current algebra is equivalent to the Kac-Moody algebra

$$[J^a_m, J^b_n] = if^{abc}J^c_{m+n} + km\delta^{ab}\delta_{m,-n} \tag{5.4}$$

for each of the mutually commuting currents. The zero modes J_0^a form a subalgebra equivalent to the original $SU(N)$ Lie algebra.

The light-cone singlet vacuum $|0\rangle$, defined for each chiral algebra, is the unique highest weight state which satisfies

$$J_n^a|0\rangle = 0 \ , n > 0 \ , \quad J_0^a|0\rangle = 0 \ . \tag{5.5}$$

The definition of normal ordering is with respect to this vacuum state,

$$\begin{aligned} :J_m^a J_n^b: =& J_m^a J_n^b \ \ m < 0 \ , \\ =& J_n^a J_m^b \ \ m \geq 0 \ . \end{aligned} \tag{5.6}$$

For the even link WZW models, the Lorentz generators $P^+ = H + P$ and $P^- = H - P$ are given by

$$\begin{aligned} P^+_{\rm WZW} =& \frac{1}{2L(2k+N)} \sum_n :J_n^a J_{-n}^a: = \frac{1}{2L} L_0 \ , \\ P^-_{\rm WZW} =& \frac{1}{2L(2k+N)} \sum_n :\bar{J}_n^a \bar{J}_{-n}^a: = \frac{1}{2L} \bar{L}_0 \ ; \end{aligned} \tag{5.7}$$

the odd link Lorentz generators are the same with $(J, \bar{J}) \to (K, \bar{K})$. The modes of the currents are diagonal with respect to the light-cone momenta, and have non-vanishing commutation relations $[L_0, J_{-m}^a] = m J_{-m}^a$ and $[\bar{L}_0, \bar{J}_{-m}^a] = m \bar{J}_{-m}^a$.

The space of states for each link is built up by applying a product of raising operators $\{J_{-m}^a\}\{\bar{J}_{-m}^a\}$ upon a highest weight (vacuum) state $|\ell_0\rangle \otimes \overline{|\ell_0\rangle}$ for the right-mover and left-mover sectors. This construction is analogous to the Fock space basis of the linear sigma model states. For the right-mover sector, these states satisfy

$$\begin{aligned} &J_m^a|\ell_0\rangle = 0, \ \ m > 0 \ , \\ &E_0^\alpha|\ell_0\rangle = 0 \ , \end{aligned} \tag{5.8}$$

where $E_0^{\pm\alpha}$ and H_0^i are the zero mode currents J_0^a in the Cartan-Weyl basis of the algebra. Similar results apply for the left-mover sector. The vacuum states are the highest weights of finite dimensional $SU(N)$ representations $|\ell\rangle$ generated by applying zero modes $E_0^{-\alpha}$. This will be referred to as the zero mode sub-sector of the full space of states. The zero modes have non-vanishing zero point momenta $L_0|\ell\rangle = \Delta_\ell|\ell\rangle$ and $\bar{L}_0\overline{|\ell\rangle} = \Delta_\ell\overline{|\ell\rangle}$ where $\Delta_\ell = C_\ell/2(N+k)$, and C_ℓ is the quadratic Casimir of the representation ℓ. The representations of the full Kac-Moody algebra for each highest weight are unitary if the highest weights have Young tableaux with the number of columns less than or equal to k[18].

The representations $|\ell\rangle$ are the zero modes of dimensionless primary fields $\phi_\ell(v)$ of the chiral algebras. The primary fields have simple commutation relations with the currents[13][14]. For the even links,

$$\begin{aligned}[J_n^a, \phi_\ell(x^+)] =& e^{i\pi n x^+/L}\ \phi_\ell(x^+)\ t_\ell^a\ , \\ [\bar{J}_n^a, \bar{\phi}_\ell(x^-)] =& e^{i\pi n x^-/L}\ t_\ell^a\ \bar{\phi}_\ell(x^-)\ ,\end{aligned} \tag{5.9}$$

and for the odd links,

$$\begin{aligned}[K_n^a, \phi_\ell(x^+)] =& e^{i\pi n x^+/L}\ t_\ell^a\ \phi_\ell(x^+)\ , \\ [\bar{K}_n^a, \bar{\phi}_\ell(x^-)] =& e^{i\pi n x^-/L}\ \bar{\phi}_\ell(x^-)\ t_\ell^a\ ,\end{aligned} \tag{5.10}$$

where t_ℓ^a is a generator of $SU(N)$ in the irreducible representation ℓ. Note the ordering difference here between even and odd links. The primary fields are intertwining operators of the space of states, since they interpolate between vacuum sectors. Let $|1\rangle$ denote the fundamental representation of $SU(N)$. The products of left-mover and right-mover primary fields $\bar{\phi}_1\phi_1$ for even links, and $\phi_1\bar{\phi}_1$ for odd links, are the quantum fields which correspond to the classical unitary chiral field U in the classical action. Since the classical unitary field transforms on the left and right in the same representation, we will consider only the diagonal[18] products of the highest weight fields as the vacuum states in the transverse lattice theory. The reader is referred to refs. [9][14][18] and in particular the review article ref.[17] for further information on the WZW model.

The gauge fixing proceedure of the previous section left the global symmetry at each site untouched. The generators of these gauge transformations in the quantum theory are constructed from the zero modes of the currents

$$\begin{aligned}\mathcal{J}_0^a(\vec{x}_\perp^+) &= \sum_\alpha \Big[\bar{J}_0^a(\vec{x}_\perp^+, \alpha) - \bar{K}_0^a(\vec{x}_\perp^+ - \vec{\alpha}, \alpha) \Big]\ , \\ \mathcal{J}_0^a(\vec{x}_\perp^-) &= \sum_\alpha \Big[K_0^a(\vec{x}_\perp^-, \alpha) - J_0^a(\vec{x}_\perp^- - \vec{\alpha}, \alpha) \Big]\ .\end{aligned} \tag{5.11}$$

In the classical theory, all physical states are gauge invariant. In the quantum theory, this restriction can be relaxed somewhat, since it is the gauge invariance of expectation values of states (operators) that is required[4]. Following the previous

[4] A modern example of this treatment of gauge symmetries is found in string theory, where conformal invariance is a crucial property of the quantum theory. The Virasoro modes L_n generate conformal transformations, and physical states need to be annihilated by only the modes with $n \geq 0$.

Fig. 5. The two cases encountered when gluing together the zero modes of two links to satisfy the global gauge invariance constraints at a site. Figure 5(a) (5(b)) denotes two links of the same (opposite) lattice parity.

treatment of this point for the the 1-D transverse lattice[10], we require

$$\begin{aligned} \mathcal{E}_0^{\alpha}|\text{physical}\rangle &= 0 \ , \\ \mathcal{H}_0^{i}|\text{physical}\rangle &= 0 \ , \end{aligned} \tag{5.12}$$

where $\mathcal{E}_0^{\pm\alpha}, \mathcal{H}_0^i$ denote $\mathcal{J}_0^a$ in the Cartan-Weyl basis. In the 1-D transverse lattice case, these constraints led to the correct set of physical states.

This criterion can first be applied to the subspace of states obtained by taking products of zero mode states on the lattice. In the 1-D transverse lattice construction, non-trivial states of this type were found. They were interpreted as zero modes of Wilson loops on the lattice. Each link is associated with a product of left-mover zero modes $|\ell\rangle$ and right-mover zero modes $\overline{|\ell\rangle}$. Even and odd links at $U_\alpha(\vec{x}_\perp)$ have zero mode structure

$$\begin{aligned} &\overline{|\ell\rangle}_{\vec{x}_\perp,\alpha} \times |\ell\rangle_{\vec{x}_\perp,\alpha} \quad \text{, even link} \ , \\ &|\ell\rangle_{\vec{x}_\perp,\alpha} \times \overline{|\ell\rangle}_{\vec{x}_\perp,\alpha} \quad \text{, odd link} \ . \end{aligned} \tag{5.13}$$

Consider gluing together two links with non-trivial zero mode structure at a site $\vec{x}_\perp$, such that (5.12) is satisfied. There are two basic cases to consider, as shown in figure 5. In figure 5(a) (5(b)), the two links have the same (opposite) lattice parity. In figure 5(a), two right-mover zero modes $\overline{|m\rangle} \times \overline{|\ell\rangle}$ need to be glued together. The constraints (5.12) are satisfied if we project onto the singlet sector of the tensor product: $P^0_{m,\ell}\{\overline{|m\rangle} \otimes \overline{|\ell\rangle}\}$. This is because the constraint involves the sum of currents for each link, i.e. the diagonal subalgebra of the link zero mode algebras. For the other case denoted in figure 5(b), the constrains involve the difference between the currents for each link. This was the situation encountered in the 1-D transverse lattice analysis. The solution to the constraints is to project onto the highest weights of the same representation: $|\ell_0\rangle \times |m_0\rangle \ \delta_{\ell,m}$. These two rules for gluing together zero modes at a site can be used to construct a wide variety of solutions to the gauge invariance conditions. For example, a plaquette solution is of the form

$$\begin{aligned} &P^0\left\{\overline{|\ell^\dagger\rangle}_{\vec{x}^+_\perp,y} \otimes \overline{|\ell\rangle}_{\vec{x}^+_\perp,x}\right\} \ |\ell_0\rangle_{\vec{x}^+_\perp,x} \ |\ell_0\rangle_{\vec{x}^+_\perp+\hat{x},y} \\ &\times P^0\left\{\overline{|\ell\rangle}_{\vec{x}^+_\perp+\hat{x},y} \otimes \overline{|\ell^\dagger\rangle}_{\vec{x}^+_\perp+\hat{y},x}\right\} \ |\ell_0^\dagger\rangle_{\vec{x}^+_\perp+\hat{y},x} \ |\ell_0^\dagger\rangle_{\vec{x}^+_\perp,y} \ , \end{aligned} \tag{5.14}$$

where $\ell^\dagger$ is the conjugate representation of ℓ. This plaquette state is the zero mode of a Wilson loop with flux in representation ℓ, flowing in the counterclockwise

orientation. It has zero point energy $(P^+_{\rm WZW}+P^-_{\rm WZW})/2 = 2\Delta_\ell/L$ and momentum $(P^+_{\rm WZW}=P^-_{\rm WZW})/2=0$. This is the part of a Wilson loop that can never be gauged away. While in the linear sigma model transverse lattice theory[1][2] these type of states are presumably soliton excitations, in the new non-linear theory they arise quite naturally as vacuum sectors. The space of states for the new transverse lattice model breaks up into different vacuum sectors of Wilson loop zero modes, and the intertwining operator, which takes states from one sector to another, is the plaquette operator discussed below.

The contribution of the Coulomb interactions to the Lorentz generators is obtained by inserting the mode expansions (5.3) into the effective action (4.12) and normal ordering with respect to the vacuum state (5.5),

$$\begin{aligned} P^+_{\rm C} =& 2L\Big(\frac{g_1}{8\pi a}\Big)^2 \sum_{\vec{x}^-_\perp} \sum_{n=-\infty}^{\infty} \frac{1}{n^2} :\mathcal{J}^a_n(\vec{x}^-_\perp)\mathcal{J}^a_n(\vec{x}^-_\perp): \ , \\ P^-_{\rm C} =& 2L\Big(\frac{g_1}{8\pi a}\Big)^2 \sum_{\vec{x}^+_\perp} \sum_{n=-\infty}^{\infty} \frac{1}{n^2} :\mathcal{J}^a_n(\vec{x}^+_\perp)\mathcal{J}^a_n(\vec{x}^+_\perp): \ , \end{aligned} \tag{5.15}$$

where

$$\begin{aligned} \mathcal{J}^a_n(\vec{x}^+_\perp) =& \sum_\alpha K^a_n(\vec{x}^+_\perp,\alpha) - J^a_n(\vec{x}^+_\perp-\vec{\alpha},\alpha) \ , \\ \mathcal{J}^a_n(\vec{x}^-_\perp) =& \sum_\alpha \bar{J}^a_n(\vec{x}^-_\perp,\alpha) - \bar{K}^a_n(\vec{x}^-_\perp-\vec{\alpha},\alpha) \ . \end{aligned} \tag{5.16}$$

Note that the light-cone momenta $P^\pm_{\rm C}$ are proportional to the infared cutoff L. The contribution to the mass operator from the Coulomb potential $M^2_{\rm C} = P^+_{\rm WZW}P^-_{\rm C} + P^-_{\rm WZW}P^+_{\rm C}$ is therefore independent of L. Diagonalization of $M^2_{\rm C}$ on a basis of states defined in the cutoff theory is therefore the exact (cutoff independent) result in that basis. The masses are finite only if the potentially infared divergent $n=0$ coefficients in $P^\pm_{\rm C}$ vanish

$$:\mathcal{J}^a_0(\vec{x}_\perp)\mathcal{J}^a_0(\vec{x}_\perp):=0 \ , \quad \forall\ \vec{x}_\perp \ . \tag{5.17}$$

This is a statement of charge neutrality for each site. The incoming charge must equal the outgoing charge, where the direction is defined by the arrows associated with each link (See figure 4). The zero mode states discussed above, and in particular the state given by eqn. (5.14), satisfy this constraint by construction. In fact, the constraints (5.17) and (5.12) are equivalent. For a localized state, such as a link-antilink excitation, the constraint requires that physical states be gauge singlets at each site.

The structure of the Coulomb term for each site $\vec{x}_\perp$ is similar to that obtained in $1+1$ QCD with massless fermions quantized following the same approach[19]. In that case, it is known that there is no mass gap[20] generated by the Coulomb interactions, because states can be constructed from the $U(1)$ fermion number current which commutes with the non-Abelian currents that make up the Coulomb potential. This is not the case for the new transverse lattice model, since this current does not exist in this case. Analysis of the 'tHooft equation[21] for link-antilink bound states can make this result quantitative by determining the bare mass gap for these states. Preliminary calculations show that a linear Regge trajectory for the mass spectrum is obtained in the large N limit.

The plaquette interaction in (2.4) explicitly mixes left-mover and right-mover modes, like a mass term. In principle, then, one could eliminate left-movers in terms of right-movers by solving a constraint equation for each equal light-cone time surface $x^+ = \text{const}$. However, in practice this is very difficult because of the complicated form of the interaction in terms of the WZW currents. Therefore, the plaquette term will be treated as an interaction in the quantum theory with independent commutation relations for both left- and right-movers. The relevant Cauchy surface on which to fix initial conditions is an equal time surface. The WZW model current algebras are equal time commutation relations written in light-cone coordinates (see in particular sec. 2 of ref. [22].).

For fixed time quantization, the link fields are functions of the single variable x, and for $t=0$ it is useful to define the complex variable $z = e^{i\pi x/\sqrt{2}L}$. The quantum link fields are products of left-mover and right-mover highest weights

$$\begin{aligned} U(z) &= :\bar{\phi}_1(z^*)\phi_1(z): , \quad \text{even link} , \\ U(z) &= :\phi_1(z)\bar{\phi}_1(z^*): , \quad \text{odd link} , \end{aligned} \tag{5.18}$$

where ϕ_1 is the primary field in the fundamental representation of $SU(N)$. The conjugate $U^\dagger$ can be similarly defined in terms of the conjugate primary fields $\phi_1^\dagger$. The contribution of the plaquette interaction to the Lorentz group generators is then

$$P_{\rm P}^{\pm} = \frac{L}{\sqrt{2}\pi g_2^2 a^2} \oint dz \sum_{\beta\neq\alpha} \text{Tr}\left[1 - U_\alpha(\vec{x}_\perp)U_\beta(\vec{x}_\perp+\vec{\alpha})U_\alpha^{-1}(\vec{x}_\perp+\vec{\beta})U_\beta^{-1}(\vec{x}_\perp)\right] . \tag{5.19}$$

The trace in eqn. (5.19) denotes a projection onto products of highest weights for each site such that the gauge invariance constraints are satisfied. The plaquette interaction does not contribute to the spatial momentum $(P^+ - P^-)/2$.

6. Discussion

Like any other theory which describes non-perturbative behavior of QCD, the transverse lattice construction outlined above is very complicated. The basic advantage over 4-D lattice simulations is that a continuum analysis is used to describe local link dynamics. To take advantage of the continuum description, one has to find a suitable truncation of the full model that still contains the desired physics. For the calculation of glueball masses in in the Bardeen Pearson model[2], the basic degrees of freedom were truncated to link-antilink and four-link bound states. These states mix under the plaquette interaction, which also provides for transverse motion. It is essential to complete this basic analysis for the new transverse lattice model and verify that in this case the link number expansion is a valid one, i.e., that link number violation is strong enough to allow for transverse motion of the states, yet small enough to validate the link number expansion.

The physical state of the link-antilink and four-link truncated basis is a Wilson loop smeared over the lattice. The link-antilink states correspond to Wilson loops which extend in the longitudinal directions. This type of state in the gauge fixed transverse lattice model is the bilinear $U(x)U^\dagger(y)$ integrated over a wavefunction. The link-antilink bound state is *not* a bilinear J^aJ^a of WZW currents. Although the currents are the linear degrees of freedom of the WZW model, the real degrees of freedom of the lattice theory are the link fields.

The spectrum of bound states that can be constructed by the above formalism is degenerate because the zero mode of the state can be shifted and boosted by Lorentz transformations. We want to select a basis of bound states that does not contain copies of the same state. Moreover, we want a basis of states for which one of the momentum operators is manifestly diagonal. For instance, in the linear sigma model analysis[2], P^- was manifestly diagonal. This type of analysis in the bilocal model is complicated, because as discussed above both the left- and right-mover currents are treated as dynamical. There is no simple constraint equation which can be used to solve for one set in terms of the other.

There is a basis of states one can use to study, in the simplest way, the bound state spectrum. It is defined by allowing zero-mode excitations for both left- and right-movers, but truncating all non-zero mode excitations of the left-movers. (The parity conjugate basis is equally simple.) There is no proof that all physical states have a representative is this truncated basis. However, this truncation is similar to

the analysis of ref. [23], where the spectrum of the massless Schwinger model was studied. There, bound states were constructed from fermions of only one chirality. In this chiral Schwinger model, there exists only one copy of the free massive scalar boson, instead of the infinite number of degenerate copies which exist in the full Schwinger model[16]. Nevertheless, the single copy has the correct mass. Again, the idea here is to suggest a starting point for the explicit calculation of the mass spectrum.

In the truncated basis, a local link-antilink state is given by

$$|P\rangle = \oint dz \oint dy \sum_{k=0}^{P} w(k)\ z^{-k} y^{k-P}\ \mathrm{Tr} : \phi_1(z)\phi_1^\dagger(y) : |0\rangle \otimes \overline{|0\rangle}\ . \tag{6.1}$$

For the link-antilink state, the only possible right-mover singlet state is the vacuum. Periodicity in the z and y variables quantizes the momenta P and k to be integers. The diagonal contribution to the light-cone momentum P^+ is from the WZW model. Using the conformal field theory commutator $[L_0, \phi_1(z)] = (z\partial_z + \Delta_1)\phi_1$, one finds

$$P^+_{\mathrm{WZW}}|P\rangle = \frac{P + 2\Delta_1}{2L}|P\rangle\ . \tag{6.2}$$

In the truncated basis, the contribution P^+_C from the Coulomb interactions always vanishes. And as discussed in the previous section, the contribution P^+_{P} from the plaquette interation mixes this state with four link states. A physical two link state is a local state(6.1) smeared over the entire lattice with a wavefunction specifying the transverse momentum distribution.

The simplest non-trivial local four link state is obtained by replacing the right-mover zero modes in equation (5.14) with integrals over the link fields. There are actually a number of local four link configurations to consider (see fig. 2 of ref. [2]). The reader is referred to ref. [2] for a complete description of the method for determining the mass spectrum of the $3+1$ Lorentz multiplets in the two and four link basis.

The $U(1)$ case avoids all of the complications developed in sections 3 and 4, and therefore may be a good laboratory to probe the recovery of 4-D Lorentz invariance from the transverse lattice construction. The non-linear sigma model action is then the Gaussian model for the fields θ, where $U = \exp i\theta$. The plaquette interaction is a product of normal ordered $U(1)$ vertex operators. However, it may be difficult, if not impossible, to approach the continuum limit for the Abelian case, since link number violation will be large for a deconfined theory.

Unlike 4-D lattice gauge theory, the transverse lattice construction may be able to generate structure functions of relativistic bound states, since the wavefunctions of the states are explicitly calculated when diagonalizing the mass spectrum. Only when such a problem, impossible to work out by conventional techniques, is solved via the transverse lattice construction, will this new approach be fully accepted as a tool for probing non-perturbative physics.

The transverse lattice construction connects 2-D physics to more realistic higher dimensional models. Since string theory has motivated a great deal of recent progress in 2-D field theory, there are surely many more connections that can be made, to the benefit of both mathematically- and phenomenologically-oriented physicists.

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