

Loop Quantization of Polarized Gowdy Model on T^3 : Classical Theory

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Abstract

The vacuum Gowdy models provide much studied, non-trivial midi-superspace examples. Various technical issues within Loop Quantum Gravity can be studied in these models as well as one can hope to understand singularities and their resolution in the loop quantization. The first step in this program is to reformulate the model in real connection variables in a manner that is amenable to loop quantization. We begin with the unpolarized model and carry out a consistent reduction to the polarized case. Carrying out complete gauge fixing, the known solutions are recovered.

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I. INTRODUCTION

Gowdy space-times [1] are solutions of vacuum Einstein equation admitting two commuting, space-like isometries and having closed spatial hypersurfaces. Such models have essentially one of the three topologies for the spatial slices: $T^3, S^3, S^1 \times S^2$. The case of T^3 is the simplest of these. A further simplification is possible. One can restrict to the so-called *polarized* models in which the two Killing vectors are orthogonal. This simplest case of polarized, T^3 vacuum Gowdy model is the focus of this series of works. In this case, the complete set of exact solutions is known [2] which generically have *initial singularity*. But there is also an infinite sub-family of solutions for which all curvature invariants are finite. The approach to classical singularity is well studied and is known to follow a special case of the BKL scenario known as asymptotically velocity term dominated near singularity (AVTDS) [3]. At late times, the model is known to be asymptotically homogeneous [4].

These models have been analysed in the canonical framework in both metric variables as well as in terms of the complex Ashtekar variables. The first attempts of quantization, were carried out in ADM variables in [5]. Another approach which has been more successful was based on an interesting property of the model. After a suitable (partial) gauge fixing, these models can be described by (modulo a remaining global constraint) a “point particle” degree of freedom and by a scalar field ϕ which is subject to the same equations of motion as a massless, rotationally symmetric, free scalar field propagating in a fictitious two dimensional expanding torus. This equivalence was used in the quantization carried out in [6]. Subsequent analysis has been carried out in a large number of works some of which are listed in [7]. However in these quantizations, the evolution turned out to be non-unitary and in [8] a new parametrization was introduced which implemented unitary evolution in quantum theory.

Canonical description of *unpolarised* Gowdy T^3 model in terms of the complex Ashtekar variables has been given in [9]. A complete set of Dirac observables is also known [10]. The canonical quantization of this model was carried out in [11] and the physical Hilbert space was obtained. Although quantization has been carried out, the difficult issue of singularities has not been fully addressed (see however [12] for preliminary attempts).

In this series of works, we aim to carry out *loop quantization* of the polarized, T^3 Gowdy model, obtained via symmetry reduction in terms of real connection variables. In this paper,

we report the first step of recasting the Gowdy model in the (real) connection formulation, including the restriction to the polarized model. In subsequent papers the quantization will be carried out and issue of singularities will be addressed.

In section II we discuss, in brief, Gowdy models and then restrict attention to the polarised T^3 case in metric variables. The form of the metric and the space-time solutions are discussed. In section III the unpolarised model is described in terms of real Ashtekar variables and a *consistent* reduction is carried out to obtain the polarised model. Here consistency refers to preservation of the diagonal form of the metric under the Hamiltonian evolution. In section IV, we explain a gauge fixing leading to recovery of the standard form of the solutions. Section V summarizes the results and includes preliminary comments.

II. POLARISED GOWDY T^3 MODEL IN METRIC VARIABLES

Gowdy space-times [1] are globally hyperbolic solutions of the vacuum Einstein's equations which are isometric under the action of the Abelian group T^2 which acts on the spatial slices assumed to be closed. This means that there are two independent commuting spatial Killing vectors. This condition along with the Einstein's equations restricts the allowable choices of spatial topology to be only T^3 , S^3 and $S^2 \times S^1$ (or certain manifolds with one of these as cover). If, in addition we make an additional assumption that the Killing vector fields which generate the T^2 isometry can be chosen to be mutually orthogonal everywhere, we get the so-called *polarized Gowdy model*. The metric, is then diagonal and can be written as [2]:

$$ds^2 = e^{2a}(-dT^2 + d\theta^2) + T(e^{2W}dx^2 + e^{-2W}dy^2) \quad (1)$$

where $\partial/\partial x$ and $\partial/\partial y$ are the two Killing vectors and a and W are functions of T and periodic functions of θ .

The solutions for W can be obtained from the second order differential equation:

$$\frac{\partial^2 W}{\partial T^2} + \frac{1}{T} \frac{\partial W}{\partial T} - \frac{\partial^2 W}{\partial \theta^2} = 0 \quad (2)$$

Given a solution, $W(T, \theta)$, the function $a(T, \theta)$ has to satisfy:

$$\begin{aligned} \frac{\partial a}{\partial \theta} &= 2T \frac{\partial W}{\partial T} \frac{\partial W}{\partial \theta} \\ \frac{\partial a}{\partial T} &= -\frac{1}{4T} + T \left[\left(\frac{\partial W}{\partial T} \right)^2 + \left(\frac{\partial W}{\partial \theta} \right)^2 \right] \end{aligned} \quad (3)$$

The W equation (2) encodes the dynamics while the (3) encodes the constraints. Incidentally, this makes the initial value problem and the problem of preservation of constraints in numerical relativity trivial for this model. The initial values of the dynamical variable W can be freely specified and given a W the constraint a can be trivially determined [13]. The requirement of a being a periodic function of θ imposes a condition of the solutions of the W equation, namely, $\int d\theta \partial_T W \partial_\theta W = 0$. The general solution given below, does satisfy this condition [2].

The general solution to (2) is given by

$$W = \alpha + \beta \ln T + \sum_{n=1}^{\infty} \left[a_n J_0(nT) \sin(n\theta + \gamma_n) + b_n N_0(nT) \sin(n\theta + \delta_n) \right] \quad (4)$$

where α , β , a_n , b_n , γ_n and δ_n are real constants and J_0 and N_0 are regular and irregular Bessel functions of the zeroth order. The special case of *homogeneous* model is given by $\beta = \frac{1}{2}$ and $a_n = 0 = b_n$ and corresponds to the flat Kasner solution described as:

$$ds^2 = -dT^2 + d\theta^2 + T^2 dx^2 + dy^2 \quad (5)$$

It can be shown that the curvature invariant $C \equiv R_{abcd} R^{abcd}$ blows up almost everywhere as $T \rightarrow 0_+$. The solutions are therefore generically singular. However for the special choice,

$$b_n = 0, \quad \beta = \frac{1}{2} \quad (6)$$

the curvature invariant remains bounded and all components of R_{abcd} have finite limit as $T \rightarrow 0_+$. It can also be shown that these nonsingular solutions are analytically extendible [2] but are causally ill-behaved (have closed time-like curves) in the extended portion. Thus there exists a infinite number of nonsingular solutions which however form a set of measure zero in the space of solutions. Curvature unboundedness is the generic behaviour.

III. POLARISED GOWDY T^3 MODEL IN REAL ASHTEKAR VARIABLES

A. Unpolarised Case

We begin with the connection formulation in the notation of [14], with $P_a^i = E_a^i / (\kappa \gamma)$ substituted. The classical symmetry reduction to the unpolarized Gowdy T^3 model in terms of the complex connection has been studied by [9, 11]. Translated into real variables, the

symmetry reduction is achieved by setting to zero the following components of the densitized triad and connection [15]:

$$E_I^\theta = 0 = E_3^\rho, \quad A_\theta^I = 0 = A_\rho^3; \quad \rho = x, y; \quad I = 1, 2. \quad (7)$$

In these variables, the Gauss, the diffeomorphism and the Hamiltonian constraints [14], are given by ($\kappa := 8\pi G_{\text{Newton}}$, $G := G_3$, $C := C_\theta$):

$$G = \frac{1}{\kappa\gamma} [\partial_\theta E_3^\theta + \epsilon_J^K A_\rho^J E_K^\rho]; \quad \epsilon_J^K := \epsilon_{3J}^K \quad (8)$$

$$C = \frac{1}{\kappa\gamma} [(\partial_\theta A_\rho^I) E_I^\rho + \epsilon_J^K A_\rho^J E_K^\rho A_\theta^3 - \kappa\gamma A_\theta^3 G_3]; \quad (9)$$

$$H = \frac{1}{2\kappa} \frac{1}{\sqrt{|\det E|}} [2A_\theta^3 E_3^\theta A_\rho^J E_J^\rho + A_\rho^J E_J^\rho A_\sigma^K E_K^\sigma - A_\rho^K E_J^\rho A_\sigma^J E_K^\sigma - 2\epsilon_J^K (\partial_\theta A_\rho^J) E_K^\rho E_3^\theta - (1 + \gamma^2) (2K_\theta^3 E_3^\theta K_\rho^J E_J^\rho + K_\rho^J E_J^\rho K_\sigma^K E_K^\sigma - K_\rho^K E_J^\rho K_\sigma^J E_K^\sigma)] . \quad (10)$$

In the above, K_a^i are the components of the extrinsic curvature which are related to the gravitational connection A_a^i and the torsion-free spin connection, Γ_a^i , as: $K_a^i = \gamma^{-1}(A_a^i - \Gamma_a^i)$ and γ is the Barbero-Immirzi parameter. The spin-connection is defined in eqn. (A.2).

Since none of the quantities depend on x or y we can integrate over the T^2 and write the symplectic structure and the total Hamiltonian as:

$$\Omega = \frac{4\pi^2}{\kappa\gamma} \int d\theta (dA_\theta^3 \wedge dE_3^\theta + dA_\rho^I \wedge dE_I^\rho) \quad (11)$$

$$H_{\text{tot}} = 4\pi^2 \int d\theta \{ \lambda^3 G + N^\theta C + NH \} \quad (12)$$

Under the θ coordinate transformation E_3^θ transforms as a scalar, E_I^ρ 's transform as scalar densities of weight 1, A_θ^3 transforms as a scalar density of weight 1 and A_ρ^I 's transform as scalars ¹.

For each ρ , the A_ρ^I and E_I^ρ , rotate among themselves under the $U(1)$ gauge transformations generated by the Gauss constraint. It is however possible to choose variables which are gauge invariant and will turn out to be more suitable for loop quantization (see section V). These are introduced through the following definitions:

$$E_1^x = E^x \cos\beta \quad ; \quad E_2^x = E^x \sin\beta \quad (13)$$

¹ In one dimension, under orientation preserving coordinate transformations, a tensor density of contravariant rank p , covariant rank q and weight w , can be thought of as a scalar density of weight $= w + q - p$.

$$E_1^y = -E^y \sin \bar{\beta} \quad ; \quad E_2^y = E^y \cos \bar{\beta} \quad (14)$$

$$A_x^1 = A_x \cos(\alpha + \beta) \quad ; \quad A_x^2 = A_x \sin(\alpha + \beta) \quad (15)$$

$$A_y^1 = -A_y \sin(\bar{\alpha} + \bar{\beta}) \quad ; \quad A_y^2 = A_y \cos(\bar{\alpha} + \bar{\beta}) \quad (16)$$

The angles for the connection components are introduced in a particular fashion for later convenience.

The radial coordinates, E^x, E^y, A_x, A_y , are gauge invariant and always strictly positive (vanishing radial coordinates correspond to trivial symmetry orbit which is ignored).

In terms of these variables, the symplectic structure (11) gets expressed as:

$$\Omega = \frac{4\pi^2}{\kappa\gamma} \int d\theta \left[dA_\theta^3 \wedge dE_3^\theta + dX \wedge dE^x + dY \wedge dE^y + d\beta \wedge dP^\beta + d\bar{\beta} \wedge d\bar{P}^\beta \right] \quad (17)$$

where:

$$X := A_x \cos(\alpha) \quad ; \quad Y := A_y \cos(\bar{\alpha}) \quad (18)$$

$$P^\beta := -E^x A_x \sin(\alpha) \quad ; \quad \bar{P}^\beta := -E^y A_y \sin(\bar{\alpha}) \quad (19)$$

The gauge transformations generated by the Gauss constraint shift $\beta, \bar{\beta}$ rendering α and $\bar{\alpha}$ gauge invariant. *From now on we will absorb the $4\pi^2$ and use $\kappa' := \frac{\kappa}{4\pi^2} = \frac{2G_{\text{Newton}}}{\pi}$.*

It is convenient to make a further canonical transformation:

$$\xi = \beta - \bar{\beta} \quad ; \quad \eta = \beta + \bar{\beta} \quad (20)$$

$$P^\xi = \frac{P^\beta - \bar{P}^\beta}{2} \quad ; \quad P^\eta = \frac{P^\beta + \bar{P}^\beta}{2} \quad (21)$$

In terms of these variables the Gauss and the diffeomorphism constraints can be written as:

$$G = \frac{1}{\kappa\gamma} \left[\partial_\theta E_3^\theta + 2P^\eta \right] \quad (22)$$

$$C = \frac{1}{\kappa\gamma} \left[(\partial_\theta X) E^x + (\partial_\theta Y) E^y - (\partial_\theta E_3^\theta) A_\theta^3 + (\partial_\theta \eta) P^\eta + (\partial_\theta \xi) P^\xi \right] \quad (23)$$

The Hamiltonian constraint is complicated but after putting $K_i^a = (A_i^a - \Gamma_i^a)/\gamma$, substituting the explicit expressions of Γ_i^a , and further simplification, turns out to be:

$$\begin{aligned} H = & - \frac{\gamma^{-2}}{2\kappa} \frac{1}{\sqrt{E}} \left[E_3^\theta \left\{ (X E^x + Y E^y) \partial_\theta \eta + (X E^x - Y E^y) \partial_\theta \xi - 2P^\xi \partial_\theta \left(\ln \frac{E^y}{E^x} \right) \right. \right. \\ & \left. \left. + 2P^\eta (\partial_\theta \ln E_3^\theta + (\tan \xi) \partial_\theta \xi) \right\} + 2 \left\{ (\cos^2 \xi) (X E^x Y E^y + (P^\eta)^2 - (P^\xi)^2) \right. \right. \end{aligned}$$

$$\begin{aligned}
& + (XE^x + YE^y)E_3^\theta A_\theta^3 \Big\} + (\sin 2\xi) \Big\{ (XE^x + YE^y)P^\xi - (XE^x - YE^y)P^\eta \Big\} \\
& + \left(\frac{1 + \gamma^2}{2} \right) \left\{ (\partial_\theta E_3^\theta)^2 - \left(\frac{E_3^\theta \partial_\theta \xi}{\cos \xi} \right)^2 - \left(\frac{E_3^\theta \partial_\theta (\ln(E^y/E^x))}{(\cos \xi)} \right)^2 \right\} \Big] \\
& - \frac{1}{2\kappa} \partial_\theta \left(\frac{4E_3^\theta P^\eta}{\sqrt{E}} \right) \tag{24}
\end{aligned}$$

where $E = |E_3^\theta E^x E^y (\cos \xi)|$.

Under the action of the diffeomorphism constraint X, Y, E_3^θ, η and ξ transforms as scalars while $E^x, E^y, A_\theta^3, P^\eta$ and P^ξ transform as scalar densities of weight 1.

This completes the description of the *unpolarised* Gowdy T^3 Model in the variables we have defined. The number of canonical field variables is 10 while there is a 3-fold infinity of first class constraints. There are therefore 2 field degrees of freedom. We now need to impose two second class constraints such that the number of field degrees of freedom are reduced from two to one (as it should be in the polarized case).

B. Reduction to Polarized model

The spatial 3-metric, $g_{ab} := e_a^i e_b^j \delta_{ij}$, with the co-triad e_a^i defined through $e_a^i E_j^a := \delta_j^i \sqrt{E}$, $e_a^i E_i^b := \delta_a^b \sqrt{E}$, is given by,

$$ds^2 = \cos \xi \frac{E^x E^y}{E_3^\theta} d\theta^2 + \frac{E_3^\theta}{\cos \xi} \frac{E^y}{E^x} dx^2 + \frac{E_3^\theta}{\cos \xi} \frac{E^x}{E^y} dy^2 - 2 \frac{E_3^\theta}{\cos \xi} \sin \xi \, dx dy \tag{25}$$

For the Killing vectors $\partial/\partial x$ and $\partial/\partial y$ to be orthogonal to each other, the $dx dy$ term in the metric should be zero. This implies that the polarization condition is implemented by restricting to $\xi = 0$ sub-manifold of the phase space of the unpolarized model. For getting a non-degenerate symplectic structure, one needs to have one more condition. This condition should be chosen *consistently* in the following sense.

We expect the two conditions to reduce a field degree of freedom. This can be viewed in two equivalent ways. The condition $\xi = 0$ makes the metric diagonal and this property should be preserved under evolution (i.e. the extrinsic curvature should also be diagonal). Alternatively, the unpolarized model is a constrained system and we want to impose two conditions such that one *physical* (field) degree of freedom is reduced. The extra conditions to be imposed should therefore be *first class* with respect to the constraints of the unpolarized model i.e. should weakly Poisson-commute with them.

Indeed, this can be done systematically by viewing $\xi = 0$ as a new constraint² and demanding its preservation under the evolution generated by the total Hamiltonian. Since $\xi = 0$ weakly Poisson commutes with the Gauss and the diffeomorphism constraints, only the Poisson bracket with the Hamiltonian constraint is needed.

$$\xi(\theta) \approx 0 \quad , \quad \left\{ \xi(\theta), \int d\theta' N(\theta') H(\theta') \right\} \approx 0 . \quad (26)$$

It follows that,

$$\dot{\xi}(\theta) \approx 0 \quad \Rightarrow \quad \chi(\theta) := 2P^\xi + E_3^\theta \partial_\theta (\ln E^y / E^x) \approx 0 \quad (27)$$

The Poisson Bracket of χ with the Hamiltonian turns out to be zero on the constraint surface i.e. $\dot{\chi} \approx \chi \approx 0$. Thus, the reduction to the Polarized model is obtained by imposing the two *polarization constraints*

$$\xi \approx 0 \quad ; \quad \chi \approx 0 \quad ; \quad \left\{ \xi(\theta), \chi(\theta') \right\} = 2\kappa\gamma\delta(\theta - \theta') \quad (28)$$

Remark: To see that the $\chi \approx 0$ condition follows from preservation of $g_{xy} = 0$, note that, in the metric formulation, for the present case, it implies that $\dot{g}_{xy} \sim K_{xy} = K_x^i e_y^i (= e_x^i K_y^i) = 0$. Using the definition $K_a^i = \gamma^{-1}(A_a^i - \Gamma_a^i)$ and the expressions given in the appendix, one can check directly that $K_{xy} = 0 \Leftrightarrow \chi = 0$. Note that this is *not* equivalent to requiring orthogonality of components of the connection, $A_x^i A_y^i = 0$ which would imply $\alpha = \bar{\alpha}$ (see eqns. (15 and 16)). This condition, mentioned in the literature [15, 16], is very different from the $\chi \approx 0$ condition and is not preserved under evolution.

It follows from (22) that $\{\chi, G\} = 0$ And using (23), one can see that :

$$\left\{ \xi, \int N^\theta C_\theta \right\} = N^\theta \partial_\theta \xi \approx 0 \quad , \quad \left\{ \chi, \int N^\theta C_\theta \right\} = \partial_\theta (N^\theta \chi) \approx 0 . \quad (29)$$

We can solve the polarization constraints strongly and use Dirac brackets. Symbolically,

$$\{f, g\}^* = \{f, g\} - \{f, \xi\} \odot \{\xi, \chi\}^{-1} \odot \{\chi, g\} - \{f, \chi\} \odot \{\chi, \xi\}^{-1} \odot \{\xi, g\}$$

Here \odot denotes appropriate integrations since we have field degrees of freedom.

Since the polarization constraints weakly commute with all the other constraints, the constraint algebra in terms of Dirac brackets is same as that in terms of the Poisson brackets

² The choice $\xi = 0$ also requires $E_3^\theta > 0$ for the spatial metric to have signature $(+, +, +)$. The choice $\xi = \pi$ would require $E_3^\theta < 0$. From now on $E_3^\theta > 0$ will be assumed.

and thus remains unaffected. Furthermore, equations of motions for all the variables other than ξ, P_ξ also remain unaffected. We can thus set the polarization constraints strongly equal to zero in all the expressions and continue to use the original Poisson brackets.

The expressions of the constraints simplify greatly and in particular the Hamiltonian constraint simplifies to,

$$H = -\frac{\gamma^{-2}}{2\kappa} \frac{1}{\sqrt{E}} \left[\frac{(\kappa\gamma G)^2}{2} + (XE^x + YE^y)E_3^\theta \partial_\theta \eta + 2 \left\{ XE^x Y E^y + (XE^x + YE^y)E_3^\theta A_\theta^3 \right\} \right. \\ \left. + \frac{\gamma^2}{2} \left\{ (\partial_\theta E_3^\theta)^2 - (E_3^\theta \partial_\theta \ln(E^y/E^x))^2 \right\} \right] + \frac{1}{2\kappa} \partial_\theta \left\{ \frac{2E_3^\theta (\partial_\theta E_3^\theta - \kappa\gamma G)}{\sqrt{E}} \right\} \quad (30)$$

where, we have also eliminated P_η in terms of the Gauss constraint using $2P^\eta = (\kappa\gamma G - \partial_\theta E_3^\theta)$ and $E = |E_3^\theta| E^x E^y$.

Noting that η is translated under a gauge transformation, we can set it to any constant and fix the gauge transformation freedom. Explicitly, imposing $\eta \approx 0$ as a constraint, we can fix the λ^3 from preservation of this gauge fixing condition. Once again we can use Dirac brackets with respect to the Gauss constraint and the $\eta \approx 0$ constraint and impose these constraints strongly. With this done, the first two terms and the G dependent piece in the last term in the Hamiltonian, drop out and so do the degrees of freedom η, P_η . We are left with six canonical degrees of freedom and the two first class constraints, leaving one field degree of freedom. Thus our final variables and constraints for the *polarized* Gowdy model are (absorbing away the Immirzi parameter):

$$\kappa := \frac{8\pi G_{\text{Newton}}}{4\pi^2}, \quad \mathcal{E} := E_3^\theta, \quad \mathcal{A} := \gamma^{-1} A_\theta^3, \quad K_x := \gamma^{-1} X, \quad K_y := \gamma^{-1} Y \quad (31)$$

$$\{K_x(\theta), E^x(\theta')\} = \kappa \delta(\theta - \theta') \quad , \quad (\text{and similarly for } (K_y, E^y), (\mathcal{A}, \mathcal{E}) \text{ pairs}); \quad (32)$$

$$C = \frac{1}{\kappa} [(\partial_\theta K_x)E^x + (\partial_\theta K_y)E^y - (\partial_\theta \mathcal{E})\mathcal{A}] \quad (33)$$

$$H = \frac{1}{\kappa} \left[-\frac{1}{\sqrt{E}} \left\{ (K_x E^x K_y E^y) + (K_x E^x + K_y E^y) \mathcal{E} \mathcal{A} \right\} \right. \\ \left. - \frac{1}{4\sqrt{E}} \left\{ (\partial_\theta \mathcal{E})^2 - (\mathcal{E} \partial_\theta (\ln(E^y/E^x)))^2 \right\} + \partial_\theta \left(\frac{\mathcal{E} \partial_\theta \mathcal{E}}{\sqrt{E}} \right) \right] \quad (34)$$

The constraint algebra among the $C[N^\theta], H[N]$ is:

$$\{C[N^\theta], C[M^\theta]\} = C [N^\theta \partial_\theta M^\theta - M^\theta \partial_\theta N^\theta] \quad (35)$$

$$\{C[N^\theta], H[N]\} = H [N^\theta \partial_\theta N] \quad (36)$$

$$\{H[M], H[N]\} = C [(M \partial_\theta N - N \partial_\theta M) \mathcal{E}^2 E^{-1}] \quad (37)$$

Since each term in the Hamiltonian constraint is a scalar density of weight +1 and each term in the diffeomorphism constraint is of density weight +2, the first two brackets are easily verified. The last one also follows with a bit longer computation. We have thus verified the constraint algebra of polarized model showing the consistency of the reduction procedure.

IV. SPACE-TIME CONSTRUCTION

The next task is to find the set of gauge *inequivalent* solutions of the Hamilton's equations of motion, satisfying the two sets of constraints and obtain the space-time interpretation. The total Hamiltonian being a constraint, the Lagrange multipliers – the lapse function and the shift vector – also enter in the Hamilton's equations of motion. These need to be either *prescribed* or deduced via a gauge-fixing procedure. Once this is done, one can obtain the solution curves in the phase space with “initial points” lying on the constrained surface. The space-time metric, solving Einstein equation is then given by,

$$ds^2 = -N^2(t, x^i)dt^2 + g_{ij}(t, x^i) (dx^i - N^i(t, x^i)dt) (dx^j - N^j(t, x^i)dt) \quad (38)$$

For our case, the metric is diagonal, $(x^1, x^2, x^3) \leftrightarrow (\theta, x, y)$, $N^i \leftrightarrow (N^\theta, 0, 0)$ and the metric is independent of the coordinates (x, y) . The $t = \text{constant}$, hyper-surfaces are diffeomorphic to the 3-torus. The metric components are given by: $g_{\theta\theta} = E^x E^y \mathcal{E}^{-1} = E \mathcal{E}^{-2}$, $g_{xx} = \mathcal{E} E^y / E^x$, $g_{yy} = \mathcal{E} E^x / E^y$ (eq. (25)). The Gowdy form of the metric (1) is realized if one *prescribes* $N^\theta = 0$ and $N^2 = g_{\theta\theta}$.

Such a prescription is eminently consistent since any metric on a two dimensional manifold (coordinatized by (t, θ)), can always be (locally) chosen to be conformally flat. This however does *not* fix the coordinates t, θ completely – one can still make the conformal diffeomorphisms: $t \rightarrow t' = t + \xi^t(t, \theta)$, $\theta \rightarrow \theta' = \theta + \xi^\theta(t, \theta)$ with ξ satisfying the conformal Killing equations: $\partial_t \xi^t - \partial_\theta \xi^\theta = 0 = \partial_\theta \xi^t - \partial_t \xi^\theta$.

We will first take the above prescription for the lapse and the shift, obtain the Hamilton's equations of motion, use the freedom of conformal diffeomorphisms and reduce the equations to those given in section II. Subsequently, we will also exhibit gauge fixing functions to arrive at the same result. This will complete the identification of inequivalent solutions of the Einstein equation.

With the choices $N^\theta = 0$, $N = \sqrt{E}\mathcal{E}^{-1}$, the space-time metric (38) is

$$ds^2 = E\mathcal{E}^{-2}(-dt^2 + d\theta^2) + \mathcal{E} \left(\frac{E^y}{E^x} dx^2 + \frac{E^x}{E^y} dy^2 \right), \quad (39)$$

and the time evolution is governed by the Hamiltonian alone which is given by,

$$H[\mathcal{E}^{-1}\sqrt{E}] = \frac{1}{\kappa} \int d\theta \left[-\frac{1}{\mathcal{E}} \left\{ (K_x E^x K_y E^y) + (K_x E^x + K_y E^y) \mathcal{E} \mathcal{A} \right\} - \frac{1}{4\mathcal{E}} \left\{ (\partial_\theta \mathcal{E})^2 - (\mathcal{E} \partial_\theta (\ln(E^y/E^x)))^2 \right\} + \frac{\sqrt{E}}{\mathcal{E}} \partial_\theta \left(\frac{\mathcal{E} \partial_\theta \mathcal{E}}{\sqrt{E}} \right) \right] \quad (40)$$

In anticipation let us define $2W := \ln(E^y/E^x)$ and $2a := \ln(E^x E^y/\mathcal{E})$. One obtains,

$$\frac{\dot{E}^x}{E^x} = \mathcal{E}^{-1}(K_y E^y + \mathcal{A}\mathcal{E}) \quad , \quad \frac{\dot{E}^y}{E^y} = \mathcal{E}^{-1}(K_x E^x + \mathcal{A}\mathcal{E}) \quad , \quad \dot{\mathcal{E}} = (K_x E^x + K_y E^y) \quad (41)$$

$$2\partial_t W := \partial_t \ln \frac{E^y}{E^x} = \frac{(K_x E^x - K_y E^y)}{\mathcal{E}} \quad (42)$$

$$2\partial_t a := \partial_t \ln \frac{E^x E^y}{\mathcal{E}} = 2\mathcal{A} \quad (43)$$

The Poisson brackets of $K_x E^x$, $K_y E^y$ with the Hamiltonian are given by,

$$\{K_x E^x, H[\mathcal{E}^{-1}\sqrt{E}]\} = \frac{1}{2} \partial_\theta \left(\mathcal{E} \partial_\theta \ln \frac{E^y}{E^x} \right) + \frac{1}{2} \partial_\theta^2 \mathcal{E} \quad (44)$$

$$\{K_y E^y, H[\mathcal{E}^{-1}\sqrt{E}]\} = -\frac{1}{2} \partial_\theta \left(\mathcal{E} \partial_\theta \ln \frac{E^y}{E^x} \right) + \frac{1}{2} \partial_\theta^2 \mathcal{E} \quad (45)$$

$$\{K_x E^x - K_y E^y, H[\mathcal{E}^{-1}\sqrt{E}]\} = \partial_\theta \left(\mathcal{E} \partial_\theta \ln \frac{E^y}{E^x} \right) \quad (46)$$

$$\{K_x E^x + K_y E^y, H[\mathcal{E}^{-1}\sqrt{E}]\} = \partial_\theta^2 \mathcal{E} \quad (47)$$

From these, we get second order equations for \mathcal{E} , W as,

$$\partial_t^2 \mathcal{E} = \partial_\theta^2 \mathcal{E} \quad (48)$$

$$\partial_t^2 W = \frac{1}{\mathcal{E}} \partial_\theta (\mathcal{E} \partial_\theta W) - \left(\frac{1}{\mathcal{E}} \partial_t \mathcal{E} \right) \partial_t W \quad (49)$$

The equation for \mathcal{E} is a simple wave equation and given a solution of this, the equation for W can be solved determining W or the ratio E^y/E^x . From the first order equations, one determines the $K_x E^x \pm K_y E^y$ as well. The Hamiltonian constraint then determines \mathcal{A} in terms of known quantities, \mathcal{E} , W and the θ -derivatives of a . Using the equation $\partial_t a = \mathcal{A}$, one obtains,

$$\partial_t a = \mathcal{A} = -\frac{1}{4} \frac{\partial_t \mathcal{E}}{\mathcal{E}} + \frac{\mathcal{E}}{\partial_t \mathcal{E}} \left((\partial_t W)^2 + (\partial_\theta W)^2 \right) - \frac{\partial_\theta \mathcal{E}}{\partial_t \mathcal{E}} \partial_\theta a - \frac{1}{4} \frac{(\partial_\theta \mathcal{E})^2}{\mathcal{E} \partial_t \mathcal{E}} + \frac{\partial_\theta^2 \mathcal{E}}{\partial_t \mathcal{E}} \quad (50)$$

One can also obtain, by direct computation and using the diffeomorphism constraint,

$$\partial_\theta a = \frac{\mathcal{E}}{\partial_t \mathcal{E}} \left[2\partial_t W \partial_\theta W - \frac{\partial_\theta \mathcal{E}}{\mathcal{E}} \partial_t a + \frac{\partial_{t\theta}^2 \mathcal{E}}{\mathcal{E}} - \frac{\partial_t \mathcal{E} \partial_\theta \mathcal{E}}{2\mathcal{E}^2} \right] \quad (51)$$

From these two equations, one can obtain $\partial_\theta a, \partial_t a$ in terms of \mathcal{E} and W which can be integrated. Thus the metric can be completely determined starting from a solution for \mathcal{E} .

However, all these solutions are not gauge inequivalent corresponding to the fact that the coordinates can still be subjected to conformal diffeomorphisms. Under these coordinate transformations, \mathcal{E} which is the determinant of the metric on the symmetry torus, is a scalar. Under conformal diffeomorphisms, the wave operator gets scaled by a prefactor. Hence, under the transformations generated by conformal Killing vectors, solutions of the wave equation transform among themselves. In fact the conformal Killing vectors also satisfy the wave equation and on the cylinder (t, θ) , both \mathcal{E} and ξ satisfy the same boundary conditions. Thus, their general solutions are linear combinations of $\exp\{in(t \pm \theta)\}, n \neq 0$ and a solution of the form $A + Bt$. The Killing vectors however satisfy first order coupled equations. This removes the θ -independent, linear in t piece from the general solution. Consequently, one can use conformal diffeomorphisms to remove the θ -dependence from the solutions for \mathcal{E} as well as the constant piece. In other words, all solutions for \mathcal{E} , except $\mathcal{E} = \#t$ are related to each other by conformal diffeomorphisms. The gauge inequivalent solutions are thus obtained from the choice $\mathcal{E} = t$. Equivalently, one has finally fixed the (t, θ) coordinates completely. The time coordinate so fixed will be denoted by T .

With this choice, $\mathcal{E} = T$, the constraints also simplify to,

$$0 = \mathcal{E}(\partial_\theta W)^2 - \mathcal{E}^{-1} \{K_x E^x K_y E^y + (K_x E^x + K_y E^y) \mathcal{A} \mathcal{E}\} \quad (52)$$

$$0 = E^x \partial_\theta K_x + E^y \partial_\theta K_y, \quad (53)$$

one gets $K_x E^x + K_y E^y = 1$ and the equations (49, 50, 51) go over to the equations (2, 3). From these one recovers the usual solutions listed in section II.

Remark: Up to the derivation of the equations for \mathcal{E} and W , the constraints are not used. The T^3 topology has also not been used! Thus these expressions are also valid for polarized versions of Gowdy models with other topologies. In Gowdy's original analysis, the three allowed topologies are distinguished by different choices of solutions of the equation for \mathcal{E} (R , the determinant of the two metric on the T^2 orbits, in Gowdy's notation). The different topologies get distinguished by the boundary conditions on \mathcal{E} and on the conformal

Killing vectors. For non- T^3 topologies, $\theta \in [0, \pi]$ and \mathcal{E}, ξ^θ have to vanish at the end-points. With these taken into account, the gauge inequivalent solutions are obtained by choosing $\mathcal{E} = \sin(t) \sin(\theta)$ [1].

We reproduced the known results by obtaining the solutions of the Hamilton's equations with chosen lapse and shift, motivated by comparison with the space-time form of Gowdy model, *and* invoking the 'residual' freedom in the space-time coordinates to obtain the gauge inequivalent solutions. Thus we used the canonical structure as well as the anticipated form of space-time geometry to arrive at the distinct solutions. We would like to see if the same result can also be derived by using only the phase space view.

Within a phase space view, the lapse and the shift are to be determined by doing an explicit gauge fixing. To do this, we will keep the lapse and the shift as unspecified and look at the evolution generated by the *total* Hamiltonian,

$$\begin{aligned}
H_{\text{tot}}[N^\theta, N] = & \frac{1}{\kappa} \int d\theta N^\theta \{E^x \partial_\theta K_x + E^y \partial_\theta K_y - \mathcal{A} \partial_\theta \mathcal{E}\} \\
& + \frac{1}{\kappa} \int d\theta N \left[-\frac{1}{\sqrt{E}} \left\{ (K_x E^x K_y E^y) + (K_x E^x + K_y E^y) \mathcal{E} \mathcal{A} \right\} \right. \\
& \quad \left. - \frac{1}{4\sqrt{E}} \left\{ (\partial_\theta \mathcal{E})^2 - (\mathcal{E} \partial_\theta (\ln(E^y/E^x)))^2 \right\} + \partial_\theta \left(\frac{\mathcal{E} \partial_\theta \mathcal{E}}{\sqrt{E}} \right) \right] \quad (54)
\end{aligned}$$

Denoting by over-dots, the Poisson brackets with the total Hamiltonian, it is straight forward to see,

$$\frac{\dot{E}^x}{E^x} = \frac{N}{\sqrt{E}} (K_y E^y + \mathcal{A} \mathcal{E}) + \frac{\partial_\theta (N^\theta E^x)}{E^x} \quad (55)$$

$$\frac{\dot{E}^y}{E^y} = \frac{N}{\sqrt{E}} (K_x E^x + \mathcal{A} \mathcal{E}) + \frac{\partial_\theta (N^\theta E^y)}{E^y} \quad (56)$$

$$\frac{\dot{\mathcal{E}}}{\mathcal{E}} = \frac{N}{\sqrt{E}} (K_x E^x + K_y E^y) + \frac{N^\theta \partial_\theta \mathcal{E}}{\mathcal{E}} \quad (57)$$

$$(K_x \dot{E}^x) = \frac{1}{2} \partial_\theta \left\{ \frac{N \mathcal{E}}{\sqrt{E}} \left(\mathcal{E} \partial_\theta \ln \frac{E^y}{E^x} + \partial_\theta \mathcal{E} \right) \right\} + \partial_\theta (N^\theta K_x E^x) \quad (58)$$

$$(K_y \dot{E}^y) = \frac{1}{2} \partial_\theta \left\{ \frac{N \mathcal{E}}{\sqrt{E}} \left(-\mathcal{E} \partial_\theta \ln \frac{E^y}{E^x} + \partial_\theta \mathcal{E} \right) \right\} + \partial_\theta (N^\theta K_y E^y) \quad (59)$$

The following combinations are convenient for looking at gauge fixing.

$$(K_x E^x + K_y E^y) \dot{} = \partial_\theta \left\{ \frac{N \mathcal{E}}{\sqrt{E}} \partial_\theta \mathcal{E} \right\} + \partial_\theta \{ N^\theta (K_x E^x + K_y E^y) \} \quad (60)$$

$$\dot{\mathcal{E}} = \frac{N \mathcal{E}}{\sqrt{E}} (K_x E^x + K_y E^y) + N^\theta \partial_\theta \mathcal{E} \quad (61)$$

$$(K_x E^x - K_y E^y) = \partial_\theta \left\{ \frac{N\mathcal{E}}{\sqrt{E}} \mathcal{E} \partial_\theta \ln \frac{E^y}{E^x} \right\} + \partial_\theta \{ N^\theta (K_x E^x - K_y E^y) \} \quad (62)$$

$$\left(\ln \frac{E^y}{E^x} \right) = \frac{N}{\sqrt{E}} (K_x E^x - K_y E^y) + N^\theta \partial_\theta \ln \frac{E^y}{E^x} \quad (63)$$

The first two equations above show that we can consistently impose $K_x E^x + K_y E^y = C_1$, a constant, and $\partial_\theta \mathcal{E} = 0$ as two gauge fixing conditions. Preservation of the first leads to $N^\theta = f(t)$ while that of the second leads to $N\mathcal{E}/\sqrt{E} = g(t)$. Since $\partial_\theta \mathcal{E} = 0$ already requires \mathcal{E} to be a function of t alone, we can strengthen the gauge fixing condition by specifying $\mathcal{E} = t$. This determines $N = C_1 \sqrt{E} \mathcal{E}^{-1}$. Evidently, we must have a non-zero lapse and therefore $C_1 \neq 0$ must be chosen. The sign of C_1 will determine if \mathcal{E} increases or decreases with t and by convention we can take the sign to be positive and without any loss of generality, we choose $C_1 = +1$ and denote the t by T as before.

The shift is however determined to be a function of T alone. With such a shift, $C[N^\theta] = f(T) \int C$ generates T -dependent *translations* of the θ -coordinate. All tensor densities on the spatial slice, transform as scalars under these translations, and there is no way to fix the left over constraint $\int C$, by any gauge fixing condition. However, we can always redefine the θ -coordinate such that $d\theta - f(t)dt =: d\theta'$. This means that solutions inequivalent with respect to translations, can be determined by effectively choosing shift = 0. Incidentally, for other admissible topologies, the shift has to vanish at $\theta = 0, \pi$ and hence $f(t) = 0$ is the only admissible solution. We have thus achieved our goal of determining the *same* lapse and shift, by explicit gauge fixing. The inequivalent solutions are then obtained as in section II.

One can make the physical degrees of freedom explicit by noting that $2W = \ln(E^y/E^x)$ and $\pi_W := K_x E^x - K_y E^y$ are canonically conjugate. Similarly, $2\bar{a} := -\ln(E^x E^y)$ and $\pi_{\bar{a}} := K_x E^x + K_y E^y$ are also conjugate variables. The gauge fixing conditions are: $\mathcal{E} = T$, $\pi_{\bar{a}} = 1$ while the gauge-fixed form of constraints become:

$$C = \frac{1}{\kappa} [\pi_W \partial_\theta W + \partial_\theta \bar{a}] \quad (64)$$

$$(T^{-1} \sqrt{E}) H = \frac{1}{\kappa} \left[-\frac{1 - \pi_W^2}{4T} - \mathcal{A} + T(\partial_\theta W)^2 \right] \quad (65)$$

The Hamiltonian constraint determines \mathcal{A} completely in terms of W, π_W while the diffeomorphism constraints determines the \bar{a} *except* for the homogeneous (θ -independent) part. The periodicity of \bar{a} also requires the $\int \pi_W \partial_\theta W = 0$ which is a constraint on the W, π_W . The physical degrees of freedom are thus described by W, π_W together with one constraint

and the homogeneous pieces of $\bar{a}, \pi_{\bar{a}}$. Our gauge fixing has fixed the homogeneous part of $\pi_{\bar{a}}$ to be 1. These are of course the well known results [11].

Observe that in the homogeneous limit (all variables independent of θ), one gets the Bianchi I model. The Hamiltonian constraint, for each θ looks like a Bianchi model with a potential and is highly suggestive of the BKL scenario and has been explored numerically as well [17].

This completes the canonical formulation of the polarized Gowdy model on T^3 in terms of the real connection variables.

V. DISCUSSION

In this paper, two main reformulations of the polarized Gowdy model in real connection variables have been done. First is the choice of the gauge invariant variables: $A_x, A_y, E^x, E^y, \alpha, \bar{a}$ and the subsequent canonical transformation to the variables X, Y, P^ξ, P^η . This has already been done in the case of spherical symmetry and also mentioned for cylindrical waves in [15]. The main advantages of these variables are that the volume becomes a functional of the momenta variables alone and the components of the connection along the homogeneous directions are separated neatly and gauge invariantly, into extrinsic curvature components (X, Y) and the spin-connection components (Γ_x, Γ_y) . In the quantum theory, both the features allow a simpler choice of edge and point holonomies, simpler form for the volume operator and also a more tractable form of the Hamiltonian constraint [18].

The second aspect, obtains the polarized model from the unpolarized by a simple systematic reduction (Dirac procedure) ensuring a consistent reduction at the level of *physical degrees of freedom*. Getting this reduction consistently is important since the form of the reduced constraints, depend on the reducing conditions. In contrast to the second polarization condition mentioned in the literature, namely orthogonality of the connection components in analogy with that of the triad components, our $\chi \approx 0$ condition, (27) is consistent with dynamics. The consistency is seen in three ways: from a systematic derivation, verifying the constraint algebra of the reduced constraints and finally reproducing the known spacetimes, obtained by directly solving the Einstein equations for polarized ansatz. We are thus confident to use these constraint expressions in the passage to quantization.

We would also like to note that in the reduction to polarized model, we had two options: $\xi = 0$ ($E_3^\theta > 0$) or $\xi = \pi$ ($E_3^\theta < 0$). In the metric variables and classically, either one of these suffices. (In the triad variables, these two correspond to opposite orientations.) The subsequent gauge fixing was also naturally restricted to one of these choices (we chose the former). At this stage, one could imagine doing a “loop quantization” of the gauge-fixed model which now has a true Hamiltonian and explore the fate of the singularity. In a quantum theory however, one could have an extension across the degenerate triad and this will be missed in a quantization of the gauge-fixed model.

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APPENDIX

In this appendix, we collect some of the useful expressions to help reproduce the computations in the main text.

The symmetry reduction leaves only the following non-zero components of the gravitational connection and the densitized triad:

$$\begin{aligned} A_a^i &\longrightarrow A_\theta^3, A_\rho^I, \quad I = 1, 2 \quad \rho = x, y, \\ E_i^a &\longrightarrow E_3^\theta, E_I^\rho, \quad I = 1, 2 \quad \rho = x, y. \end{aligned} \quad (\text{A.1})$$

The triad has the components: $e_i^a = E_i^a E^{-1/2}$ with $E := \det E_i^a = E_3^\theta \Delta$, $\Delta := E_1^x E_2^y - E_2^x E_1^y$. The co-triad (inverse triad), e_a^i , has the components: $e_\theta^3 = \sqrt{\Delta/E_3^\theta}$, $e_x^1 = \sqrt{E_3^\theta/\Delta} E_2^y$, $e_y^1 = -\sqrt{E_3^\theta/\Delta} E_2^x$, $e_x^2 = -\sqrt{E_3^\theta/\Delta} E_1^y$, $e_y^2 = \sqrt{E_3^\theta/\Delta} E_1^x$.

The spin connection is defined by,

$$\Gamma_a^i := -\epsilon^{ijk} e_j^b \left(\partial_{[a} e_{b]}^k + \frac{1}{2} e_k^c e_a^l \partial_{[c} e_{b]}^l \right). \quad (\text{A.2})$$

Of these, $\Gamma_\rho^3 = 0 = \Gamma_\theta^I$ are identically zero. The remaining components are given by,

$$\begin{aligned} \Gamma_\theta^3 &= \frac{1}{2\Delta} (E_1^x \partial_\theta E_1^y - E_1^y \partial_\theta E_1^x + E_2^x \partial_\theta E_2^y - E_2^y \partial_\theta E_2^x) \\ \Gamma_x^1 &= \frac{1}{2} \sqrt{\frac{E_3^\theta}{\Delta}} \left[\partial_\theta \left(\sqrt{\frac{E_3^\theta}{\Delta}} E_1^y \right) + \frac{E_3^\theta}{\Delta} \vec{E}^y \cdot \vec{E}^y \partial_\theta \left(\frac{E_2^x}{\sqrt{E}} \right) - \frac{E_3^\theta}{\Delta} \vec{E}^x \cdot \vec{E}^y \partial_\theta \left(\frac{E_2^y}{\sqrt{E}} \right) \right] \end{aligned}$$

$$\begin{aligned}
\Gamma_x^2 &= \frac{1}{2} \sqrt{\frac{E_3^\theta}{\Delta}} \left[\partial_\theta \left(\sqrt{\frac{E_3^\theta}{\Delta}} E_2^y \right) - \frac{E_3^\theta}{\Delta} \vec{E}^y \cdot \vec{E}^y \partial_\theta \left(\frac{E_1^x}{\sqrt{E}} \right) + \frac{E_3^\theta}{\Delta} \vec{E}^x \cdot \vec{E}^y \partial_\theta \left(\frac{E_1^y}{\sqrt{E}} \right) \right] \quad (\text{A.3}) \\
\Gamma_y^1 &= \frac{1}{2} \sqrt{\frac{E_3^\theta}{\Delta}} \left[-\partial_\theta \left(\sqrt{\frac{E_3^\theta}{\Delta}} E_1^x \right) - \frac{E_3^\theta}{\Delta} \vec{E}^x \cdot \vec{E}^y \partial_\theta \left(\frac{E_2^x}{\sqrt{E}} \right) + \frac{E_3^\theta}{\Delta} \vec{E}^x \cdot \vec{E}^x \partial_\theta \left(\frac{E_2^y}{\sqrt{E}} \right) \right] \\
\Gamma_y^2 &= \frac{1}{2} \sqrt{\frac{E_3^\theta}{\Delta}} \left[-\partial_\theta \left(\sqrt{\frac{E_3^\theta}{\Delta}} E_2^x \right) + \frac{E_3^\theta}{\Delta} \vec{E}^x \cdot \vec{E}^y \partial_\theta \left(\frac{E_1^x}{\sqrt{E}} \right) - \frac{E_3^\theta}{\Delta} \vec{E}^x \cdot \vec{E}^x \partial_\theta \left(\frac{E_1^y}{\sqrt{E}} \right) \right]
\end{aligned}$$

where, $\vec{E}^x \cdot \vec{E}^y := E_1^x E_1^y + E_2^x E_2^y$ etc.

In terms of the radial and angular variables $E^x, E^y, E_3^\theta (= \mathcal{E}), \xi, \eta$ given in equations (13, 14, 20), one has $\Delta = E^x E^y \cos \xi$, $\vec{E}^x \cdot \vec{E}^x = (E^x)^2$, $\vec{E}^y \cdot \vec{E}^y = (E^y)^2$, $\vec{E}^x \cdot \vec{E}^y = E^x E^y \sin \xi$.

In the computation of the Hamiltonian constraint, one needs the combinations: $E_i^a \Gamma_b^i$ and $E_i^a A_b^i$. The non-zero ones are given by,

$$\begin{aligned}
E_i^\theta \Gamma_\theta^i &= \frac{1}{2} E_3^\theta \left[\tan \xi \partial_\theta \left(\ln \frac{E^y}{E^x} \right) - \partial_\theta \eta \right] \\
E_x^x \Gamma_x^i &= \frac{1}{2} \partial_\theta (E_3^\theta \tan \xi) \quad , \quad E_y^y \Gamma_y^i = -\frac{1}{2} \partial_\theta (E_3^\theta \tan \xi) \\
E_x^x \Gamma_y^i &= -\frac{1}{2} \partial_\theta \left(\frac{E^x}{E^y} \frac{E_3^\theta}{\cos \xi} \right) \quad , \quad E_y^y \Gamma_x^i = \frac{1}{2} \partial_\theta \left(\frac{E^y}{E^x} \frac{E_3^\theta}{\cos \xi} \right) \quad (\text{A.4}) \\
E_x^x A_x^i &= K_x E^x \quad , \quad E_y^y A_y^i = K_y E^y \quad , \quad E_\theta^\theta A_\theta^i = E_3^\theta A_\theta^3 \quad , \\
E_x^x A_y^i &= \left(\frac{E^y}{E^x} \right) \{ -(P^\xi + P^\eta) \cos \xi + K_x E^x \sin \xi \} \\
E_y^y A_x^i &= \left(\frac{E^x}{E^y} \right) \{ (P^\eta - P^\xi) \cos \xi + K_y E^y \sin \xi \} \quad (\text{A.5})
\end{aligned}$$

The Hamiltonian constraint in eqn (10) is simplified by eliminating the extrinsic curvature in terms of the gravitational connection and the spin-connection, $K_a^i := \gamma^{-1}(A_a^i - \Gamma_a^i)$, and using the above equations. One begins with the expression:

$$H = \frac{1}{2\kappa} \frac{1}{\sqrt{E}} \left[\epsilon_{ijk} E_i^a E_j^b (\partial_a A_b^k - \partial_b A_a^k) + \epsilon_{ijk} \epsilon^k{}_{jk} E_i^a E_j^b A_a^m A_b^n \right. \quad (\text{A.6}) \\
\left. - (1 + \gamma^{-2}) \{ E_i^a (A_a^i - \Gamma_a^i) E_j^b (A_b^j - \Gamma_b^j) - E_i^a (A_a^j - \Gamma_a^j) E_j^b (A_b^i - \Gamma_b^i) \} \right]$$

$$\begin{aligned}
&= \frac{1}{2\kappa} \frac{1}{\sqrt{E}} \left[\{ \epsilon_{ijk} E_i^a E_j^b (\partial_a A_b^k - \partial_b A_a^k) \} \quad (\text{A.7}) \right. \\
&\quad - \gamma^{-2} \{ (E_i^a A_a^i) (E_j^b A_b^j) - (E_i^a A_b^i) (E_j^b A_a^j) \} \\
&\quad - (1 + \gamma^{-2}) \{ (E_i^a \Gamma_a^i) (E_j^b \Gamma_b^j) - (E_i^a \Gamma_b^i) (E_j^b \Gamma_a^j) \} \\
&\quad \left. - 2(1 + \gamma^{-2}) \{ -(E_i^a A_a^i) (E_j^b \Gamma_b^j) + (E_i^a \Gamma_b^i) (E_j^b A_a^j) \} \right]
\end{aligned}$$

The terms quadratic in A 's combine to get γ^{-2} , while terms linear in A and Γ get a prefactor of $2(1 + \gamma^{-2})$. In terms of the angular and radial variables given in (13), the terms in the

braces in eqn (A.7) become:

$$\begin{aligned}
\text{First} &= 2\epsilon_{JK} E_3^\theta E_J^\rho \partial_\theta A_\rho^K \\
&= E_3^\theta \left[(K_x E^x + K_y E^y) \partial_\theta \eta + (K_x E^x - K_y E^y) \partial_\theta \xi - 4\partial_\theta P^\eta \right. \\
&\quad \left. + 2P^\eta \partial_\theta \ln(E^x E^y) - 2P^\xi \partial_\theta \ln(E^y/E^x) \right] \tag{A.8}
\end{aligned}$$

$$\begin{aligned}
\text{Second} &= (E_3^\theta A_\theta^3 + K_x E^x + K_y E^y)^2 - (E_3^\theta A_\theta^3)^2 - (E_I^\rho A_\sigma^I)(E_J^\sigma A_\rho^J) \\
&= 2 \cos^2(\xi) (K_x E^x K_y E^y + (P^\eta)^2 - (P^\xi)^2) + 2 E_3^\theta A_\theta^3 (K_x E^x + K_y E^y) \\
&\quad + 2 \sin(\xi) \cos(\xi) (P^\xi (K_x E^x + K_y E^y) - P^\eta (K_x E^x - K_y E^y)) \tag{A.9}
\end{aligned}$$

$$\begin{aligned}
\text{Third} &= - (E_I^\rho \Gamma_\sigma^I) (E_J^\sigma \Gamma_\rho^J) \\
&= \frac{1}{2} \left[(\partial_\theta E_3^\theta)^2 - \left(\frac{E_3^\theta}{\cos(\xi)} \partial_\theta(\xi) \right)^2 - \left(\frac{E_3^\theta}{\cos(\xi)} \partial_\theta \ln(E^y/E^x) \right)^2 \right] \tag{A.10}
\end{aligned}$$

$$\begin{aligned}
\text{Fourth} &= -(E_3^\theta \Gamma_\theta^3)(E_3^\theta A_\theta^3 + K_x E^x + K_y E^y) + (E_3^\theta \Gamma_\theta^3)(E_3^\theta A_\theta^3) + (E_I^\rho \Gamma_\sigma^I)(E_J^\sigma A_\rho^J) \\
&= \frac{E_3^\theta}{2} \left[-2P^\xi \partial_\theta \ln(E^y/E^x) + 2P^\eta \partial_\theta \ln E_3^\theta + 2P^\eta \tan(\xi) \partial_\theta \xi \right. \\
&\quad \left. + (K_x E^x + K_y E^y) \partial_\theta \eta + (K_x E^x - K_y E^y) \partial_\theta \xi \right] \tag{A.11}
\end{aligned}$$

Using these equations and with further rearrangement of terms, leads finally to the Hamiltonian constraint of eqn. (24).

The polarization conditions are $\xi = 0 = 2P^\xi + E_3^\theta \partial_\theta \ln(E^y/E^x)$. With $\xi = \beta - \bar{\beta} = 0$ alone, the spin-connection components simplify as: $\Gamma_\theta^3 = -(1/2)\partial_\theta \eta$,

$$\Gamma_x^1 = -\sin\beta \Gamma_x, \quad \Gamma_x^2 = \cos\beta \Gamma_x \quad ; \quad \Gamma_y^1 = \cos\beta \Gamma_y, \quad \Gamma_y^2 = \sin\beta \Gamma_y \tag{A.12}$$

$$\Gamma_x := \frac{1}{2} \frac{E_3^\theta}{E^x} \partial_\theta \ln \left(\frac{E_3^\theta E^y}{E^x} \right), \quad \Gamma_y := \frac{1}{2} \frac{E_3^\theta}{E^y} \partial_\theta \ln \left(\frac{1}{E_3^\theta} \frac{E^y}{E^x} \right) \tag{A.13}$$

Using the Gauss constraint and the polarization constraint, $\chi = 0$, it follows that,

$$A_x \sin \alpha := -\frac{P^\beta}{E^x} = \Gamma_x, \quad A_y \sin \bar{\alpha} := -\frac{P^{\bar{\beta}}}{E^y} = -\Gamma_y. \tag{A.14}$$

Notice that Γ_x, Γ_y are *gauge invariant*.

In checking preservation of various constraints as well as verifying the constraint algebra, the following Poisson bracket is useful,

$$\left\{ K_x E^x(\theta), \frac{\partial_{\theta'} E^x(\theta')}{E^x(\theta')} \right\} = \kappa \partial_{\theta'} \left(\frac{E^x(\theta)}{E^x(\theta')} \delta(\theta - \theta') \right), \quad \kappa := \frac{2G_{\text{Newton}}}{\pi}. \tag{A.15}$$

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