NILPOTENCY IN THE HOMOTOPY OF SIMPLICIAL COMMUTATIVE ALGEBRAS

JAMES M. TURNER

Abstract. In this paper, we continue a study of simplicial commutative algebras with finite André-Quillen homology, that was begun in [19]. Here we restrict our focus to simplicial algebras having characteristic 2. Our aim is to find a generalization of the main theorem in [19]. In particular, we replace the finiteness condition on homotopy with a weaker condition expressed in terms of nilpotency for the action of the homotopy operations. Coupled with the finiteness assumption on André-Quillen homology, this nilpotency condition provides a way to bound the height at which the homology vanishes. As a consequence, we establish a special case of an open conjecture of Quillen.

Introduction

Throughout this paper, unless otherwise stated, all rings and algebras are commutative.

Given a simplicial supplemented ℓ-algebra A, with ℓ a field having non-zero characteristic, it was shown, in [19], that if its total André-Quillen homology $H_*^Q(A)$ is *finite* (as a graded ℓ-module) then its homotopy π_*A being finite as well implies that $H_*^Q(A)$ is concentrated in degree 1. In this paper, we seek to find a generalization of this result by weakening the finiteness condition on homotopy. Thus we need to focus more on its internal structure. As such we restrict our attention to the case where $\operatorname{char} \ell = 2$ in order to take advantage the rich theory available in [10] and [11].

To be more specific, given such a simplicial algebra A having characteristic 2, M. André [2] showed that the homotopy groups π_*A have the structure of a divided power algebra. Furthermore, W. Dwyer [10] showed that there are natural maps

$$\delta_i : \pi_m A \to \pi_{m+i} A, \qquad 2 \le i \le m,$$

which are homomorphisms for $i < m$ and $\delta_m = \gamma_2$, the divided square. All resulting primary operations can now be described in terms of linear combinations of composites of the δ_is. Furthermore, there are Adem relations which allow any such composite to be described in terms of admissible composites. This gives the set $\mathcal{B}$ of operations a non-commutative ring structure. π_*A then becomes an algebroid over $\mathcal{B}$. Moreover, the module of indecomposables $Q\pi_*A$ inherits the structure of an unstable $\mathcal{B}$-module.

Date: February 1, 2008.

1991 *Mathematics Subject Classification.* Primary: 13D03; Secondary: 13D07, 13H10, 18G30, 55U35.

Key words and phrases. simplicial commutative algebras, André-Quillen homology, homotopy operations.

Partially supported by a grant from the National Science Foundation (USA).

A different perspective of Dwyer's operations can be taken in the following way. For $0 \leq i \leq n-2$ define

$$\alpha_i : \pi_n A \longrightarrow \pi_{2n-i} A$$

by $\alpha_i(x) = \delta_{n-i}(x)$. So, for example, $\alpha_0(x) = \gamma_2(x)$. Written in this way, iteration of these reindexed Dwyer operations need not be nilpotent. Nevertheless, the main theorem of this paper shows a connection between the vanishing of André-Quillen homology and the nilpotency of the Dwyer operations.

Theorem A: *Let ℓ be a field of characteristic 2 and let A be a connected simplicial supplemented ℓ-algebra such that the total André-Quillen homology $H^Q_*(A)$ is finite. Then a nilpotent action of α_{n-2} on $Q\pi_* A$ implies that $H^Q_s(A) = 0$ for $s \geq n$.*

As a consequence, the following strengthens the main theorem of [19] and resolves a conjecture posed in [20, 4.7] at the prime 2:

Corollary: *Let A be as in Theorem A and suppose that $\pi_* A$ is locally nilpotent as a divided power algebra. Then $H^Q_s(A) = 0$ for $s \geq 2$.*

Note: The restriction to characteristic 2 is due to the need for an Adem relations among the homotopy operations δ_i which insures that arbitrary composites can be written in terms of admissible operations. At the prime 2, this was established in the work of Dwyer [10] and Goerss-Lada [12]. At odd primes, Bousfield's work [7] gives a preliminary version of Adem relations, but a final version still awaits to be produced.

Connection to conjectures of Quillen. For a simplicial algebra A over a ring R the *André-Quillen homology*, $D_*(A|R; M)$, of A over R with coefficients in an A-module M was first defined by M. André and D. Quillen [1, 16, 17]. In particular, for a simplicial supplemented ℓ-algebra A, we write

$$H^Q_*(A) := D_*(A|\ell; \ell).$$

Next, recall that a homomorphism $\varphi : R \to S$ of Noetherian rings is *essentially of finite type* if for each $\mathfrak{n} \in \operatorname{Spec} S$ there is a factorization

$$R \xrightarrow{\tau} R[X]_{\mathfrak{N}} \xrightarrow{\sigma} S_{\mathfrak{n}} \tag{0.1}$$

where $R[X] = R[X_1, \ldots, X_n]$ is a polynomial ring, $\mathfrak{N}$ is a prime ideal in $R[X]$ lying over $\mathfrak{n}$, the homomorphism $\tau : R \to R[X]_{\mathfrak{N}}$ is the localization map, and the homomorphism σ is surjective. Furthermore, we call such a homomorphism a *locally complete intersection* if, for each $\mathfrak{n} \in \operatorname{Spec} S$, $\operatorname{Ker}(\sigma)$ is generated by a regular sequence.

In [16, (5.6, 5.7)], Quillen formulated the following two conjectures on the vanishing of André-Quillen homology:

Conjecture: *Let $\varphi : R \to S$ be a homomorphism essentially of finite type between Noetherian rings and assume further that $D_s(S|R; -) = 0$ for $s \gg 0$. Then:*

I. $D_s(S|R; -) = 0$ *for* $s \geq 3$*;*

II. *If, additionally, the flat dimension* $\mathrm{fd}_R S$ *is finite, then* φ *is a locally complete intersection homomorphism.*

In [4], L. Avramov generalized the notion of local complete intersections to arbitrary homomorphisms of Noetherian rings. He further proved a generalization of Conjecture II. to such homomorphisms. See P. Roberts review [18] of this paper for an excellent summary of these results and the history behind them. A proof of Conjecture II. was also given in [19] for homomorphisms with target S having non-zero characteristic.

We now indicate how Theorem A bears on providing a resolution to the above Conjecture. To formulate this, let $R \to (S, \ell)$ be a homomorphism of local rings with $\mathrm{char}\, \ell = 2$. Let

$$\vartheta : (Q\,\mathrm{Tor}_*^R(S, \ell))_m \to (Q\,\mathrm{Tor}_*^R(S, \ell))_{2m-1}$$

be the operation induced by α_1. We call this operation the *André operation* since it generalizes the operation studied by M. André [3] when $S = \ell$ and $m = 3$.

Theorem B: *Let* $\varphi : R \to (S, \ell)$ *be a surjective homomorphism of local rings with* $\mathrm{char}\, \ell = 2$ *and assume further that* $D_s(S|R; \ell) = 0$ *for* $s \gg 0$*. Then*

1. $D_s(S|R; \ell) = 0$ *for* $s \geq 3$ *if and only if the André operation* ϑ *acts nilpotently on* $Q\,\mathrm{Tor}_*^R(S, \ell)$*;*
2. φ *is a complete intersection if and only if the divided square* γ_2 *acts nilpotently on* $Q\,\mathrm{Tor}_*^R(S, \ell)$*.*

As an application of Theorem B, the following proves the vanishing portions of the conjecture for certain homomorphisms in characteristic 2:

Theorem C: *Let* $R \to S$ *be a homomorphism essentially of finite type between Noetherian rings of characteristic 2 and assume further that* $D_s(S|R; -) = 0$ *for* $s \gg 0$*. Then:*

1. $D_s(S|R; -) = 0$ *for* $s \geq 3$ *provided* $R \to S$ *is a homomorphism of Cohen-Macaulay rings;*
2. *If the flat dimension* $\mathrm{fd}_R S$ *is finite, then* φ *is a locally complete intersection.*

Note: L. Avramov showed in [4] that Conjecture I. holds when either R or S is a locally complete intersection. More recently, L. Avramov and S. Iyengar [5] have strengthened this special case of Conjecture I. by showing that it holds for homomorphisms $R \to S$ for which there exists a composite $Q \to R \to S$ which is a local complete intersection homomorphism. See [6] for a more leisurely discussion of their results.

Organization of this paper. The first section reviews the properties of the homotopy and André-Quillen homology for simplicial commutative algebras and the methods for computing them, particularly in characteristic 2. The next section then focuses on a device called the character map associated to simplicial algebras having finite André-Quillen homology. After showing that Dwyer's operations possess certain annihilation properties, we show that the character map can be highly non-trivial. From this Theorem A easily follows. This enables us, in the third section, to prove Theorem B. Finally, the last section begins with establishing a chain level criterion for the nilpotency of André's operation. After a brief excursion into commutative algebra, we prove a special case of Theorem C and then show how the general case follows.

Acknowledgements. The author would like to thank Lucho Avramov for sharing his expertise and insights and for a thorough and critical reading of an earlier draft of this paper. He would also like to thank Paul Goerss and Jean Lannes for helpful advice and discussions during work on this project.

1. Homotopy and homology of simplicial commutative algebras in characteristic 2

1.1. **Operations on chains and homotopy.** Again let A be a simplicial algebra of characteristic 2. We describe the algebra structure associated to A at two levels: on the associated chain complex and on the associated homotopy groups. While only the latter will be needed in the process of proving Theorem A, the former description will be needed in the subsequent applications.

Let V be a simplicial vector space, over the field $\mathbb{F}_2$, and let $C(V)$ denote its associated chain complex. In [10], Dwyer constructs natural chain maps

$$\Delta^k : (C(V) \otimes C(V))_{n+k} \to C(V \otimes V)_n \tag{1.2}$$

for all $0 \le k \le n$. They satisfy the relations

$$\Delta^0 + T\Delta^0 T + \phi_0 = \Delta \tag{1.3}$$

and

$$\Delta^k + T\Delta^k T + \phi_k = \partial\Delta^{k-1} + \Delta^{k-1}\partial. \tag{1.4}$$

Here $T : C(V) \otimes C(V) \to C(V) \otimes C(V)$ and $T : C(V \otimes V) \to C(V \otimes V)$ are the twist maps. Also,

$$\phi_k : C(V) \otimes C(V) \to C(V \otimes V), \quad k \ge 0$$

is the degree $-k$ map that is zero on $[C(V) \otimes C(V)]_m$ for $m \ne 2k$, and, in degree $2k$, is the projection on one factor:

$$[C(V) \otimes C(V)]_{2k} = \oplus_{p+q=2k} V_p \otimes V_q \to V_k \otimes V_k.$$

Finally $\Delta : C(V) \otimes C(V) \to C(V \otimes V)$ is the Eilenberg-Zilber map.

Now given a simplicial $\mathbb{F}_2$-algebra (A, μ), define the maps

$$\alpha_i : C(A)_n \to C(A)_{2n-i} \quad \textit{for} \quad 0 \le i \le n-1 \tag{1.5}$$

by

$$x \to \mu\Delta^i(x \otimes x) + \mu\Delta^{i-1}(x \otimes \partial x). \tag{1.6}$$

To describe the algebra structure on the associated chains of A, recall that a *dg* Γ-*algebra* is a non-negatively differentially graded $\mathbb{F}_2$-algebra (Λ, ∂) together with maps

$$\gamma_k : \Lambda_n \to \Lambda_{kn} \quad for \quad k \geq 0 \quad and \quad n \geq 2,$$

satisfying the following relations

1. $\gamma_0(x) = 1$ and $\gamma_1(x) = x$
2. $\gamma_h(x)\gamma_k(x) = \binom{h+k}{h}\gamma_{h+k}(x)$
3. $\gamma_k(x+y) = \sum_{r+s=k} \gamma_r(x)\gamma_s(x)$
4. $\gamma_k(xy) = 0$ for $k \geq 2$ and $x, y \in \Lambda_{\geq 1}$
5. $\gamma_k(xy) = x^k\gamma_k(y)$ for $x \in \Lambda_0$ and $y \in \Lambda_{\geq 2}$
6. $\gamma_k(\gamma_2(x)) = \gamma_{2k}(x)$
7. $\partial\gamma_k(x) = (\partial x)\gamma_{k-1}(x)$.

Proposition 1.1. *Let A be a simplicial $\mathbb{F}_2$-algebra.*

1. *The chain complex $C(A)$ possesses a dg Γ-algebra with $\alpha_0 = \gamma_2$;*
2. *For $x \in C(A)_n$ and $0 < i < n-1$,*

$$\partial(\alpha_i(x)) = \alpha_{i-1}(\partial(x)), \quad for \quad 0 < i < n-1,$$

$$\partial(\alpha_0(x)) = x \cdot \partial(x), \quad and \quad \partial(\alpha_{n-1}(x)) = \alpha_{n-2}(\partial x) + x^2.$$

Proof: 1. See [2] and [11, §2 and 3].

2. By 1.4, 1.6, and the Leibniz rule for ∂,

$$\begin{aligned} \partial\alpha_i(x) &= \mu\partial\Delta^i(x \otimes x) + \mu\partial\Delta^{i-1}(x \otimes \partial x) \\ &= \mu\Delta^i\partial(x \otimes x) + \mu\Delta^{i-1}\partial(x \otimes \partial x) + \mu\Delta^i(\partial x \otimes x) + \mu\Delta^i(x \otimes \partial x) \\ &= \mu\Delta^{i-1}(\partial x \otimes \partial x) \\ &= \alpha_{i-1}(\partial x). \end{aligned}$$

The calculation of $\partial\alpha_0(x)$ and $\partial\alpha_{n-1}(x)$ from (1.3) and (1.4) is similar. □

Note: For a divided power algebra Λ of characteristic 2, it is enough to specify the action of divided square γ_2 to determine all divided powers. Specifically, for $x \in \Lambda$

$$\gamma_k = \gamma_2^{s_1}(x) \cdot \gamma_2^{s_2}(x) \dots \gamma_2^{s_r}(x),$$

where $k = 2^{s_1} + \ldots + 2^{s_r}$. Cf. [2] and [11, §2].

Now, the α_i induce the homotopy operations

$$\alpha_i : \pi_n A \to \pi_{2n-i} A, \quad 0 \leq i \leq n-2.$$

Furthermore, Dwyer's *higher divided squares*

$$\delta_i : \pi_n A \to \pi_{n+i} A, \quad 2 \leq i \leq n$$

are now defined as

$$\delta_i[x] = [\alpha_{n-i}(x)].$$

In particular,

$$\delta_n[x] = [\alpha_0(x)] = \gamma_2[x].$$

We now summarize the properties of the higher divided squares, as established in [10, 12] (see also [11, §2 and 3]).

Proposition 1.2. *The higher divided squares possess the following properties:*

1. *Adem relations:*
   (a) *For $i < 2j$,*

$$\delta_i\delta_j = \sum_{\frac{i+1}{2} \le s \le \frac{i+j}{3}} \binom{j-i+s-1}{j-s} \delta_{i+j-s}\delta_s$$

   (b) *For $j < i$,*

$$\alpha_i\alpha_j = \sum_{\frac{i+2j}{3} \le s \le \frac{i+j-1}{2}} \binom{i-s-1}{s-j} \alpha_{i+2j-2s}\alpha_s$$

2. *Cartan formula: for $x, y \in \pi_* A$,*

$$\delta_i(xy) = \begin{cases} x^2\delta_i(y) & |x| = 0; \\ y^2\delta_i(x) & |y| = 0; \\ 0 & |x| > 0, |y| > 0. \end{cases}$$

Let $I = (i_1, \ldots, i_s)$ be a sequence of positive integers. Then call I *admissible* provided $i_t \ge 2i_{t+1}$ for all $1 \le t < s$. Furthermore, define for I its *excess* to be the integer

$$e(I) = (i_1 - 2i_2) + (i_2 - 2i_3) + \ldots + (i_{s-1} - 2i_s) + i_s = i_1 - i_2 - \ldots - i_s,$$

its *length* to be the integer $\lambda(I) = s$, and its *degree* to be the integer

$$d(I) = i_1 + \ldots + i_s.$$

Also write

$$\delta_I = \delta_{i_1} \ldots \delta_{i_s} \qquad \alpha^I = \alpha_1^{i_1} \ldots \alpha_s^{i_s}.$$

As an application of the Adem relations, given any sequence I, δ_I can be written as a sum of δ_J's, with each J an admissible sequence. Similarly, by another application of the Adem relations, any α_I can be written as a sum of α^J's, with J not necessarily admissible.

Finally, denote by $\mathcal{B}$ the algebra spanned by $\{\delta_I | I \text{ admissible}\}$. A $\mathcal{B}$-module M is then called *unstable* provided $\delta_I x = 0$ for any $x \in M_n$, whenever I is admissible with $e(I) > n$. For example, given a simplicial supplemented ℓ-algebra A then $Q\pi_* A$, the module of indecomposables, is an unstable $\mathcal{B}$-module.

1.2. **The homotopy of symmetric algebras.** Let ℓ be a field having characteristic 2. We provide description of the homotopy groups of a very important type of simplicial ℓ-algebra, namely, the homotopy of $S_\ell(V)$ - the symmetric algebra, or free commutative algebra, generated by a simplicial vector space V over ℓ. When $\ell = \mathbb{F}_2$, we simply write S for S_ℓ.

If W is a vector space, let $K(W,n)$ denote the simplicial vector space with homotopy $\pi_* K(W,n) \cong W$, concentrated in degree n. Then write

$$\begin{aligned} S_\ell(W,n) &= S_\ell(K(W,n)) \\ S_\ell(n) &= S_\ell(\ell, n) \end{aligned}$$

By a theorem of Dold [9], there is a functor of graded vector spaces $\mathcal{S}_\ell$ and a natural isomorphism

$$\pi_* S_\ell(V) \cong \mathcal{S}_\ell(\pi_* V). \tag{1.7}$$

Again, we simply write $\mathcal{S}$ for $\mathcal{S}_\ell$ when $\ell = \mathbb{F}_2$.

To describe the functor $\mathcal{S}_\ell$ on graded vector spaces, we first note that it commutes with colimits. Thus we need only describe $\mathcal{S}_\ell(F_\ell(n))$ where $F_\ell(n) \cong \ell\langle x_n \rangle$. We now recall the description of this functor when $\ell = \mathbb{F}_2$. A proof of the following can be found, for example, in [10]:

Proposition 1.3.

$$\begin{aligned} \mathcal{S}(F(n)) &\cong \Gamma[\alpha^I(x_n) : \sigma(I) < n-1] \\ &\cong \Gamma[\delta_I(x_n) : I \textit{ admissible}, \ e(I) < n]. \end{aligned}$$

Here $\Gamma[-]$ denotes the free divided power algebra functor. Note that $Q\mathcal{S}(W)$ is a free unstable $\mathcal{B}$-module, for any positively graded vector space W.

The functor $\mathcal{S}$ can be further decomposed as

$$\mathcal{S}(-) = \bigoplus_{m \geq 0} \mathcal{S}_m(-).$$

We review its description because of its importance below.

First, define the *weight* of an element of $\mathcal{S}(W)$ as follows:

$$\mathrm{wt}(u) = 1, \quad \mathrm{wt}(uv) = \mathrm{wt}(u) + \mathrm{wt}(v),$$
$$\mathrm{wt}(\delta_i(u)) = 2\,\mathrm{wt}(u) \quad \text{for} \quad u, v \in W.$$

Proposition 1.4. *Let W be a graded vector space. Then $\mathcal{S}_m(W)$ is the subspace $\mathcal{S}(W)$ spanned by elements of weight m.*

Now, to describe $\mathcal{S}_\ell$, note first that, the uniqueness of adjoint funtors, there is a natural isomorphism of ℓ-algebras

$$\eta : S_\ell(V \otimes_{\mathbb{F}_2} \ell) \to S(V) \otimes_{\mathbb{F}_2} \ell$$

where V is any $\mathbb{F}_2$-vector space.

Proposition 1.5. *The natural isomorphism* η *in turn induces a natural isomorphism*

$$\eta_* : \mathcal{S}_\ell((-) \otimes_{\mathbb{F}_2} \ell) \xrightarrow{\cong} \mathcal{S}(-) \otimes_{\mathbb{F}_2} \ell$$

of functors to the category of Γ*-algebras. Moreover, for each* $i, m \geq 2$ *and graded* $\mathbb{F}_2$*-vector space* W*, there is a commutative diagram*

$$\begin{array}{ccc} \mathcal{S}_\ell(W \otimes_{\mathbb{F}_2} \ell)_n & \xrightarrow{\delta_i} & \mathcal{S}_\ell(W \otimes_{\mathbb{F}_2} \ell)_{n+i} \\ \downarrow & & \downarrow \\ \mathcal{S}(W)_n \otimes_{\mathbb{F}_2} \ell & \xrightarrow{\delta_i \otimes F} & \mathcal{S}(W)_{n+i} \otimes_{\mathbb{F}_2} \ell \end{array}$$

where F *denotes the Frobenius map on* ℓ*.*

Proof: Since $\pi_* S_\ell(-) \cong \mathcal{S}_\ell(-)$, by Dold's theorem, then the first point regarding η_* follows from a Kunneth theorem argument. To prove the second part, it suffices to prove it for the graded vector space $F_\ell(n) = F(n) \otimes_{\mathbb{F}_2} \ell$, by a standard argument utilizing universal examples. In this case, the map η extends the map $\ell\langle x_n \rangle \to \mathbb{F}_2\langle x_n \rangle \otimes_{\mathbb{F}_2} \ell$ sending ax_n to $x_n \otimes a$. By naturality of η_* and the properties of Dwyer's operations, we have

$$\begin{aligned} \delta_i(\eta_*(ax_n)) &= \delta_i(x_n \otimes a) = \delta_i((x_n \otimes 1)(1 \otimes a)) \\ &= \delta_i(x_n \otimes 1)(1 \otimes a)^2 = (\delta_i(x_n) \otimes 1)(1 \otimes a^2) \\ &= \delta_i(x_n) \otimes a^2 \end{aligned}$$

which is the desired result. □

It now follows that, for a simplicial ℓ-vector space V, the generators and relations for $\pi_* S_\ell(V)$ are completely determined by Dwyer's result Proposition 1.2 and Dold's theorem.

1.3. **André-Quillen homology and the fundamental spectral sequence.** We now provide a brief review of André-Quillen homology, for simplicial supplemented ℓ-algebras, and the main computational device for relating homotopy and homology - Quillen's fundamental spectral sequence. Our primary source for this material is [11]. Cf. also [16, 17, 14].

Let A be a simplicial supplemented ℓ-algebra. Then the *André-Quillen homology* of A is defined as the graded vector space

$$H_*^Q(A) = \pi_* QX,$$

where X is a cofibrant replacement of A, in the closed simplicial model structure on simplicial supplemented ℓ-algebras [14, 11].

Some standard properties of André-Quillen homology are summarized in the following:

Proposition 1.6. [11, §4] *Let* A *be a simplicial supplemented* ℓ*-algebra.*

1. *If* $A = S_\ell(V)$*, for some simplicial vector space* V*, then* $H_*^Q(A) \cong \pi_* V$*.*

2. *Let* $A \xrightarrow{f} B \xrightarrow{g} C$ *be a cofibration sequence in the homotopy category of simplicial supplemented algebras. Then there is a long exact sequence*

$$\cdots \to H^Q_{s+1}(C) \xrightarrow{\partial} H^Q_s(A) \xrightarrow{H^Q_*(f)} H^Q_s(B)$$

$$\xrightarrow{H^Q_*(g)} H^Q_s(C) \xrightarrow{\partial} H^Q_{s-1}(C) \to \cdots$$

3. *If* V *is a vector space and* $[\ \ ,\ \ ]$ *denotes morphisms in the homotopy category of simplicial supplemented algebras, then the map*

$$[S_\ell(V,n), A] \to \mathrm{Hom}(V, I\pi_n A)$$

is an isomorphism. In particular, $\pi_n A = [S_\ell(n), A]$.

Another important tool we will need is the notion of *connected envelopes* for a simplicial supplemented algebra A. Cf. [19, §2]. These are defined as a sequence of cofibrations

$$A = A(0) \xrightarrow{j_1} A(1) \xrightarrow{j_2} \cdots \xrightarrow{j_n} A(n) \xrightarrow{j_{n+1}} \cdots$$

with the following properties:

(1) For each $n \geq 1$, $A(n)$ is a n-connected.

(2) For $s > n$,

$$H^Q_s A(n) \cong H^Q_s A.$$

(3) There is a cofibration sequence

$$S_\ell(H^Q_n A, n) \xrightarrow{f_n} A(n-1) \xrightarrow{j_n} A(n).$$

The following is proved in [19]:

Lemma 1.7. *If* $H^Q_s(A) = 0$ *for* $s > n$ *then* $A(n-1) \cong S_\ell(H^Q_n(A), n)$ *in the homotopy category of simplicial supplemented algebras.*

Finally, a very important tool for bridging the André-Quillen homology to the homotopy of a simplicial algebra is provided by the *fundamental spectral sequence* of Quillen [16, 17]. For simplicial supplemented ℓ-algebras in characteristic 2, we will need certain properties of this spectral sequence which can be described by combining the results of [11, §6] with Proposition 1.5. The following summarizes those features that we will need.

Proposition 1.8. *Let* A *be a simplicial supplemented* ℓ*-algebra. Then there is a spectral sequence of algebras*

$$E^1_{s,t}A = \mathcal{S}_s(H^Q_*(A))_t \otimes_{\mathbb{F}_2} \ell \Longrightarrow \pi_t A$$

with the following properties:

1. *For* $\pi_0 A \cong \ell$, $E^1_{s,t}A = 0$ *for* $s > t$ *and, hence, the spectral sequence converges;*
2. *The differentials act as* $d_r : E^r_{s,t} \to E^r_{s+r,t-1}$*;*
3. *The Dwyer operations*

$$\delta_i : E^r_{s,t} \to E^r_{2s,t+i}, \quad 2 \leq i \leq t$$

have indeterminacy 2r-1 and satisfy the following properties:

(a) *Up to determinacy, the Adem relations and Cartan formula holds;*

(b) *If* $x \in E^r A$ *and* $2 \leq i < t$, *then* $\delta_i(x)$ *survives to* $E^{2r}A$ *and*

$$\begin{aligned} d_{2r}\delta_i(x) &= \delta_i(d_r x) \\ d_r\delta_t(x) &= x d_r x \end{aligned}$$

modulo indeterminacy;

(c) *The operations on* $E^r A$, $r \geq 2$, *are induced by the operations on* $E^{r-1}A$. *The operations on* $E^\infty A$ *are induced by the operations on* $E^r A$ *with* $r < \infty$*; and*

(d) *The operations on* $E^\infty A$ *are also induced by the operations on* $\pi_* A$.

Recall that, if

$$B^q_{s,t} \subseteq E^r_{s,t}, \quad q \geq r$$

is the vector space of elements that survive to $E^q_{s,t}$ but have zero residue class in $E^q_{s,t}$, then $y \in E^r_{s,t}$ is defined *up to indeterminacy* q if y is a coset representitive for a particular element in $E^r_{s,t}/B^q_{s,t}$.

2. Proof of Theorem A

2.1. **The character map for simplicial algebras with finite homology.** Fix a connected simplicial supplemented ℓ-algebra A with $H^Q_*(A)$ finite as a graded ℓ-module. Define the *André-Quillen dimension* of A [6] to be

$$\text{AQ-dim}\, A = \max\{s : H^Q_s(A) \neq 0\}$$

and define the *connectivity* of A to be

$$\text{conn}\, A = \min\{s : H^Q_s(A) \neq 0\} - 1.$$

We assume that $\text{AQ-dim}\, A \geq 2$.

Let $n = \text{AQ-dim}\, A$. Then the (n-1)-connected envelope $A(n-1)$ has the property that

$$A(n-1) \cong S_\ell(H^Q_n(A), n)$$

in the homotopy category. Cf. Lemma 1.7. Thus we have a map $A \to S_\ell(H^Q_n(A), n)$ in the homotopy category with the property that it is an H^Q_n-isomorphism.

We now define the *character map* of A to be the resulting induced map of unstable $\mathcal{B}$-modules

$$\Phi_A : Q\pi_* A \to Q\pi_* S_\ell(H^Q_n(A), n) \tag{2.8}$$

The importance of the character map is established by the following:

Theorem 2.1. *Let* A *be a connected simplicial supplemented* ℓ*-algebra having finite André-Quillen dimension* n. *Then the character map* Φ_A *is non-trivial. Furthermore,* $y \in Q\pi_* A$ *can be chosen so that*

$$\Phi_A(y) = \alpha^s_{n-2}(x),$$

for some non-trivial $x \in H^Q_n(A)$ *and some* $s > 0$. *Thus* α_{t-2} *acts non-nilpotently on* y*, for all* $2 \leq t \leq n$.

As an immediate consequence, we are now in a position to supply the following:

Proof of Theorem A: This follows immediately from Theorem 2.1 since if α_{n-2} acts nilpotently on any $x \in Q\pi_* A$, the same must also hold for $\Phi(x)$. Hence it follows that $n > \text{AQ-dim}\, A$. □

2.2. **Annihilation properties among some homotopy operations.** Before we prove Theorem 2.1, we need to pin down specific annihilators of elements of $\mathcal{B}$. To this end, define, for $s, t \geq 0$, the operation

$$\theta(s,t) = \delta_{2^{s+t}} \delta_{2^{s+t-1}} \dots \delta_{2^{t+1}}.$$

Proposition 2.2. *Let J be a finite subset of $\{j | j > 2^t\}$ and let*

$$\xi = \sum_{j \in J} a_j \delta_j w_j$$

be a linear combination of admissible operations, with each $a_j \in \ell$. Then $\theta(s,t)\xi = 0$ for $s \gg 0$.

Proof: It is sufficient to prove the result for $\xi = \delta_j$ with $j > 2^t$. Write $j = 2^t + n$ with $n \geq 1$. Note first that an application of the Adem relations shows that, for any t,

$$\delta_{2^{t+1}} \delta_{2^t+1} = \delta_{2^{t+1}} \delta_{2^t+2} = 0.$$

We thus assume, by induction, that, for each t and $0 < i < n$, there exists $s \gg 0$ such that

$$\theta(s,t)\delta_{2^t+i} = 0.$$

By another application of the Adem relations, we have

$$\delta_{2^{t+1}} \delta_{2^t+n} = \sum_{1 \leq r \leq \frac{n}{3}} \binom{n+r-1}{n-r} \delta_{2^{t+1}+n-r} \delta_{2^t+r}.$$

Notice that, for each such r, $2^{t+1} < 2^{t+1} + n - r < 2^{t+1} + n$. Thus, by induction, we can find $s \gg 0$ so that

$$\theta(s,t+1) \Big(\sum_{1 \leq r \leq \frac{n}{3}} \binom{n+r-1}{n-r} \delta_{2^{t+1}+n-r} \delta_{2^t+r} \Big) = 0.$$

We conclude that

$$\theta(s,t)\delta_{2^t+n} = \theta(s,t+1)\delta_{2^{t+1}}\delta_{2^t+n} = 0.$$

□

2.3. **Proof of Theorem 2.1.** As we noted previously, it is sufficient to show that, for some $t > 0$, $\alpha^t_{n-2}(x) = \delta_{2^t}\delta_{2^{t-1}}\ldots\delta_2(x) \in E^1_{2^t}$ survives non-trivially to E^∞ for some $x \in H^Q_n(A)$, where $n = \text{AQ-dim}\, A$. Such an element will map non-trivially under Φ with the desired properties, by Proposition 1.3.

The strategy is to examine the induced map of spectral sequences

$$\{E^r A\} \to \{E^r S(H^Q_n(A), n)\}$$

which is split surjective at E^1. The goal is to show that the image of the splitting on the indecomposables contains a non-trivial infinite cycle with the requisite specifications.

Now, assume $n \geq 2$. Then the result holds when $n = m$ and $n = m+1$, where $m = \text{conn}\, A + 1$, because in these cases Quillen's spectral sequence collapses [19] and, hence, Φ_A is a split surjection. Thus we can now induct on $n - m$. This further reduces to an induction on $\dim_\ell H^Q_m(A)$, which is finite by assumption.

By the Hurewicz theorem [11, (8.3)], $\pi_m A \cong H^Q_m(A)$. By Proposition 1.6.3, a choice of a basis element $y \in H^Q_m(A)$ is represented by a map $\sigma : S(m) \to A$ of simplicial supplemented ℓ-algebras, by Proposition 1.6.3. Let B be the homotopy cofibre (aka mapping cone [11, (4.5)]) of σ and let $f : A \to B$ be the induced map. Note that there is an identity:

$$\Phi_A = \Phi_B(Qf_*) \tag{2.9}$$

Then $\dim_\ell H^Q_m(B) = \dim_\ell H^Q_m(A) - 1$. By induction, we assume that, for $x \in H^Q_n(A)$, $\theta(b,0)x \in E^1_{2^b,t}B$ survives non-trivially to $E^\infty B$ and determines an element $y' \in \pi_t B$ such that

$$\Phi_B(y') = \alpha^b_{n-2}(x). \tag{2.10}$$

Now, in the spectral sequence for A, Proposition 1.8.3 (b) tells us, by induction, that $\theta(b,0)x$ survives to some $E^r_{2^b,t}A$ with $r \geq 2^b$. Furthermore, by Proposition 1.4, the differential satisfies:

$$d_r\,[\theta(b,0)x] = \begin{cases} [uy + \xi y] & r = 2^a - 2^b,\ a > b; \\ [uy] & \text{otherwise,} \end{cases}$$

with $u \in E^1_{2^b+r-1}$ and ξ a linear combination of admissible Dwyer operations. Our goal is to show that there exists $s \gg b$ so that $\theta(s,0)x$ is an infinite cycle. We examine the above cases on r in reverse order.

$\mathbf{r \neq 2^a - 2^b}$: We use induction on $t - 2^b - r$, by first noting that for $t - 2^b - r = 0$, $d_r[\theta(b,0)x] \in E^r_{2^b+r,t-1} = 0$. Since $\theta(b+1,0) = \delta_{2^{b+1}}\theta(b,0) \neq 0$ then $\delta_{2^{b+1}}\theta(b,0)x$ survives to E^{2r} by Proposition 1.8.3 (b). By Proposition 1.8.1 and Proposition 1.8.3 (a) and (d),

$$d_{2r}[\delta_{2^{b+1}}\theta(b,0)x] = \delta_{2^{b+1}}[d_r\theta(b,0)x] = 0.$$

Hence $\delta_{2^{b+1}}\theta(b,0)x$ survives to $E^{2r+1}_{2^{b+1},t+2^{b+1}}$. By assumption, we have

$$(t + 2^{b+1}) - 2^{b+1} - (2r+1) \leq t + 2^b + r - 2^{b+1} - 2r - 1 < t - 2^b - r.$$

Thus, by induction, $\theta(s,0)x$ is an infinite cycle for some $s \gg b+1$.

$\mathbf{r = 2^a - 2^b}$: Write $\xi = \sum a_I \delta_I$ as a homogeneous linear combination of admissible operations, with each $a_I \in \ell$. Then a typical indexing I can be written as $I = (i_1, \ldots, i_a)$. Admissibility implies that

$$i_1 \geq 2i_2 \geq \ldots 2^{a-1} i_a \geq 2^a > 2^b.$$

Proposition 2.2 now applies to tell us that $\theta(s, b)\xi = 0$ for some $s \gg 0$. Thus $\theta(s, 0)x = \theta(s, b)\theta(b, 0)x$ survives to $E^{2^e r}$, where $e = s - b$, and

$$d_{2^e r}[\theta(s, b)\theta(b, 0)x] = \theta(s, b)[d_r \theta(b, 0)x] = 0 \in E^{2^e r}.$$

Thus $\theta(s, 0)x = \theta(s, b)\theta(b, 0)x$ survives to $E^{2^e r+1}$, so proceed as per the previous case.

Let $y \in \pi_* A$ be the element determined by $[\theta(s, 0)x] \in E^\infty A$. To see that it is non-trivial, suppose that $[\theta(s, 0)x] \in E^r A$ is a boundary. Then $E^r(f)([\theta(s, 0)x]) = [\theta(s, 0)x]$ is also a boundary in $E^r B$. But, since $s \geq b$, this contradicts the induction hypothesis.

We conclude, by induction and Equation 2.10, that

$$\Phi_A(y) = \Phi_B(\alpha_{n-2}^{s-b}(y')) = \alpha_{n-2}^{s-b}\Phi_B(y') = \alpha_{n-2}^{s-b}\alpha^b(x) = \alpha_{n-2}^s(x).$$

□

3. Proof of Theorem B

Theorem 3.1. *Let $R \to S \to \ell$ be a surjective homomorphisms of Noetherian rings, with ℓ a field of characteristic 2, such that $D_s(S|R; \ell) = 0$ for $s \gg 0$. Then the following hold:*

1. *If the divided square γ_2 acts nilpotently on $Q\,\mathrm{Tor}_*^R(S, \ell)$ it follows that $D_{\geq 2}(S|R; \ell) = 0$.*
2. *If the André operation ϑ acts nilpotently on $Q\,\mathrm{Tor}_*^R(S, \ell)$ it follows that $D_{\geq 3}(S|R; \ell) = 0$.*

Proof: Using the simplicial model structure for simplicial commutative algebras [15, §II], let $R \hookrightarrow \mathfrak{R} \xrightarrow{\sim} S$ be a factorization of $R \to S$ as a cofibration followed by an acyclic fibration. Let $A = \mathfrak{R} \otimes_R \ell$. Since $\mathfrak{R}$ can be chosen to be a degreewise free R-algebra, A is a connected simplicial supplemented ℓ-algebra with the properties

$$H_*^Q(A) \cong D_*(S|R; \ell) \tag{3.11}$$

and

$$\pi_* A \cong \mathrm{Tor}_*^R(S, \ell). \tag{3.12}$$

which follows from [17, (4.7)] and [11, (4.7)]. The result now follows from Theorem A. □

Proof of Theorem B: First, if $D_{\geq 3}(S|R; \ell) = 0$ then a Quillen spectral sequence argument shows that $\mathrm{Tor}_*^R(S, \ell) \cong \Gamma[D_*(S|R; \ell)]$. It follows from the Adem relations that ϑ acts trivially, hence nilpotently, on the indecomposables. The converse follows from Theorem 3.1.

Next, φ is a complete intersection if and only if $D_2(S|R;\ell) = 0$, by [1, VI.25]. Thus γ_2 acts trivially, and hence nilpotently, on the indecomposables if φ is a complete intersection. Conversely, if γ_2 acts nilpotently on the indecomposables, then φ is a complete intersection, by Theorem 3.1. □

4. Proof of Theorem C

4.1. **A chain condition for the nilpotency of the André operation.** Let $(R, \mathfrak{m}, \ell) \to S$ be a surjective homomorphism of local rings of characteristic 2. Using the simplicial model structure for simplicial commutative algebras [15, §II], let

$$R \hookrightarrow \Re \xrightarrow{\sim} S \tag{4.13}$$

be a factorization of $R \to S$ as a cofibration followed by an acyclic fibration. Then the chains $C(\Re)$ provides a free resolution of S under R. Furthermore, we have

$$C(\Re \otimes_R \ell) = C(\Re) \otimes_R \ell \cong \frac{C(\Re)}{\mathfrak{m}C(\Re)} \tag{4.14}$$

for which the homology is isomorphic to $\operatorname{Tor}^R_*(S, \ell)$.

We now provide a chain level condition which insures the nilpotency of the André operation on Tor.

Lemma 4.1. *If the divided square γ_2 acts nilpotently on $\mathfrak{m}C_{\geq 2}(\Re)$, then the André operation ϑ acts nilpotently on $Q\operatorname{Tor}^R_*(S, \ell)$.*

Proof: Let $x \in C_{\geq 2}(\Re) \otimes_R \ell$ be a cycle which represents a non-trivial element in homology. Let $y \in C(\Re)$ be a pre-image of x under the canonical projection. Since $C(\Re)$ is acyclic then $\partial y \in \mathfrak{m}C(\Re)$ by Equation 4.14 and is non-trivial.

By Proposition 1.1 and an induction argument, we have

$$\partial \alpha_1^s(y) = \gamma_2^s(\partial y).$$

By assumption $\gamma^s(\partial y) = 0$ for $s \gg 0$. Thus $\alpha_1^s(y)$ is a boundary for $s \gg 0$ by the acyclicity of $C(\Re)$. Hence, by the naturality of α_1, $\alpha_1^s(x)$ represents the trivial element in the homology of $C(\Re \otimes_R \ell)$. □

4.2. **Some properties of local rings and their homomorphisms.** We now summarize some important properties of certain local rings and homomorphisms between them. First, recall that a *Cohen-Macaulay ring* is a Noetherian ring B such that $\operatorname{depth} B = \dim B$. Cf. [8].

Lemma 4.2.
1. *R is a Cohen-Macaulay ring if and only if $R_\wp$ and $\widehat{R}_\wp$ are Cohen-Macaulay rings for any $\wp \in \operatorname{Spec} R$.*
2. *If R is a Cohen-Macaulay ring, then any polynomial ring $R[x_1, \ldots, x_n]$ over R is Cohen-Macaulay.*

3. *If $(R, \mathfrak{m}) \to (S, \mathfrak{n})$ is a surjective homomorphism of local rings there exists a commutative diagram of local rings*

$$\begin{array}{ccc} R & \overset{\phi}{\twoheadrightarrow} & (S, \mathfrak{n}) \\ \downarrow & & \downarrow \\ R' & \overset{\phi'}{\twoheadrightarrow} & (S', \mathfrak{n}') \end{array}$$

with the following properties:

(a) *For $s \geq 2$, there exists an injection $D_s(S|R; -) \to D_s(S'|R'; -)$ which is an isomorphism for $s > 2$.*

(b) *If S is Cohen-Macaulay then S' is Artin and* $\operatorname{depth} R' = 0$.

(c) *R and S are both Cohen-Macaulay if and only if R' and S' are both Artin local rings.*

Proof: For 1., see [8, 2.1.3 and 2.1.8]. For 2., see [8, 2.1.9].

For 3., form a surjection $S \to (S', \mathfrak{n}')$ by quotienting out by the ideal generated by the maximal regular sequence in a minimal generating set for $\mathfrak{n}$. Let $I \subset R$ be the kernel of $R \to S \to S'$. Form a surjection $R \to (R', \mathfrak{m}')$ by, again, quotienting out by an ideal generated by the maximal regular sequence in a minimal generating set for I. Then there is a resulting commuting diagram for which the vertical maps are complete intersections (i.e the kernels are generated by regular sequences).

Now, applying the Jacobi-Zariski sequence [1, V.1] to the diagram above, we get two long exact sequences

$$\ldots \to D_s(S|R; -) \to D_s(S'|R; -) \to D_s(S'|S; -) \to D_{s-1}(S|R; -) \to \ldots$$

and

$$\ldots \to D_s(R'|R; -) \to D_s(S'|R; -) \to D_s(S'|R'; -) \to D_{s-1}(R'|R; -) \to \ldots$$

From [1, VI.26], the long exact sequences reduce to two injections

$$D_s(S|R; -) \to D_s(S'|R; -)$$

and

$$D_s(S'|R; -) \to D_s(S'|R'; -)$$

for $s \geq 2$. Furthermore, the first map is an isomorphism in the same range, as is the second map for $s > 2$. Composing the two gives the desired map.

Next, by 1., S is a Cohen-Macaulay ring if and only if S' is Cohen-Macaulay. Since $\operatorname{depth} S' = 0$ [8, 1.2] we have $\dim S' = 0$, which occurs if and only if S' is Artinian. Cf. [13, §5]. Thus $\mathfrak{n}'$ is nilpotent as an ideal.

Now, if $x \in \mathfrak{m}'$ then $\phi'(x) \in \mathfrak{n}'$ satisfies

$$\phi'(x^t) = \phi'(x)^t = 0 \quad \text{for} \quad t \gg 0.$$

Thus $x^t \in \operatorname{Ker} \phi'$ for $t \gg 0$. From the construction, $\operatorname{depth}(\operatorname{Ker} \phi') = 0$ so we can choose $y \neq 0$ in $\operatorname{Ker} \phi'$ such that $yx^t = 0$ for $t \gg 0$. Choose the smallest $t \geq 1$ such that $yx^t = 0$. Then $u = yx^{t-1} \neq 0$ satisfies $ux = 0$. We conclude that $\operatorname{depth} R' = 0$.

Finally, if both R and S are both Cohen-Macaulay then we can conclude, from 3. (b), that S' is Artin and that $\dim R' = 0$, hence R' is Artin. The converse follows from [8, 2.1.3]. □

4.3. **Proof of Theorem C.** We are now in a position to prove Theorem C. We first give a result which connects the Artinian property on a local ring to the nilpotency of the André operation on Tor.

Theorem 4.3. *Let $(R, \mathfrak{m}, \ell) \to S$ be a surjective homomorphism of local rings of characteristic 2. Then R being Artin implies that the André operation ϑ acts nilpotently on $Q\operatorname{Tor}_*^R(S, \ell)$.*

Proof: Factor $R \to S$ as per (4.13). Let $w \in \mathfrak{m}C_{\geq 2}(\Re)$. Then

$$w = t_1x_1 + \ldots + t_nx_n \quad \text{with} \quad t_1, \ldots, t_n \in \mathfrak{m},\ x_1, \ldots, x_n \in C_{\geq 2}(\Re).$$

From the properties of divided squares, we have

$$\gamma_2^s(w) = t_1^{2^s}\gamma_2^s(x_1) + \ldots t_n^{2^s}\gamma_2^s(x_n) \text{ modulo decomposables.}$$

Since R is an Artin local ring, then $\mathfrak{m}^s = 0$ for $s \gg 0$, by [13, 2.3], hence

$$\gamma_2^s(w) = 0 \text{ modulo decomposables,} \quad s \gg 0.$$

Since γ_2 kills decomposables in positive degrees, we conclude that γ_2 acts nilpotently on $\mathfrak{m}C_{\geq 2}(\Re)$. The result now follows from Lemma 4.1. □

Proof of Theorem C: By [1, S.29], it is enough to show that $D_{\geq 3}(S|R; \ell) = 0$, for any residue field $\ell = S_\mathfrak{n}/\mathfrak{n}S_\mathfrak{n}$, $\mathfrak{n} \in \operatorname{Spec} S$. By the stability of André-Quillen homology under localization [1, V.27] we may assume that $\varphi : R \to (S, \mathfrak{n}, \ell)$ is a homomorphism of local rings. Further, since we are assuming that φ is essentially of finite type, there is a factorization

$$R \xrightarrow{\tau} R[X]_\mathfrak{N} = T \xrightarrow{\sigma} S.$$

as per (0.1). Since τ is faithfully flat, flat base change [1, IV.54] tells us that

$$D_s(T|R; \ell) \cong D_s(\ell[X]|\ell; \ell) = 0 \quad \text{for} \quad s \geq 1.$$

Applying the Jacobi-Zariski sequence [1, V.1], we conclude that

$$D_s(S|R; \ell) \cong D_s(S|T; \ell) \quad \text{for} \quad s \geq 2.$$

Note further that $\operatorname{fd}_T S = \operatorname{fd}_R S$, by a base change spectral sequence argument, and if R is Cohen-Macaulay, then T is also Cohen-Macaulay, by Lemma 4.2.1 and 4.2.2. Thus we may assume that $\varphi = \sigma$.

Now suppose R and S are both Cohen-Maclaulay. Then Lemma 4.2.3 allows us to assume that R is an Artin local ring. Thus 1. follows from Theorem 4.3 and Theorem B.

Finally, if $\operatorname{fd}_R S$ is finite, then $Q\operatorname{Tor}_*^R(S, \ell)$ is finite and, hence, possesses a nilpotent action of γ_2. Thus φ is a complete intersection, by Theorem B, giving us 2. □

References

[1] M. André, *Homologie des algèbres commutatives*, Die Grundlehren der Mathematischen Wissenschaften 206, Springer-Verlag, 1974.

[2] __________, "Puissances divisees des algebres simpliciales en caracteristique deux et series de Poincare de certains anneaux locaux," Manuscripta Math. 18 (1976), 83-108.

[3] __________, "La (2p+1)-ème déviation d'un anneau local," Enseignement Math. (2) 23 (1977), 239-248.

[4] L. Avramov, "Locally complete intersection homomorphisms and a conjecture of Quillen on the vanishing of cotangent homology," Annals of Math. (2) 150 (1999), 455-487.

[5] L. Avramov and S. Iyengar, "André-Quillen homology of algebra retracts," preprint, Purdue University (2001).

[6] L. Avramov and S. Iyengar, "Homological criteria for regular homomorphisms and for locally complete intersection homomorphisms," preprint, Purdue University (2000).

[7] A.K. Bousfield, "Operations on derived functors of non-additive functors," manuscript, Brandeis University 1967.

[8] W. Bruns and J. Herzog, *Cohen-Macaulay rings,* Cambridge University Press, 1998 (revised).

[9] A. Dold, "Homology of symmetric products and other functors of complexes," Ann. of Math. (2) 68 (1958), 40-80.

[10] W. Dwyer, "Homotopy operations for simplicial commutative algebras," Trans. A.M.S. 260 (1980), 421-435.

[11] P. Goerss, *On the André-Quillen cohomology of commutative $\mathbb{F}_2$-algebras*, Astérique 186 (1990).

[12] P. Goerss and T. Lada, "Relations among homotopy operations for simplicial commutative algebras," Proc. A.M.S. 123 (1995), 2637-2641.

[13] H. Matsumura, *Commutative ring theory,* Cambridge University Press, 1986.

[14] H. Miller, "The Sullivan conjecture on maps from classifying spaces," Annals of Math. 120 (1984), 39–87.

[15] D. Quillen, *Homotopical algebra*, Lecture Notes in Mathematics 43, Springer-Verlag, 1967.

[16] __________, "On the (co)homology of commutative rings," Proc. Symp. Pure Math. 17 (1970), 65–87.

[17] __________, "On the homology of commutative rings," Mimeographed Notes, M.I.T.

[18] P. Roberts, Review of "Locally complete intersection homomorphisms and a conjecture of Quillen on the vanishing of cotangent homology," Math. Reviews 2001a:13024.

[19] J. M. Turner, "On simplicial commutative algebras with vanishing André-Quillen homology," Invent. Math. 142 (3) (2000) 547-558.

[20] __________, "Simplicial Commutative F_p-Algebras Through the Looking Glass of F_p-Homotopy Theory", in *Homotopy Invariant Algebraic Structures*, Contemporary Mathematics, Meyer et. al. editors (1999)

Department of Mathematics, Calvin College, 3201 Burton Street, S.E., Grand Rapids, MI 49546

E-mail address: jturner@calvin.edu