Characterization of association schemes by equitable partitions

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Abstract
In this paper we deal with equitable partitions of association schemes. We try to generalize a result in group theory and show examples that a generalization of a certain property conjectured for permutation groups does not hold for association schemes.

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1. Introduction
Let \( \Gamma \) be a transitive permutation group on a finite set \( X \). Then \( \Gamma \) acts on \( X \times X \) by \( \gamma \cdot (x, y) := (\gamma(x), \gamma(y)) \) for \( (x, y) \in X \times X \) and \( \gamma \in \Gamma \). We call an orbit of this action an orbital. We denote the set of orbitals of \( \Gamma \) by \( \text{Orb}_2(\Gamma) \), and the set of permutations of \( X \) which preserves each orbital of \( \Gamma \) by \( \overline{\Gamma} \). It is well known that \( \text{Orb}_2(\Gamma) \) forms an association scheme if \( \Gamma \) is transitive, and \( \Gamma \leq \overline{\Gamma} \). According to [3] we say that \( \Gamma \) is 2-closed if \( \Gamma = \overline{\Gamma} \). For example, any proper doubly transitive permutation group is not 2-closed, and each regular permutation group is 2-closed. In [3] it is conjectured that any 2-closed transitive permutation group contains an element of prime order without any fixed point. Now, we consider a combinatorial analogy of this conjecture. Note that such a
permutation partitions $X$ into cells of the same size by taking the orbits; furthermore, the cells form an equitable partition of $\text{Orb}_2(\Gamma')$ in the sense of [4]. This fact indicates that $\text{Orb}_2(\Gamma')$ has an equitable partition with the cells of the same size if the above conjecture is true.

We say that an equitable partition of an association scheme is homogeneous of degree $d$ if the cells of the partition have the same size $d$, and we denote such a partition by a $d$-HEP.

We say that an equitable partition of an association scheme is relatively prime if $|C|$ is relatively prime to $|D|$ for all $C, D \in \pi$ with $C \neq D$.

In this paper we study two properties of equitable partitions of association schemes. First, we generalize a result in group theory to the class of association schemes; namely, we will show that, if a primitive association scheme has a nontrivial relatively prime equitable partition, then it is derived from a complete graph. Second, we study a $p$-HEP to give two essentially different examples of association schemes which have no $p$-HEP for each prime $p$, showing that an analogous conjecture for permutation groups does not immediately generalize to association schemes. Moreover, we show that a strongly regular graph with parameter $(25, 12, 5, 6)$ has a 5-HEP if and only if it is the Paley graph. Hence, the property having a 5-HEP characterizes the Paley graph among the strongly regular graphs with parameter $(25, 12, 5, 6)$.

2. Preliminaries

Let $r$ be a binary relation on a finite set $X$, i.e., $r \subseteq X \times X$. We say that $\sigma \in \text{Sym}(X)$ is an automorphism of $(X, r)$ if $(\sigma(x), \sigma(y)) \in r$ for each $(x, y) \in r$ where we denote by $\text{Sym}(X)$ the symmetric group on $X$. We denote the set of automorphisms of $(X, r)$ by $\text{Aut}(X, r)$. Let $F$ be a set of binary relations on $X$. We denote by $\text{Aut}(X, F)$ the intersection of $\text{Aut}(X, f)$ where $f$ ranges over the elements of $F$.

According to [10] we give several terminologies related to association schemes. We set $r^* := \{(x, y) \mid (y, x) \in r\}$ and $xr := \{y \in X \mid (x, y) \in r\}$ where $x \in X$. We denote the cardinality of a set $\Omega$ by $|\Omega|$.

**Definition 2.1.** Let $G$ be a set of nonempty disjoint binary relations on $X$ whose union covers $X \times X$. We say that the pair $(X, G)$ is an association scheme (or simply, scheme) if it satisfies the following conditions:

(i) $1_X := \{(x, x) \mid x \in X\}$ is a member of $G$.

(ii) For each $g \in G$, $g^*$ is a member of $G$.

(iii) For all $d, e, f \in G \mid xd \cap ye^*$ is constant whenever $(x, y) \in f$. The constant is denoted by $a_{def}$, and $\{a_{def}\}_{d, e, f \in G}$ are called the intersection numbers of $G$. For each $g \in G$ we abbreviate $a_g g^* 1_X$ as $n_g$, which is called the valency of $g$.

Let $(X, G)$ be an association scheme. Then, for each $(x, y) \in X$ there exists a unique element in $G$ which contains $(x, y)$. We shall write such a unique element as $r(x, y)$, and for all subset $Y, Z \subseteq X$ we set $r(Y, Z) := \{r(x, y) \mid x \in Y, y \in Z\}$ and $r(Y) := r(Y, Y)$. For each $Y \subseteq X$ and $D \subseteq G$ we set $YD := \bigcup_{y \in Y, d \in D} yd$ and $D^x := D - \{1_X\}$. In [10]
the **complex product** $DE$ of two subsets $D, E \subseteq G$ is defined as follows:

$$DE := \left\{ f \in G \mid \sum_{d \in D, e \in E} a_{de} > 0 \right\}.$$  

**Remark 2.1.**

(i) Suppose that $\Gamma \subseteq \text{Sym}(X)$ is transitive. Then, as described in Section 1, $\text{Orb}_2(\Gamma)$ forms an association scheme, say, $(X, G)$, so $\Gamma \subseteq \text{Aut}(X, G)$.

(ii) By definition, $\sigma \in \text{Aut}(X, G)$ if and only if $\sigma \in \text{Sym}(X)$ such that $r(x, y) = r(\sigma(x), \sigma(y))$ for all $x, y \in X$.

(iii) The complex product is an associative binary operation on the power set of $G$, and we have $Y(DE) = (YD)E$ for each $Y \subseteq X$ and $D, E \subseteq G$.

A subset $H$ of $G$ is called **closed** if $HH \subseteq H$, **normal** if $gH = Hg$ for each $g \in G$, and **strongly normal** if $g^*Hg \subseteq H$ for each $g \in G$. The concept of closed subsets corresponds to that of subgroups in group theory. It is shown in [10] that a strongly normal closed subset is normal, but the converse does not hold in general. We say that a subset $D$ of $G$ is **thin** if $n_d = 1$ for each $d \in D$.

For each $g \in G$ and $Y, Z \subseteq X$ we set $g_{Y,Z} := g \cap (Y \times Z)$, $g_Y := g_{Y,Y}$, and $G_Y := \{g_Y \mid g \in G, g_Y \neq \emptyset\}$.

Let $H$ be a closed subset of $G$. According to [8] we say that $Y \subseteq X$ is a **transversal** of $H$ in $X$ if $|xH \cap Y| = 1$ for each $x \in X$.

**Lemma 2.1.** Let $(X, G)$ be an association scheme and $H$ a thin closed subset of $G$. If $Y$ is a transversal of $H$ in $X$, then each $\sigma \in \text{Aut}(Y, G_Y)$ extends $\bar{\sigma} \in \text{Aut}(X, G)$ defined by $\bar{\sigma}(yt) := \sigma(y)t$, $y \in Y$ and $t \in H$, where we denote any singleton $\{\alpha\}$ by $\alpha$.

**Proof.** Note that each $t \in H$ induces a permutation on $G$ such that $g$ is mapped to $gt$ (or $tg$) for each $g \in G$ (see [6] for the proof, which can be easily proved from definitions of association schemes and complex products). For all $y_1, y_2 \in Y$ and $t_1, t_2 \in T$ we have

$$r(\bar{\sigma}(y_1t_1), \bar{\sigma}(y_2t_2)) = r(\sigma(y_1)t_1, \sigma(y_2)t_2) = t_1^*r(\sigma(y_1), \sigma(y_2))t_2$$

$$= t_1^*r(y_1, y_2)t_2 = r(y_1t_1, y_2t_2).$$

Here we remark that each of $\{y_it_t, \sigma(y_it_i) \mid i = 1, 2\}$ is a singleton in $X$, and each of $\{t_1^*r(\sigma(y_1), \sigma(y_2))t_2, t_1^*r(y_1, y_2)t_2\}$ is a singleton in $G$. \hfill $\Box$

We say that an association scheme $(X, G)$ is **primitive** if $G$ has no closed subset other than $\{1_X\}$ or $G$, which is equivalent to the condition that there is no subset $Y$ of $X$ such that $1 < |Y| < |X|$ and $Yg \subseteq Y$ for some $g \in G^\times$.

**Remark 2.2.** If $\Gamma \subseteq \text{Sym}(X)$ is primitive in the sense of [9], then $\text{Orb}_2(\Gamma)$ forms a primitive association scheme.

Let $(X, G)$ be an association scheme. For each $g \in G$ we define a matrix $A_g$ called the **adjacency matrix** of $g$ as follows:

$$(A_g)_{x,y} := \begin{cases} 1 & \text{if } (x, y) \in g \\ 0 & \text{otherwise} \end{cases}$$
where the rows and columns of $A_g$ are indexed by the elements of $X$. It is well known that the vector space spanned by $\{A_g \mid g \in G\}$ forms a sub-algebra of the full matrix algebra of degree $|X|$.

According to [4] we say that a partition $\pi$ is an equitable partition of $(X, G)$ if, for all $U, V \in \pi$ and $g \in G$, $|xg \cap V|$ is constant whenever $x \in U$. We denote the constant by $[UgV]$. Customarily, we call an element of $\pi$ a cell. We note that $\pi$ is equitable if and only if $(A_g)_{U, V}$ has a constant row-sum $[UgV]$ for all $U, V \in \pi$ and $g \in G$.

Originally, the terminology “equitable” is used for a partition of the vertex set of a simple graph or digraph. Since $G$ is closed under the transposition map, it follows that $\pi$ is an equitable partition of $(X, G)$ if and only if $\pi$ is an equitable partition of $(X, g)$ for each $g \in G$.

In [4] it is shown that, if $\pi$ is an equitable partition of $(X, G)$, then the linear map defined by $A_g \mapsto A_g/\pi$ for each $g \in G$ is an algebra homomorphism from the span of $\{A_g \mid g \in G\}$ over the complex field to the full matrix algebra of degree $|\pi|$ where $A_g/\pi$ is a square matrix of degree $|\pi|$ defined by $(A_g/\pi)_{U, V} := [UgV]$.

**Example 2.3.** (i) Clearly, $\{X\}$ and $\{|x\} \mid x \in X\}$ are equitable partitions of $(X, G)$, and they are trivial.

(ii) For each $y \in X$, $\{yg \mid g \in G\}$ forms an equitable partition of $(X, G)$, since $|xg \cap yf| = a_{gf}e^x$ whenever $x \in ye$.

(iii) For each closed subset $H$ of $G$, $\{xH \mid x \in X\}$ is a $|xH|$-HEP of $(X, G)$ if and only if $H$ is normal.

(iv) If $\Gamma \leq \text{Aut}(X, G)$, then $\Gamma$ forms an equitable partition of $(X, G)$. If $\Gamma \leq \text{Aut}(X, G)$ is semi-regular, then the set of orbits of $\Gamma$ forms a $|\Gamma|$-HEP of $(X, G)$.

For each partition $\pi$ we set

$$\text{Aut}(X, G)_\pi := \{\sigma \in \text{Sym}(X) \mid \forall U \in \pi, \sigma_U \in \text{Aut}(U, G_U)\}$$

where $\sigma_U$ is the restriction of $\sigma$ on $U$, so $\text{Aut}(X, G)_\pi \simeq \prod_{U \in \pi} \text{Aut}(U, G_U)$ as groups.

**Lemma 2.2.** Let $(X, G)$ be an association scheme and $\pi$ an equitable partition of $(X, G)$. Suppose that $U, V \in \pi$ with $(|U|, |V|) = 1$. Then, for each $g \in G$, $g_{U, V} = U \times V$ if and only if $g_{U, V} \neq \emptyset$.

**Proof.** Counting the elements of $g_{U, V}$ in two ways we obtain that

$$|U|| UgV| = |V| [V g^*U].$$

It follows from the assumption $(|U|, |V|) = 1$ and $[UgV] \leq |V|$ that $[UgV] = |V|$ and $[V g^*U] = |U|$. Hence, $g_{U, V} = U \times V$ if and only if $g_{U, V} \neq \emptyset$. $\square$

**Proposition 2.3.** Let $(X, G)$ be an association scheme. If $\pi$ is a relatively prime equitable partition of $(X, G)$, then $\text{Aut}(X, G)_\pi \leq \text{Aut}(X, G)$.

**Proof.** Let $\sigma \in \text{Aut}(X, G)_\pi$. If $(x, y) \in U \times U$ and $U \in \pi$, then $r(\sigma(x), \sigma(y)) = r(x, y)$ by the definition of $\text{Aut}(X, G)_\pi$. If $(x, y) \in U \times V$ for distinct $U, V \in \pi$, then $(\sigma(x), \sigma(y)) \in U \times V = r(x, y)_{U, V}$ by Lemma 2.2. This implies that $\sigma \in \text{Aut}(X, G)$. $\square$
**Lemma 2.4.** Let \((X, G)\) be an association scheme. Suppose that \(\pi\) is an equitable partition of \((X, G)\) such that \(|U| \leq 2\) for each \(U \in \pi\). Then the cells of \(\pi\) are the orbits of an involution in \(\text{Aut}(X, G)\).

**Proof.** Define \(\sigma \in \text{Sym}(X)\) such that \(\sigma(x) := x'\) for each \(x \in X\) where \(\{x, x'\}\) is a cell of \(\pi\), i.e., \(x = x'\) if \(\{x\} \in \pi\), \(x \neq x'\) otherwise. We will show that \(\sigma \in \text{Aut}(X, G)\). For all \(x, y \in X\) we have

\[ r(\sigma(x), \sigma(y)) = r(x', y'). \]

If \(x = x'\) and \(y = y'\), then \(r(x', y') = r(x, y)\). If \(x \neq x'\) and \(y = y'\), then \(r(x', y') = r(x', y) = r(x, y)\) since \(\pi\) is an equitable partition. Similarly, \(r(x', y') = r(x, y)\) if \(x = x'\) and \(y \neq y'\). If \(x \neq x'\) and \(y \neq y'\), then we divide our considerations into the following cases:

(i) \(r(x, y) = r(x, y')\);
(ii) \(r(x, y) \neq r(x, y')\).

If \(r(x, y) = r(x, y')\), then \(r(x', y') = r(x', y')\) since \(\pi\) is an equitable partition. If \(r(x, y) \neq r(x, y')\), then \(r(x', y') \neq r(x', y')\), and, hence, \(r(x', y') = r(x, y)\).

Thus, \(r(x', y') = r(x, y)\) for each case, and, hence, \(\sigma \in \text{Aut}(X, G)\). \(\square\)

### 3. Characterization by equitable partitions

#### 3.1. Primitivity and relatively prime equitable partitions

Let \(\Gamma\) be a primitive permutation group on a finite set \(X\). It is well known (see Theorem 13.1, 13.3 in [9] for example) that, if \(\Gamma\) contains a cycle of length two, then \(\Gamma = \text{Sym}(X)\), and, if there exists \(U \subseteq X\) such that \(1 < |U| < |X|\) and \(\bigcap_{x \in X - U} \Gamma_x\) is transitive on \(U\), then \(\Gamma\) is doubly transitive.

In this section we aim to generalize these results to association schemes to obtain the following theorem:

**Theorem 3.1.** Suppose that \((X, G)\) is a primitive association scheme with a nontrivial relatively prime equitable partition. Then \(|G| = 2\).

Let \(\pi\) be an equitable partition of \((X, G)\). We set \(\pi^* := \{C \in \pi \mid |C| > 1\}\). Note that an equitable partition with \(|\pi^*| = 1\) is relatively prime. **Theorem 3.1** implies that, if \(\text{Aut}(X, G)\) contains a subgroup \(H\) whose orbits are \(\{C, \{x\} \mid x \in X - C\}\) for some \(C \subseteq X\) with \(1 < |C| < |X|\), then \(|G| = 2\), and, hence, \(\text{Aut}(X, G) = \text{Sym}(X)\). This shows a connection between **Theorem 3.1** and the above two results.

For the remainder of this subsection we assume that \((X, G)\) is a primitive association scheme and \(\pi\) is a nontrivial relatively prime equitable partition of \((X, G)\), and we fix a cell \(U \in \pi\) such that \(1 < |U| < |X|\).

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1The assumption of the theorem can be weakened to the condition that \(\pi\) is an equitable partition of \((X, G)\) which contains a cell \(C\) such that \(|C| > 1\) and \(|\{C, \{D\}\}| = 1\) for each \(D \in \pi\) with \(C \neq D\). The reader will see that the proof given in this paper also works with the statement replaced by the weakened assumption. We were informed of this by Ponomarenko in August of 2004 (see [7]).
Lemma 3.2. For each \( x \in U \) and \( g \in r(U)^\times \) we have \( xg - U \neq \emptyset \).

**Proof.** If \( xg \subseteq U \), then \(|yg \cap U| = |xg \cap U| = |xg| = n_g \) for each \( y \in U \) by the definition of equitable partitions. This implies that \( Ug \subseteq U \), which contradicts \((X, G)\) being primitive. \( \square \)

Lemma 3.3. For all \( x, y \in U \) and \( g \in r(U) \) we have \( xg - U = yg - U \).

**Proof.** For each \( V \in \pi \) with \( g_{U,V} \neq \emptyset \) we have \( g_{U,V} = U \times V \) by Lemma 2.2. This implies that \( xg - U \) is a union of cells, which is independent of the choice of \( x \in U \). \( \square \)

Lemma 3.4. For each \( g \in r(U) \), if \( g \neq g^* \), then \( g_{Ug - U, U} = \emptyset \).

**Proof.** Suppose \((z, w) \in g_{Ug - U, U} \). Since \( z \in (Ug - U) \) and \( w \in U \), it follows from Lemma 3.3 that \( z \in Ug - U = wg - U \). Therefore, \((z, w) \in g \cap g^* \), which contradicts \( g \neq g^* \). \( \square \)

Lemma 3.5. For each \( g \in r(U) \) we have \( g = g^* \).

**Proof.** Suppose \( g \neq g^* \) for some \( g \in r(U) \), and so \( g \in r(U)^\times \). Let \( x, y \in U \) with \((x, y) \in g \). By Lemma 3.2, we can take \( z \in Ug - U \). Since \((x, z) \in g \), we have \( a_{gg^* g} = |xg \cap zg| \). Since \( g_{(Ug - U), U} = \emptyset \) by Lemma 3.4, \( zg \cap U = \emptyset \), and, hence,

\[
|xg \cap zg| = |(xg \cap U) \cap zg| + |(Ug - U) \cap zg| = |(Ug - U) \cap zg|.
\]

On the other hand, by Lemma 3.3, \( a_{gg^* g} = |xg \cap yg| \geq |Ug - U| \). These equations imply that \( |(Ug - U) \cap zg| \geq |Ug - U| \), which contradicts \( z \notin zg \). \( \square \)

Lemma 3.6. We have \( |r(U)^\times| = 1 \).

**Proof.** Assume that there exist \( f, g \in r(U)^\times \) such that \( f \neq g \). For convenience we set \( Y := Ug - U \) and \( Z := Uf - U \), so \( Y \cap Z = \emptyset \) since \( f \cap g = \emptyset \). Let \((x, u) \in f_U, (y, z) \in Y \times Z \). Then, by Lemma 3.3, \( a_{gg^* f} = |xg \cap ug| \geq |Y| \). On the other hand, \( a_{gg^* f} = |xg \cap zg| = |(xg \cap U) \cap zg| + |Y \cap zg| \).

We claim that \((xg \cap U) \cap zg = \emptyset \). If \( v \in xg \cap U \cap zg \), then \( v \in U \) and \((z, v) \in g \). By Lemma 3.5, \((v, z) \in g^* = g \). It follows from Lemma 3.3 that \( z \in Y \), a contradiction to \( z \in Z \).

Thus, \( |Y \cap zg| = a_{gg^* f} \geq |Y| \) by the claim. Since \((y, z) \in Y \times Z \) is taken arbitrarily, we conclude that \( g_Y Z = Y \times Z \), and, hence, \( n_g = |yg| \geq |U| + |Z| \geq n_f + 1 \). Replacing \( f \) by \( g \) we obtain from the symmetric argument that \( n_f \geq n_g + 1 \), a contradiction. \( \square \)

**Proof of Theorem 3.1.** For convenience we set \( Z := Ug - U \). By Lemmas 3.5 and 3.6, \( r(U) = \{1_X, g\} \) with \( g = g^* \). For all \( x, y \in U \) with \( x \neq y \) and \( z \in Z \) we have

\[
|Z \cap zg| + |(U \times xg) \cap zg| = |xg \cap zg| = |xg \cap yg| = |Z| + |U| - 2.
\]

Since \( |U \cap xg \cap zg| = |U| - 1 \), it follows that \( |Z \cap zg| = |Z| - 1 \). This implies that \( zg \subseteq (U \cup Z) - \{z\} \) and \((U \cup Z)g \subseteq U \cup Z \). Thus, we conclude from primitivity that \( U \cup Z = X \), and, hence, \( G = \{1_X, g\} \).
4. p-HEP in strongly regular graphs

4.1. 2-HEP in strongly regular graphs

The next result shows that any 2-HEP in a certain type of strongly-regular graph is completely regular in the sense of [2]. We note that the latter part of the proof to show the resulting parameter is obtained as a corollary of 11.1.8 in [2].

**Proposition 4.1.** Suppose that $\Gamma$ is a strongly regular graph with parameter $(v, k, \lambda, \mu) = (\mu' - 2, k, 0, 2\mu' - 1)$ for some $\nu', \mu' \in \mathbb{N}$, and $\pi$ is a 2-HEP of $\Gamma$. Then there exists a strongly regular graph with parameter $(v/2, k - 1, \mu - 2, 2\mu)$.

**Proof.** Let $(X, G)$ be the association scheme derived from $\Gamma$, i.e., $X$ is the vertex set of $\Gamma$ and $G = \{1, x, g, h\}$ where $(x, g) = \Gamma$.

We claim that $r(U)^x = \{g\}$ for each $U \subseteq \pi$. Suppose not, i.e., there exists $U \in \pi$ such that $r(U)^x = \{h\}$. Set $U := \{x, y\}$ and $V := \{z, w\} \in \pi$. If $z \in xg \cap yg$, then $|wg \cap U| = |zg \cap U| = 2$, and, hence, $g_{U,V} = U \times V$. This implies that $xg \cap yg$ is a disjoint union of cells, and, hence, $|xg \cap yg|$ is even, which contradicts $\mu = |xg \cap yg|$ being odd.

We claim that $g_{U,V} \neq U \times V$ for all distinct $U, V \in \pi$. Suppose not, i.e., there exist distinct $U, V \in \pi$ such that $g_{U,V} = U \times V$. It follows from the above claim that there exists a triangle among the elements of $U \cup V$, a contradiction to $\lambda = 0$.

Therefore, $[UgV] \in \{0, 1\}$ for all $U, V \in \pi$, and, hence, $B := A_\pi / \pi$ is a $(0, 1)$-matrix. Setting $C := B - I$ we obtain that $C$ is a $(0,1)$-matrix whose diagonal entries are zero. Since

$$B^2 = (A_\pi / \pi)^2 = k(A_{1x} / \pi) + \lambda(A_\pi / \pi) + \mu(A_h / \pi)$$

$$= kI + \lambda B + \mu(B - I + 2(J - B)),$$

we have

$$(C + I)^2 = kI + \mu(-C - 2I + 2J).$$

Thus, $C^2 = (k - 1)I + (\mu - 2)C + 2\mu(J - I - C).$ This implies that $C$ is the adjacency matrix of a strongly regular graph with parameter $(v/2, k - 1, \mu - 2, 2\mu)$. \hfill $\Box$

**Remark 4.1.** The Petersen graph and the Hoffman–Singleton graph are the strongly regular graphs with parameters $(10, 3, 0, 1)$ and $(50, 7, 0, 1)$, respectively (see [1] or [2]). Note that these parameters satisfy the assumption of Proposition 4.1, but the stated parameters never occur since $\mu - 2$ is negative. This implies that each of them has no 2-HEP. It follows from Lemma 2.4 that there is no regular permutation of order two in the automorphism group for each of the two graphs.

4.2. Strongly regular graphs with parameter $(25, 12, 5, 6)$

According to the classification of association schemes with at most 28 points given in [5] there are exactly eight isomorphism classes of association schemes such that one of the relations forms a strong regular graph with parameter $(25, 12, 5, 6)$. Let $(X, G)$ be one of the eight association schemes. Then, for each $g \in G^\times$, $(X, g)$ forms a strongly regular
graph with parameter \((25, 12, 5, 6)\). In this section we will prove that only one of the eight schemes has a 5-HEP.

For the remainder of this section we assume that \((X, G)\) is an association scheme such that \((X, g)\) is a strongly regular graph with parameter \((25, 12, 5, 6)\) where \(G = \{1_X, g, h\}\).

**Lemma 4.2.** If \(\pi\) is a 5-HEP of \((X, G)\), then either \(A_g/\pi = 2I + 2J\) or \(-3I + 3J\) where \(I\) is the identity matrix of degree 5 and \(J\) is the all 1 matrix of degree 5.

**Proof.** We shall write \(A_g\) as \(A\) for short. Set \(\pi := \{X_i \mid 1 \leq i \leq 5\}\) and divide \(A\) into the sub-matrices \(\{B_{ij}\}_{1 \leq i, j \leq 5}\) where \(B_{ij}\) is the restriction of \(A\) on \(X_i \times X_j\). Since \(A^2 = 12I + 5A + 6(J - I - A)\) and \(B_{ji} = B^T_{ij}\) for all \(i, j\), we have, for each \(i\),

\[
\sum_{k=1}^{5} B_{ik} B^T_{ik} = 12I + 5B_{ii} + 6(J - I - B_{ii}).
\]  

(1)

Since \(\pi\) is equitable, \(B_{ij}\) has a constant row-sum \(\beta_{ij}\) for all \(i, j\) where \(\beta_{ij} := [X_i g X_j]\). Multiplying the all 1 matrix on both sides of (1) we obtain that

\[
\sum_{k=1}^{5} \beta_{ik}^2 = 12 + 5\beta_{ii} + 6(5 - 1 - \beta_{ii}),
\]

and, hence,

\[
\beta_{ii}^2 + \beta_{ii} + \sum_{k \neq i} \beta_{ik}^2 = 36.
\]

Note that \(0 \leq \beta_{ij} \leq 5\) and the sum of entries of \(B_{ii}\) is even for all \(i, j\), since \(B^T_{ii} = B_{ii}\). This implies that \(\beta_{ii}\) must be even. The possible cases for \((\beta_{ii}, \beta_{ij} \mid 1 \leq j \leq 5, j \neq i)\) are the following (it is an easy exercise of elementary number theory):

(i) \((0, 3, 3, 3, 3)\);
(ii) \((2, \sigma(1), \sigma(2), \sigma(3), \sigma(4))\) for some \(\sigma \in S_4\);
(iii) \((4, 2, 2, 2, 2)\).

Thus, it suffices to show that the second case never occurs since \(\beta_{ij} = \beta_{ji}\).

Suppose that the second case can occur in some rows. Then we may assume that \((\beta_{11}, \beta_{12}, \beta_{13}, \beta_{14}, \beta_{15}) = (2, 1, 2, 3, 4)\), changing the index of \(\{X_i\}\) if necessary. Since \(\beta_{ij} = \beta_{ji}\), \((\beta_{11}, \beta_{21}, \beta_{31}, \beta_{41}, \beta_{51}) = (2, 1, 2, 3, 4)\).

Here we claim that the possible choices for \(A/\pi = (\beta_{ij})\) are the following:

\[
\begin{pmatrix}
2 & 1 & 2 & 3 & 4 \\
1 & 2 & 2 & 4 & 3 \\
2 & 2 & 4 & 2 & 2 \\
3 & 4 & 2 & 2 & 1 \\
4 & 3 & 2 & 1 & 2
\end{pmatrix},
\begin{pmatrix}
2 & 1 & 2 & 3 & 4 \\
1 & 2 & 4 & 3 & 2 \\
2 & 4 & 2 & 3 & 1 \\
3 & 3 & 3 & 0 & 3 \\
4 & 2 & 1 & 3 & 2
\end{pmatrix}.
\]  

(2)

Note that \((\beta_{33}, \beta_{44}) \in \{(4, 2), (4, 0), (2, 0), (2, 2)\}\). If \(\beta_{33} = 4\), then \((\beta_{33}, \beta_{31}, \beta_{32}, \beta_{34}, \beta_{35}) = (4, 2, 2, 2, 2)\). Note that \(\beta_{44} = 2\) since \(\beta_{41} = 3\) and \(\beta_{43} = 2\). Note that \(\beta_{24} = 4\) since \((\beta_{21}, \beta_{22}, \beta_{23}) = (1, 2, 2)\) and \(\beta_{14} = 3\). This implies that \(\beta_{25} = 3\) and \(\beta_{45} = 1\). Therefore, the assumption that \(\beta_{33} = 4\) induces the first matrix in (2).
If \((\beta_{33}, \beta_{44}) = (2, 0)\), then \((\beta_{44}, \beta_{41}, \beta_{42}, \beta_{43}, \beta_{45}) = (0, 3, 3, 3, 3)\). Note that \(\beta_{23} = 4\) since \((\beta_{21}, \beta_{22}, \beta_{24}) = (1, 2, 3)\) and \(\beta_{13} = \beta_{33} = 2\). This implies that \(\beta_{25} = 2\) and \(\beta_{35} = 1\). Therefore, the assumption that \(\beta_{33} = 2\) induces the second matrix in \((2)\).

If \((\beta_{33}, \beta_{44}) = (2, 2)\), then 1 should appear exactly once in each row and each column of \((\beta_{ij})\), contradicting \((\beta_{ij})\) being symmetric.

Thus, the claim preceding \(2)\) holds. However, these two matrices do not satisfy the equation \((A/\pi)^2 = 12I + 5(A/\pi) + 6(5J - I - A/\pi)\). Therefore, we conclude that the second case never occurs. \(\Box\)

Without loss of generality we may assume that \(A_g/\pi = 2I + 2J\), changing \(g\) and \(h\) if necessary, since \(A_g/\pi + A_h/\pi + I = 5J\).

Lemma 4.3. If \(\pi\) is a 5-HEP of \((X, G)\) and \(\rho := \{x, xg, xh\}\) with \(x \in X\), then the partition \(\pi \land \rho\) induces a partition of \(xg\) into one 4-clique and four 2-cliques and that of \(xh\) into four 3-cliques, where \(\pi \land \rho\) is the greatest lower bound of \(\pi\) and \(\rho\) on the lattice consisting of all the partitions of \(X\).

Proof. Without loss of generality we may assume that \(x \in X_1\). Since \((X_1, gX_1)\) forms a 5-clique, \(xg \cap X_1\) is a 4-clique, say \(Y\).

We claim that \(|X_i \cap xg| \leq 2\) for each \(i\) with \(2 \leq i \leq 5\). If \(|X_i \cap xg| \geq 3\), then \(|X_1 g X_i| \geq 3\), a contradiction to \(A_g/\pi = 2I + 2J\).

Note that \(|xg - Y| = 8\). Combining the above claim with the pigeonholes principle we conclude that \(|X_i \cap xg| = 2\) for each \(i\) with \(2 \leq i \leq 5\). Therefore, \(xg\) is refined into \(Y\) and the four 2-cliques \(\{xg \cap X_i | 2 \leq i \leq 5\}\) and \(xh \cap X_i\) forms a 3-clique for each \(i\) with \(2 \leq i \leq 5\). \(\Box\)

Lemma 4.4. Let \(K\) be a 5-clique in \((X, g)\). If \(X_i \cap K \neq \emptyset\), then either \(|X_i \cap K| = 1\) or \(K = X_i\).

Proof. If \(X_i \cap K = \{x, y\}\) with \(x \neq y\), then \(a_{gxx} = |xg \cap yg| \leq |X_i - \{x, y\}| + |K - \{x, y\}| = 6\), a contradiction to \(a_{gxx} = 5\).

If \(3 \leq |X_i \cap K| \leq 4\), then there exists \(z \in K - X_i\). This implies that \(|zg \cap X_i| \geq 3\), which contradicts \(|X_1gX_i| = 2\) for \(j \neq i\). \(\Box\)

Lemma 4.5. If \(C_1\) and \(C_2\) are distinct 4-cliques in \(xg\), then \(xg - (C_1 \cup C_2)\) forms a 4-clique.

Proof. For short we set \(Z := xg - (C_1 \cup C_2)\). Let \(X_1 \in \pi\) with \(x \in X_1\). Since \(C_i \cup \{x\}\) forms a 5-clique, it follows from Lemma 4.4 that either \(X_1 \cap C_i = \{x\}\) or \(C_i = X_1 - \{x\}\).

If \(X_1 \cap C_i = \{x\}\) for each \(i\), then \(Z = X_1 - \{x\}\) forms a 4-clique. Suppose that \(C_i = X_1 - \{x\}\) for some \(i\). Without loss of generality we may assume that \(C_1 \subseteq X_1 - \{x\}\).

Since \(|X_1 g X_1| = 2\) for each \(i\) with \(2 \leq i \leq 5\), \(|yg \cap C_1| = 1\) for each \(y \in C_2\). Since \(|yg \cap C_2| = 3\) and \(|yg \cap xg| = 5\),

\(|yg \cap Z| = |yg \cap xg| - |yg \cap C_1| - |yg \cap C_2| = 1\).

This implies that \(\tau: C_2 \to Z\) defined by \(\tau(y) = yg \cap Z\) is well defined.

On the other hand, we claim that the above \(\tau\) is surjective. Suppose not, i.e., \(|zg \cap C_2| = 0\) for some \(z \in Z\). Since \(|zg \cap C_1| + |zg \cap C_2| + |zg \cap Z| = |zg \cap xg| = 5\), it follows
from \( |zg \cap C_1| = 1 \) that \( |zg \cap Z| = 4 = |Z| \), contradicting \( z \notin zg \). Thus, \( \phi \) is surjective, and, hence, bijective. This implies that \( |zg \cap C_2| = 1 \) for each \( z \in Z \), so 
\[
|zg \cap Z| = |zg \cap xg| - |zg \cap C_1| - |zg \cap C_2| = 3.
\]
We conclude that \( Z \) forms a 4-clique. 

**Lemma 4.6.** If \( K \) is a 5-clique in \( X - X_1 \), then \( K \in \pi \).

**Proof.** Suppose \( K \notin \pi \). Then, by Lemma 4.4, \( |X_i \cap K| \in \{0, 1\} \) for each \( i \) with \( 2 \leq i \leq 5 \). It follows that \( |K| = \sum_{i=2}^{5} |X_i \cap K| \leq 4 \), a contradiction. 

**Theorem 4.7.** Let \((X, G)\) be an association scheme with the same intersection numbers as as.25.no.11. Then \((X, G)\) has a 5-HEP if and only if \((X, G)\) is isomorphic to as.25.no.11.

**Proof.** Suppose that \((X, G)\) is isomorphic to as.25.no.11. Then there exists a semi-regular subgroup \( \Theta \) of order 5 in \( \text{Aut}(X, G) \), so the set of orbits of \( \Theta \) forms a 5-HEP.

Suppose that \((X, G)\) is not isomorphic to as.25.no.11 and \((X, G)\) has a 5-HEP \( \pi \). Fix \( x \in X \) and \( X_1 \in \pi \) with \( x \in X_1 \). We set \( X := \{x_i \mid 1 \leq i \leq 25\} \) where \( x_i \) is the point corresponding to the \( i \)-th row of the representation matrix given in [5] and \( g, h \in G \) to be the relations indexed by the letter 1, 2 in [5], respectively. We may assume \( x_1 \in X_1 \). It is not so hard to check that the subgraph \((x_1g, g_{x_1g})\) does not have three disjoint 4-cliques.

It follows from Lemmas 4.4 and 4.5 that \( X_1 \) is a unique 5-clique containing \( x_1 \), actually, 
\[
X_1 = \{x_1, x_4, x_8, x_9, x_{10}\}.
\]
We can find at most two cliques, which are the cells of \( \pi \) by Lemma 4.6. The following is the list of the another two 5-cliques according to the seven schemes:

\[
\begin{align*}
\text{as.25.no.4:} & \quad \{x_2, x_5, x_{14}, x_{17}, x_{18}\}, \{x_3, x_7, x_{22}, x_{24}, x_{25}\}; \\
\text{as.25.no.5:} & \quad \{x_2, x_5, x_{14}, x_{17}, x_{18}\}, \{x_3, x_7, x_{22}, x_{24}, x_{25}\}; \\
\text{as.25.no.6:} & \quad \{x_2, x_5, x_{14}, x_{17}, x_{18}\}, \{x_3, x_7, x_{22}, x_{24}, x_{25}\}; \\
\text{as.25.no.7:} & \quad \{x_2, x_7, x_{16}, x_{18}, x_{19}\}, \{x_3, x_6, x_{21}, x_{23}, x_{25}\}; \\
\text{as.25.no.8:} & \quad \{x_2, x_6, x_{15}, x_{17}, x_{19}\}, \{x_3, x_5, x_{20}, x_{23}, x_{24}\}; \\
\text{as.25.no.9:} & \quad \text{No other 5-cliques}; \\
\text{as.25.no.10:} & \quad \text{No other 5-cliques}.
\end{align*}
\]

We will prove that there is no equitable partition \( \pi \) such that \( A_g/\pi = -3I + 3J \), or, equivalently, that there is no equitable partition of \( \pi \) such that \( A_h/\pi = 2I + 2J \) from the argument replacing \( g \) by \( h \). The following is the list of the pairs for a fixed point \( x_i \), a unique 4-clique in \( x_i h \):

\[
\begin{align*}
\text{as.25.no.4:} & \quad \{x_1, \{x_{16}, x_{17}, x_{20}, x_{25}\}\}; \\
\text{as.25.no.5:} & \quad \{x_4, \{x_7, x_{11}, x_{17}, x_{23}\}\}; \\
\text{as.25.no.6:} & \quad \{x_1, \{x_{16}, x_{17}, x_{21}, x_{24}\}\}; \\
\text{as.25.no.7:} & \quad \{x_4, \{x_7, x_{11}, x_{17}, x_{23}\}\}; \\
\text{as.25.no.8:} & \quad \{x_{23}, \{x_1, x_{15}, x_{17}, x_{22}\}\}; \\
\text{as.25.no.9:} & \quad \{x_1, \{x_{16}, x_{17}, x_{21}, x_{24}\}\}; \\
\text{as.25.no.10:} & \quad \{x_1, \{x_{14}, x_{17}, x_{22}, x_{25}\}\}.
\end{align*}
\]

The following is the list of the another 5-cliques in \((X, h)\) which are disjoint from the unique clique containing the fixed point:
Combining Lemma 4.3 with the above list we can see that there are no other cliques for each scheme of the above. This contradicts \((X, G)\) having a 5-HEP. \(\square\)

5. Two association schemes with 28 points

Let \(AGL_1(F_8)\) denote the one-dimensional affine group over the Galois field with eight elements, i.e., \(AGL_1(F_8) = \{ t_{\alpha, \beta} \mid \alpha \in F_8^*, \beta \in F_8 \}\) where \(t_{\alpha, \beta}(x) := \alpha x + \beta\) for each \(x \in F_8\). Take any subgroup of order 4 in \(\{ t_{1, \beta} \mid \beta \in F_8 \}\), say \(H\). Then \(AGL_1(F_8)\) acts transitively on the left cosets of \(H\) in \(AGL_1(F_8)\), and the set of the orbitals forms an association scheme isomorphic to as.28.no.175 given in [5]. There is another scheme with the same intersection numbers as as.28.no.175, which is listed as as.28.no.176 in [5]. They both have a strongly normal closed subset \(T\) consisting of relations of valency 1 isomorphic to the Klein four group. We set \(T := \{1_X, a, b, c\}\) so that \(a^2 = b^2 = c^2 = 1_X\) with respect to the complex product.

We rename the elements of \(G\) as follows:

\[
G := T \cup \{d_i, e_i \mid 1 \leq i \leq 6\}
\]

where \(ad_1 = d_1b = d_1, bd_2 = d_2c = d_2, ad_3 = d_3c = d_3, d_4 = d_3^*, d_5 = d_2^*, d_6 = d_1^*,\)
\(d_1T = \{d_i, e_i\}\), and \(d_i^*T = \{d_i, e_i\}\) for each \(i\) with \(1 \leq i \leq 6\).

Throughout this section we assume that \((X, G)\) is isomorphic to as.28.no.175 or as.28.no.176.

5.1. Nonexistence of a 2-HEP

We aim to prove that \((X, G)\) has no 2-HEP by way of contradiction. Suppose that there exists a 2-HEP \(\pi\) of \((X, G)\).

**Lemma 5.1.** For each \(x \in X\) there exist exactly two cells \(U, V \in \pi\) such that \(xT = U \cup V\) and \(r(U) = r(V)\). In particular, \(\pi\) is a refinement of the partition \(\{xT \mid x \in X\}\).

**Proof.** Since \(\pi\) is a partition of \(X\), there exists a unique \(U \in \pi\) with \(x \in U\). Since \(|U| = 2\), there exists a unique \(y \in X\) such that \(U = \{x, y\}\).

Here we claim that \(r(x, y) = r(y, x)\), i.e., \(r(x, y)\) is symmetric and \(r(U)^X = \{r(x, y)\}\). Set \(g := r(x, y)\). Since \((U, gU)\) is a regular digraph, \(|yg \cap U| = |xg \cap U| = 1\), and, hence, \((x, y) \in g \cap g^*\). This implies that \(r(x, y) = r(y, x)\).

Since the set of symmetric relations in \(G\) is \(T\), it follows from the above claim that \(U \subseteq xT\).

Setting \(\{z, w\} := xT - U\) there exists a unique \(V \in \pi\) with \(z \in V\). From the same argument we conclude that \(V \subseteq zT = xT\). Since \(U \cap V = \emptyset\), it follows that \(U \cup V = xT\)

\(\square\)
and \( r(U) = \{1_X, r(x, y)\} = r(V) \). Since \( x \) is taken arbitrarily, this completes the proof of this lemma. \( \square \)

For each \( x \in X \) there exists a unique element \( t_x \in T^\times \) such that \( \{x, xt_x\} \in \pi \). We say that \( t_x \) is the content of \( x \) with respect to \( \pi \). By Lemma 5.1, \( t_x = t_y \) whenever \( y \in xT \). Thus, we can define the content of \( xT \) to be \( t_x \) with respect to \( \pi \).

**Lemma 5.2.** Under the assumption that \( \pi \) is a 2-HEP of \((X, G)\) there exist at least three elements of \( X/T \) which have the same content.

**Proof.** Since \(|X/T| = 7\) and \(|T^\times| = 3\), the lemma follows from the pigeonhole principle. \( \square \)

We say that \( t \in T \) is the right (left) stabilizer of \( g \in G \) if \( gt = \{g\} \) (resp. \( tg = \{g\} \)).

**Lemma 5.3.** If \( xT \) and \( yT \) have the same content \( t \) and \( xT \neq yT \), then there exist distinct \( s_1, s_2 \in T - \{1_X, t\} \) such that \( s_1r(x, y) = r(x, y)s_2 = r(x, y) \).

**Proof.** Set \( g := r(x, y) \), \( U \cup V = xT \), \( W \cup Z = yT \) for some \( U, V, W, Z \in \pi \).

We claim that \( tg \neq g \). Suppose not, i.e., \( tg = g \). Then \(|xg \cap (xt)g| = 2|\), since each element in \( G - T \) has valency 2. Since \( \{y, yt\} \in \{Z, W\} \), \(|yg^* \cap \{x, xt\}| = |ytg^* \cap \{x, xt\}| = 2| \) by the definition of equitable partitions. This implies that \( tg = gt = g \), but such an element does not exist in \( G \), a contradiction.

By a similar argument to that of the above claim, we have \( gt \neq g \). Note that each element in \( G - T \) has nontrivial right and left stabilizers which have a trivial intersection. This completes the proof. \( \square \)

**Lemma 5.4.** For all \( s, t \in T^\times \) with \( s \neq t \) there exists a unique \( i \in \{1, 2, 3, 4, 5, 6\} \) such that \( sd_i = d_it = d_i \).

**Proof.** This is just an observation for the intersection numbers of \((X, G)\). \( \square \)

**Theorem 5.5.** There is no 2-HEP in \((X, G)\).

**Proof.** By Lemma 5.2, there exist at least three elements of \( X/T \) which have the same content, say, \( xT, yT, zT \).

By Lemma 5.3, there exist \( s_1, s_2 \in T \) such that \( s_1r(x, y) = r(x, y)s_2 = r(x, y) \). Note that either \( s_1r(x, z) = r(x, z)s_2 = r(x, z) \) or \( s_1r(z, x) = r(z, x)s_2 = r(z, x) \) by Lemma 5.3.

By Lemma 5.4, there exists a unique element \( i \in \{1, 2, 3, 4, 5, 6\} \) such that \( sd_i = d_is_2 = d_i \). From uniqueness we have \( r(x, y) = r(x, z) \) or \( r(z, x) \), which contradicts \( yT \cap zT = \emptyset \). \( \square \)

5.2. Nonexistence of a 7-HEP

In this subsection we aim to prove that \((X, G)\) has a 7-HEP if and only if it is as. 28. n° .175. Since as. 28. n° .175 has a regular automorphism group, it has a 7-HEP. Therefore, the “if” part is done.

Suppose that \( \pi \) is a 7-HEP in \((X, G)\).
Lemma 5.6. If $U \in \pi$, then $U$ is a transversal of $T$ in $X$ and $\pi = \{ Ut \mid t \in T \}$.

Proof. Let $x, y \in U$ with $x \neq y$.

We claim that $xT \cap yT = \emptyset$. Suppose not, i.e., $z \in xT \cap yT$. Then $xT = yT$, and, hence, $r(x, y) \in \{a, b, c\}$. Since $\pi$ is equitable and $r(x, y)^* = r(x, y)$, $(U, r(x, y)U)$ is a regular undirected graph of valency 1, which contradicts the number of arcs in any undirected graph being even.

Combining the above claim with $|U| = 7$, $|xT| = 4$, and $|X| = 28$ we conclude that $U$ is a transversal of $T$ in $X$.

For each $u \in U$, $t \in T^X$, and $V \in \pi$ we have $|ut \cap V| = 1$ since $\pi$ is equitable. This implies that there exists a one-to-one correspondence between any two cells in $\pi$ by the map $u \mapsto ut$ for $t \in T$. Hence, the second statement holds. \hfill \Box

Lemma 5.7. If $U \in \pi$, then $G_U$ is the set of orbitals of a subgroup of order 7 in $\text{Sym}(U)$.

Proof. By Lemma 5.6, $r(U) \cap T = \{ 1_U \}$.

Let $g \in G - T$. We claim that $(U, g_U)$ is a regular graph of valency 1. Suppose not, i.e., there exist distinct $x, y, z \in U$ such that $y, z \in xg$ and $y \neq z$. It follows that $r(y, z) \in g^*g \subseteq T$, a contradiction to Lemma 5.6. Therefore, $(U, g_U)$ is a regular graph of valency 1 since $U \in \pi$.

Let $x_0 \in U$ and $g \in r(U)$. Then there exists a unique $x_1 \in U \cap x_0g$. Inductively, there exists a unique $x_i \in U \cap x_{i-1}g$ for each $i = 1, 2, 3, 4, 5, 6$. Furthermore, $(x_6, x_0) \in g$; otherwise $(U, g_U^6)$ has valency 2. Observing the intersection numbers of $(X, G)$ we obtain that $x_ig^i \subseteq x_{i+j}T$ for all $i, j$ where the subscripts are read modulo 7. Therefore, we conclude from the same argument as the above that $r(x_i, x_{i+j}) = r(x_0, x_j)$ for all $i, j$. This implies that the adjacency among the elements of $U$ coincides with the orbitals of $\{(x_0, x_1, \ldots, x_6)\} \leq \text{Sym}(U)$. \hfill \Box

Theorem 5.8. The scheme as.28.no.176 has no 7-HEP.

Proof. Let $U \in \pi$. By Lemma 5.6, $U$ is a transversal of $T$ in $X$. Applying Lemma 2.1 for $U$ with Lemma 5.7 we obtain $N \leq \text{Aut}(X, G)$ with $|N| = 7$. Since $\{ xg \mid g \in G \}$ is an equitable partition of $(X, G)$ for each $x \in X$, it follows from Lemma 2.4 that there exists a subgroup of $\text{Aut}(X, G)$ acting transitively on $xT$ for $x \in X$. This implies that $\text{Aut}(X, G)$ is transitive on $X$, which contradicts that the automorphism group of as.28.no.176 is not transitive on the underlying set (see [5]). \hfill \Box

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