Elementary explicit types and polynomial time operations

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Outline

1. Introduction

2. Applicative Base

3. Introduction of theory with types, PET
   - Finite axiomatisation
   - Restricted Elementary Comprehension

4. Lower Bounds

5. Upper bounds

6. Extensions and Further Work
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Introduction

- Explicit Mathematics as introduced by Feferman.
- Weak theories exist for applicative part.
- Up to now, theories with types were of strength at least PRA.

Goal / Question

We want a theory with types, (full) type induction and of strength the polynomial time computable functions. Which types can be allowed to match these requirements?
A function $F : \mathbb{W}^n \rightarrow \mathbb{W}$ is called *provably total in an $\mathcal{L}$ theory* $T$, if there exists a closed $\mathcal{L}$ term $t_F$ such that

(i) $T \vdash t_F : \mathbb{W}^n \rightarrow \mathbb{W}$ and, in addition,

(ii) $T \vdash t_F \overline{w_1} \cdots \overline{w_n} = \overline{F(w_1, \ldots, w_n)}$ for all $w_1, \ldots, w_n$ in $\mathbb{W}$.

$\overline{w}$ for $w \in \mathbb{W}$ means the corresponding standard term.

$T$ is an applicative theory comprising combinatory algebra.
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The Applicative Base

PT was introduced by Thomas Strahm. It is based on the Logic of Partial Terms with the binary words as basic elements.

Logic of Partial Terms

Language $\mathcal{L}$ individual variables $a, b, c, \ldots$; individual constants $k, s, p, p_0, p_1, d_W, \epsilon, s_0, s_1, p_W, c_\subseteq$; constants $*$ and $\times$; binary function symbol $\cdot$, unary relation symbols $\downarrow$ and $W$, binary relation symbol $\equiv$.

Terms Inductively by $\cdot$ from constants and variables.

Formulae Inductively by the usual connectives from relations.

Axioms/Rules Axioms and rules of Hilbert Calculus with equality plus axioms about definedness.
Important Abbreviations

\[ 0 := s_0 \epsilon \]
\[ (s_1, s_2) := pst \]
\[ s \subseteq t := c \subseteq st = 0 \]
\[ s \leq t := l_W s \subseteq l_W t \]
\[ 1 := s_1 \epsilon \]
\[ (s)_i := p_i s \quad (i = 0, 1) \]

\[ W_a(s) := (W(s) \land s \leq a), \]
\[ (\exists x \leq t)A := (\exists x \in W)(x \leq t \land A), \]
\[ (\forall x \leq t)A := (\forall x \in W)(x \leq t \rightarrow A), \]
\[ (t : W \mapsto W) := (\forall x \in W)(tx \in W), \]
\[ (t : W^{m+1} \mapsto W) := (\forall x \in W)(tx : W^m \mapsto W). \]
Axioms of Base Theory B

I Partial combinatory algebra and pairing
Axioms defining the behaviour of the well-known combinators \( k \) and \( s \) and of pairing \( p \) and projections \( p_0 \) and \( p_1 \)

II Definition by cases on \( W \)
\[
d_{W}xyab = \begin{cases} 
  x & a, b \in W \land a = b \\
  y & a, b \in W \land a \neq b 
\end{cases}
\]

III Closure, binary successors and predecessor
\( W \) contains the \( \epsilon \) and is closed under successors \( s_0, s_1 \) and predecessor \( p_W \). Furthermore, \( s_0, s_1 \) and \( p_W \) behave as expected.

IV Initial subword relation
\( c \subseteq \) is a total “predicate” on \( W \). It behaves decently on \( W \), deciding whether the first word is a initial subword of the second.

V Word concatenation and multiplication
\( \ast \) concatenates two words as expected. \( \times xy = X \ast \ldots \ast X \)
\( \text{length of } y \) often
Induction

\[ f : W \leftrightarrow W \land A[\epsilon] \land (\forall x \in W)(A[p_Wx] \rightarrow A[x]) \rightarrow (\forall x \in W)A[x] \ (C\text{-}I_W) \]

where \( A[x] \) belongs to the formula class \( C \)

Definition \((\Sigma^b_W/\Sigma_W^b)\)

A formula \( A[f, x] \) belongs to \( \Sigma^b_W \) (\( \Sigma_W^b \)) if it is of the form 
(\( \exists y \leq fx \)\)\( B[f, x, y] \)) where \( B[f, x, y] \) positive and \( W \)-free (and not containing \( \forall \)).

Theories \( PT \) and \( PT^- \)

\[ PT := B + (\Sigma_W^b\text{-}I_W) \]
\[ PT^- := B + (\Sigma_W^{b^-}\text{-}I_W) \]
Important Properties of PT⁻ (and PT)

**Lemma (\(\lambda\)-Abstraction, Fixpoint)**

*In B, we have \(\lambda\)-abstraction for any term \(t\) and a term \(\text{rec}\) serving as fixed point operator.*

**Theorem**

*The provably total functions of PT⁻ coincide with the functions terminating in polynomial time.*
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Language

“Theory of Types and Names”

The Language $\mathcal{L}_T$ is $\mathcal{L}$ extended with type variables $X, Y, \ldots$, binary relation symbols $\in, \mathcal{R}$ (naming), constants $w, \text{id}, \text{dom}, \text{un}, \text{int}, \text{inv}$

**Additional Shortcuts**

\[
\mathcal{R}(a) := \exists X (\mathcal{R}(a, X))
\]

\[
a \in b := \exists X (\mathcal{R}(b, X) \land a \in X)
\]
Axioms of PET

Axioms are the axioms of B plus the following:

\[ \exists x \mathcal{R}(x, X) \]  
\[ \mathcal{R}(a, X) \land \mathcal{R}(a, Y) \rightarrow X = Y \]  
\[ \forall z (z \in X \leftrightarrow z \in Y) \rightarrow X = Y \]  

(Expl1)  
(Expl2)  
(Expl3)
Axioms of PET

Axioms are the axioms of B plus the following:

\[ \exists x \mathcal{R}(x, X) \]  
(Expl1)

\[ \mathcal{R}(a, X) \land \mathcal{R}(a, Y) \rightarrow X = Y \]  
(Expl2)

\[ \forall z (z \in X \leftrightarrow z \in Y) \rightarrow X = Y \]  
(Expl3)

\[ a \in W \rightarrow \mathcal{R}(w(a)) \land \forall x (x \in w(a) \leftrightarrow W_a(x)) \]  
(\( w_a \))
Axioms of PET

Axioms are the axioms of B plus the following:

\( \exists x R(x, X) \)  
(Expl1)

\( R(a, X) \land R(a, Y) \rightarrow X = Y \)  
(Expl2)

\( \forall z(z \in X \leftrightarrow z \in Y) \rightarrow X = Y \)  
(Expl3)

\( a \in W \rightarrow R(w(a)) \land \forall x(x \in w(a) \leftrightarrow W_a(x)) \)  
\( w_a \)

\( R(id) \land \forall x(x \in id \leftrightarrow \exists y(x = (y, y))) \)  
(id)
Axioms of PET

Axioms are the axioms of B plus the following:

1. \( \exists x \mathcal{R}(x, X) \)  
2. \( \mathcal{R}(a, X) \land \mathcal{R}(a, Y) \rightarrow X = Y \)  
3. \( \forall z (z \in X \leftrightarrow z \in Y) \rightarrow X = Y \)  
4. \( a \in W \rightarrow \mathcal{R}(w(a)) \land \forall x (x \in w(a) \leftrightarrow W_a(x)) \)  
5. \( \mathcal{R}(\text{id}) \land \forall x (x \in \text{id} \leftrightarrow \exists y (x = (y, y))) \)  
6. \( \mathcal{R}(a) \land \mathcal{R}(b) \rightarrow \mathcal{R}(\text{un}(a, b)) \land \forall x (x \in \text{un}(a, b) \leftrightarrow (x \in a \lor x \in b)) \)
Axioms of PET

Axioms are the axioms of B plus the following:

\[ \exists x \mathcal{R}(x, X) \]  
(Expl1)

\[ \mathcal{R}(a, X) \land \mathcal{R}(a, Y) \rightarrow X = Y \]  
(Expl2)

\[ \forall z (z \in X \leftrightarrow z \in Y) \rightarrow X = Y \]  
(Expl3)

\[ a \in W \rightarrow \mathcal{R}(w(a)) \land \forall x (x \in w(a) \leftrightarrow W_a(x)) \]  
(w\(_a\))

\[ \mathcal{R}(\text{id}) \land \forall x (x \in \text{id} \leftrightarrow \exists y (x = (y, y))) \]  
(id)

\[ \mathcal{R}(a) \land \mathcal{R}(b) \rightarrow \mathcal{R}(\text{un}(a, b)) \land \forall x (x \in \text{un}(a, b) \leftrightarrow (x \in a \lor x \in b)) \]  
(un)

\[ \mathcal{R}(a) \land \mathcal{R}(b) \rightarrow \mathcal{R}(\text{int}(a, b)) \land \forall x (x \in \text{int}(a, b) \leftrightarrow (x \in a \land x \in b)) \]  
(int)
Axioms of PET

Axioms are the axioms of B plus the following:

\[ \exists x \mathcal{R}(x, X) \]  \hspace{1cm} (Expl1)

\[ \mathcal{R}(a, X) \land \mathcal{R}(a, Y) \rightarrow X = Y \]  \hspace{1cm} (Expl2)

\[ \forall z (z \in X \leftrightarrow z \in Y) \rightarrow X = Y \]  \hspace{1cm} (Expl3)

\[ a \in W \rightarrow \mathcal{R}(w(a)) \land \forall x (x \in w(a) \leftrightarrow W_a(x)) \]  \hspace{1cm} (w_a)

\[ \mathcal{R}(id) \land \forall x (x \in id \leftrightarrow \exists y (x = (y, y))) \]  \hspace{1cm} (id)

\[ \mathcal{R}(a) \land \mathcal{R}(b) \rightarrow \mathcal{R}(un(a, b)) \land \forall x (x \in un(a, b) \leftrightarrow (x \in a \lor x \in b)) \]  \hspace{1cm} (un)

\[ \mathcal{R}(a) \land \mathcal{R}(b) \rightarrow \mathcal{R}(int(a, b)) \land \forall x (x \in int(a, b) \leftrightarrow (x \in a \land x \in b)) \]  \hspace{1cm} (int)

\[ \mathcal{R}(a) \rightarrow \mathcal{R}(inv(f, a)) \land \forall x (x \in inv(f, a) \leftrightarrow fx \in a) \]  \hspace{1cm} (inv)
Axioms of PET

Axioms are the axioms of B plus the following:

\[ \exists x \mathcal{R}(x, X) \]  \hspace{1cm} (Expl 1)

\[ \mathcal{R}(a, X) \land \mathcal{R}(a, Y) \rightarrow X = Y \]  \hspace{1cm} (Expl 2)

\[ \forall z (z \in X \leftrightarrow z \in Y) \rightarrow X = Y \]  \hspace{1cm} (Expl 3)

\[ a \in W \rightarrow \mathcal{R}(w(a)) \land \forall x (x \in w(a) \leftrightarrow W_a(x)) \]  \hspace{1cm} (w_a)

\[ \mathcal{R}(id) \land \forall x (x \in id \leftrightarrow \exists y (x = (y, y))) \]  \hspace{1cm} (id)

\[ \mathcal{R}(a) \land \mathcal{R}(b) \rightarrow \mathcal{R}(\text{un}(a, b)) \land \forall x (x \in \text{un}(a, b) \leftrightarrow (x \in a \lor x \in b)) \]  \hspace{1cm} (un)

\[ \mathcal{R}(a) \land \mathcal{R}(b) \rightarrow \mathcal{R}(\text{int}(a, b)) \land \forall x (x \in \text{int}(a, b) \leftrightarrow (x \in a \land x \in b)) \]  \hspace{1cm} (int)

\[ \mathcal{R}(a) \rightarrow \mathcal{R}(\text{inv}(f, a)) \land \forall x (x \in \text{inv}(f, a) \leftrightarrow fx \in a) \]  \hspace{1cm} (inv)

\[ \mathcal{R}(a) \rightarrow \mathcal{R}(\text{dom}(a)) \land \forall x (x \in \text{dom}(a) \leftrightarrow \exists y ((x, y) \in a)) \]  \hspace{1cm} (dom)
Axioms of PET

Axioms are the axioms of $B$ plus the following:

$$\exists x \mathcal{R}(x, X) \quad \text{(Expl1)}$$

$$\mathcal{R}(a, X) \land \mathcal{R}(a, Y) \rightarrow X = Y \quad \text{(Expl2)}$$

$$\forall z (z \in X \leftrightarrow z \in Y) \rightarrow X = Y \quad \text{(Expl3)}$$

$$a \in W \rightarrow \mathcal{R}(\text{w}(a)) \land \forall x (x \dot{\in} \text{w}(a) \leftrightarrow \text{W}_a(x)) \quad \text{(w}_a)$$

$$\mathcal{R}(\text{id}) \land \forall x (x \dot{\in} \text{id} \leftrightarrow \exists y (x = (y, y))) \quad \text{(id)}$$

$$\mathcal{R}(a) \land \mathcal{R}(b) \rightarrow \mathcal{R}(\text{un}(a, b)) \land \forall x (x \dot{\in} \text{un}(a, b) \leftrightarrow (x \dot{\in} a \lor x \dot{\in} b)) \quad \text{(un)}$$

$$\mathcal{R}(a) \land \mathcal{R}(b) \rightarrow \mathcal{R}(\text{int}(a, b)) \land \forall x (x \dot{\in} \text{int}(a, b) \leftrightarrow (x \dot{\in} a \land x \dot{\in} b)) \quad \text{(int)}$$

$$\mathcal{R}(a) \rightarrow \mathcal{R}(\text{inv}(f, a)) \land \forall x (x \dot{\in} \text{inv}(f, a) \leftrightarrow fx \dot{\in} a) \quad \text{(inv)}$$

$$\mathcal{R}(a) \rightarrow \mathcal{R}(\text{dom}(a)) \land \forall x (x \dot{\in} \text{dom}(a) \leftrightarrow \exists y ((x, y) \dot{\in} a)) \quad \text{(dom)}$$

$$\epsilon \in X \land (\forall x \in W)(p_W x \in X \rightarrow x \in X) \rightarrow (\forall x \in W)(x \in X) \quad \text{(T-l}_W)$$
Comprehension: Preparations

Definition (Class of $\Sigma^b_T$ formulas and set of variables $FV_w(A)$)

- $A \equiv (s = t), s \downarrow$ or $(s \in X)$: $A$ is a $\Sigma^b_T$ formula and $FV_w(A) := \emptyset$.
- $A \equiv W_a(t)$: $A$ is a $\Sigma^b_T$ formula and $FV_w(A) := \{a\}$ if $a \notin FV_I(t)$.
- $A \equiv (B \land C)$ or $(B \lor C)$ with $B$ and $C$ in $\Sigma^b_T$ and if no conflict arises between $FV_I$ and $FV_w$, then $A$ is a $\Sigma^b_T$ formula and $FV_w(A) := FV_w(B) \cup FV_w(C)$.
- $A \equiv \exists x B$ with $B \in \Sigma^b_T$ and $x \notin FV_w(B)$, then $A$ is a $\Sigma^b_T$ formula and $FV_w(A) := FV_w(B)$. 

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Comprehension: Preparations

**Definition** \((\rho_A(B, x))\)

For a \(\Sigma^b_T\) formula \(A\), we define a term \(\rho_A(B, x)\) by induction on the complexity of \(B\) in \(\Sigma^b_T\), where \(x \not\in FV_W(B)\) and \(x\) not bound in \(B\):

\[
\begin{align*}
\rho_A(s = t, x) & := \text{inv}(\lambda x.(s, t), \text{id}), \\
\rho_A(s \downarrow, x) & := \text{inv}(\lambda x.(s, s), \text{id}), \\
\rho_A(s \in W_a, x) & := \text{inv}(\lambda x.s, \text{w}(a)), \\
\rho_A(s \in X, x) & := \text{inv}(\lambda x.s, \mu_A(X)), \\
\rho_A(C \land D, x) & := \text{int}(\rho_A(C, x), \rho_A(D, x)) \\
\rho_A(C \lor D, x) & := \text{un}(\rho_A(C, x), \rho_A(D, x)) \\
\rho_A(\exists y C, x) & := \text{dom}(\rho_A(C[(x)_0/x, (x)_1/y], x)).
\end{align*}
\]

where \(\mu_A(X)\) assigns an individual variable not occurring in \(A\) to the free type variable \(X\).
Restricted Elementary Comprehension

**Theorem (Restricted elementary comprehension in PET)**

For $A$ a $\Sigma^b_T$ formula with $FV_T(A) = \{X_1, \ldots, X_n\}$ and $FV_W(A) = \{w_1, \ldots, w_m\}$. Let $z_i := \mu_A(X_i)(1 \leq i \leq n)$ and $ho_{A,x} := \rho_A(A,x)$, then we have:

1. $FV_I(\rho_{A,x}) = (FV_I(A) \setminus \{x\}) \cup \{z_1, \ldots, z_n\}$,
2. $\text{PET} \vdash W(\vec{w}) \land R(\vec{z}, \vec{X}) \rightarrow R(\rho_{A,x})$,
3. $\text{PET} \vdash W(\vec{w}) \land R(\vec{z}, \vec{X}) \rightarrow (\forall x)(x \in \rho_{A,x} \iff A)$.

**Remark**

As a consequence of comprehension and type induction, induction is available for $\Sigma^b_T$ formulae.
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Lower Bounds: Preparatory Work

Lemma (Properties of the subword relation)

The following statements are provable in PET:

1. \( x \in W \land z \in W \land x \subseteq p_W z \rightarrow x \subseteq z \),
2. \( x \in W \land y \in W \land z \in W \land x \subseteq y \land y \subseteq z \rightarrow x \subseteq z \) (Transitivity),
3. \( x \in W \land y \in W \land x \subseteq y \rightarrow x \leq y \).
Lower Bounds: “Bounded Induction”

\[ a \in W \wedge \epsilon \in X \wedge (\forall x \subseteq a)(p_{Wx} \in X \rightarrow x \in X) \rightarrow a \in X \quad (T-I^b_W) \]

**Lemma**

We have that \((T-I_W)\) and \((T-I^b_W)\) are provably equivalent in PET without \((T-I_W)\).
Bounding Functions \( f : W \mapsto W \)

**Lemma**

There is a closed term \( \text{max} \) such that PET proves:

1. \( f : W \mapsto W \mapsto \text{maxf} : W \mapsto W, \)
2. \( f : W \mapsto W \land f^* = \text{maxf} \land x \in W \land y \in W \land x \subseteq y \rightarrow f^*x \leq f^*y), \)
3. \( f : W \mapsto W \land f^* = \text{maxf} \land x \in W \rightarrow fx \leq f^*x), \)
4. \( f : W \mapsto W \land f^* = \text{maxf} \land x \in W \land y \in W \land x \subseteq y \rightarrow fx \leq f^*y). \)
Bounding Functions \( f : W \mapsto W \)

**Lemma**

There is a closed term \( \text{max} \) such that PET proves:

1. \( f : W \mapsto W \mapsto \text{max} f : W \mapsto W \),
2. \( f : W \mapsto W \land f^* = \text{max} f \land x \in W \land y \in W \land x \subseteq y \rightarrow f^* x \leq f^* y \),
3. \( f : W \mapsto W \land f^* = \text{max} f \land x \in W \rightarrow f x \leq f^* x \),
4. \( f : W \mapsto W \land f^* = \text{max} f \land x \in W \land y \in W \land x \subseteq y \rightarrow f x \leq f^* y \).

**Proof (Sketch)**

\( \text{max} := \lambda f. \lambda x. f(\text{max}_{\text{arg}} f x) \) where \( \text{max}_{\text{arg}} \) is a functional detecting the argument maximising the function \( f \) up to \( x \).

Proof of (1) by proving \( f : W \mapsto W \mapsto \text{max}_{\text{arg}} f : W \mapsto W \) by \( (T \text{-} l^b_W) \) on \( (\exists y \leq a)((\text{max}_{\text{arg}} f)x = y) \). Proof of (2) and (3) by induction.
Lower Bounds

Theorem

\( \text{PT}^- \) is contained in \( \text{PET} \).
Lower Bounds

Theorem

PT− is contained in PET.

Proof

Steps of proving PET ⊢ (ΣWb−-Iw):

Lower Bounds

Theorem

$PT^{-}$ is contained in PET.

Proof

Steps of proving PET ⊢ $(\Sigma^b_W - I_W)$:

1. Take $\Sigma^b_W$ formula $A[x] \equiv (\exists y \leq fx)B[f, x, y]$ and assume $f : W \rightarrow W \land A[\epsilon] \land (\forall x \in W)(A[x] \rightarrow A[s_0x] \land A[s_1x])$

2. Prove $(\exists y \leq fx)B[x, y] \leftrightarrow (\exists y \leq f^*c)(y \leq fx \land B[x, y])$ for $c \in W$ and $x \subseteq c$. 
**Theorem**

\(\text{PT}^-\) is contained in \(\text{PET}\).

**Proof**

Steps of proving \(\text{PET} \vdash (\Sigma^b_W - \text{I}_W)\):

1. Take \(\Sigma^b_W\) formula \(A[x] \equiv (\exists y \leq fx) B[f, x, y]\) and assume \(f : W \rightarrow W \land A[\epsilon] \land (\forall x \in W)(A[x] \rightarrow A[s_0 x] \land A[s_1 x])\).

2. Prove \((\exists y \leq fx) B[x, y] \leftrightarrow (\exists y \leq f^* c)(y \leq fx \land B[x, y])\) for \(c \in W\) and \(x \subseteq c\).

3. With Comprehension we can construct type \(X\) such that \((\forall x \subseteq c)(x \in X \leftrightarrow (\exists y \leq f^* c)(y \leq fx \land B[x, y]))\) as \(f^* c \in W\).
Lower Bounds

Theorem

\( \text{PT}^- \) is contained in \( \text{PET} \).

Proof

Steps of proving \( \text{PET} \vdash (\Sigma_W^- l_W) \):

1. Take \( \Sigma_W^- \) formula \( A[x] \equiv (\exists y \leq fx)B[f, x, y] \) and assume \( f : W \to W \land A[\epsilon] \land (\forall x \in W)(A[x] \to A[s_0x] \land A[s_1x]) \).

2. Prove \( (\exists y \leq fx)B[x, y] \leftrightarrow (\exists y \leq f^*c)(y \leq fx \land B[x, y]) \) for \( c \in W \) and \( x \subseteq c \).

3. With Comprehension we can construct type \( X \) such that \( (\forall x \subseteq c)(x \in X \leftrightarrow (\exists y \leq f^*c)(y \leq fx \land B[x, y])) \) as \( f^*c \in W \).

4. \( \epsilon \in X \land (\forall x \subseteq c)(p_Wx \in X \to x \in X) \) immediate from above.
Lower Bounds

Theorem

\( \text{PT}^- \) is contained in PET.

Proof

Steps of proving PET \( \vdash (\Sigma^b_W - \text{l}_W) \):

1. Take \( \Sigma^b_W \) formula \( A[x] \equiv (\exists y \leq fx)B[f, x, y] \) and assume \( f : W \rightarrow W \land A[\epsilon] \land (\forall x \in W)(A[x] \rightarrow A[s_0x] \land A[s_1x]) \)

2. Prove \( (\exists y \leq fx)B[x, y] \leftrightarrow (\exists y \leq f^*c)(y \leq fx \land B[x, y]) \) for \( c \in W \) and \( x \subseteq c \).

3. With Comprehension we can construct type \( X \) such that \( (\forall x \subseteq c)(x \in X \leftrightarrow (\exists y \leq f^*c)(y \leq fx \land B[x, y])) \) as \( f^*c \in W \)

4. \( \epsilon \in X \land (\forall x \subseteq c)(p_Wx \in X \rightarrow x \in X) \) immediate from above.

5. By \((T\text{-l}_W^b): c \in X\)
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Structures for PET

**Definition (\(\mathcal{L}_T\) structure)**

A \(\mathcal{L}_T\)-structure \(\mathcal{M}^*\) is a tuple

\[
(\mathcal{M}, T, E, R, w, id, dom, un, int, inv)
\]

where (i) \(\mathcal{M}\) is a \(\mathcal{L}\)-structure, (ii) \(T\) is a non-empty set of subsets of \(|\mathcal{M}|\), (iii) \(E\) is the usual \(\in\) relation on \(|\mathcal{M}| \times T\), (iv) \(R\) is a non-empty subset of \(|\mathcal{M}| \times T\), and (v) \(w, id, dom, un, int, inv\) are elements of \(|\mathcal{M}|\).
Model Construction

- Take model $M$ of $\text{PT}^\neg$.
Model Construction

- Take model $\mathcal{M}$ of $\text{PT}^\neg$.
- Choose decent interpretation for constants.
Model Construction

- Take model $\mathcal{M}$ of $\text{PT}^-$. 
- Choose decent interpretation for constants.
- Inductively define 
  \[ T_k := \{ \text{ext}(m) : m \in R_k \}, \quad R_k := \{ (m, \text{ext}(m)) : m \in R_k \}, \]
  \[ \mathcal{M}^*_k := (\mathcal{M}, T_k, R_k, w, \text{id}, \text{dom}, \text{un}, \text{int}, \text{inv}) \text{ where } R_k \subseteq |\mathcal{M}| \text{ and } \]
  \[ \text{ext}(m) \subseteq |\mathcal{M}| \text{ for } m \in R_k : \]
Model Construction

- Take model $\mathcal{M}$ of $\text{PT}^-$. 
- Choose decent interpretation for constants.
- Inductively define
  \[
  T_k := \{ \text{ext}(m) : m \in R_k \}, \quad R_k := \{ (m, \text{ext}(m)) : m \in R_k \},
  \]
  \[
  \mathcal{M}_k^* := (\mathcal{M}, T_k, R_k, w, \text{id}, \text{dom}, \text{un}, \text{int}, \text{inv}) \text{ where } R_k \subseteq |\mathcal{M}| \text{ and }
  \]
  \[
  \text{ext}(m) \subseteq |\mathcal{M}| \text{ for } m \in R_k:
  \]
  \[
  k = 0 \quad R_0 := \{ \text{id} \} \cup \{ wa : a \in W^\mathcal{M} \} \]
Model Construction

- Take model $\mathcal{M}$ of PT$^-$.  
- Choose decent interpretation for constants.  
- Inductively define

  $T_k := \{ \text{ext}(m) : m \in R_k \}, \ R_k := \{ (m, \text{ext}(m)) : m \in R_k \},$

  $M^*_k := (\mathcal{M}, T_k, R_k, w, \text{id}, \text{dom}, \text{un}, \text{int}, \text{inv})$ where $R_k \subseteq |\mathcal{M}|$ and $\text{ext}(m) \subseteq |\mathcal{M}|$ for $m \in R_k$:

  $k = 0 \quad R_0 := \{ \text{id} \} \cup \{ wa : a \in W^\mathcal{M} \}$

  $k > 0 \quad R_k := \{ \text{un}(a, b), \text{int}(a, b) : a, b \in R_{k-1} \}$

  $\cup \{ \text{inv}(f, a), \text{dom}a : a \in R_{k-1} \}$ and e.g.

  $\text{ext}(\text{un}(a, b)) := \{ m \in |\mathcal{M}| : M^*_{k-1} \models m \in a \lor m \in b \}$
Model Construction

- Take model $\mathcal{M}$ of $\text{PT}^−$.
- Choose decent interpretation for constants.
- Inductively define
  \[ T_k := \{ \text{ext}(m) : m \in R_k \}, \quad \mathcal{R}_k := \{ (m, \text{ext}(m)) : m \in R_k \}, \]
  \[ \mathcal{M}^*_k := (\mathcal{M}, T_k, \mathcal{R}_k, w, \text{id}, \text{dom}, \text{un}, \text{int}, \text{inv}) \] where $R_k \subseteq |\mathcal{M}|$ and $\text{ext}(m) \subseteq |\mathcal{M}|$ for $m \in R_k$:
  \[ k = 0 \quad R_0 := \{ \text{id} \} \cup \{ wa : a \in W^\mathcal{M} \} \]
  \[ k > 0 \quad R_k := \{ \text{un}(a, b), \text{int}(a, b) : a, b \in R_{k-1} \} \]
  \[ \cup \{ \text{inv}(f, a), \text{dom}a : a \in R_{k-1} \} \] and e.g.
  \[ \text{ext}(\text{un}(a, b)) := \{ m \in |\mathcal{M}| : \mathcal{M}^*_{k-1} \models m \in a \lor m \in b \} \]
- Set $\mathcal{T} := \bigcup_{k \in \mathbb{N}} T_k$ and $\mathcal{R} := \bigcup_{k \in \mathbb{N}} \mathcal{R}_k$. 
Upper Bounds

Theorem (Model extension)

Any model $\mathcal{M}^*$ constructed as described above from a model $\mathcal{M}$ of $\text{PT}^-$ satisfies the following conditions:

1. $\mathcal{M} \models A \iff \mathcal{M}^* \models A$ for any $\mathcal{L}$ sentence $A$,
2. $\mathcal{M}^* \models \text{T-I}_W$,
3. $\mathcal{M}^* \models \text{PET}$.
Upper Bounds

Theorem (Model extension)

Any model $\mathcal{M}^*$ constructed as described above from a model $\mathcal{M}$ of $\text{PT}^-$ satisfies the following conditions:

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2. $\mathcal{M}^* \models \text{T-l}_{W}$,
3. $\mathcal{M}^* \models \text{PET}$.

Proof of 2

We show that every type $X \in T$ is weakly $\Sigma^b_W$ definable, i.e. that $X = \{ m \in |\mathcal{M}| : \mathcal{M} \models A[m] \}$ for $A \Sigma^b_W$ formula with a fixed bound. This is proved by induction on the level $k$ when $X$ is added to $R_k$. 

Outline

1. Introduction
2. Applicative Base
3. Introduction of theory with types, PET
   - Finite axiomatisation
   - Restricted Elementary Comprehension
4. Lower Bounds
5. Upper bounds
6. Extensions and Further Work
Uniformity

Cantini showed that the Uniformity Principle $\text{UP}$ can be added to PT without strengthening the theory.

$$ (\forall x)(\exists y \in W)A(x, y) \rightarrow (\exists y \in W)(\forall x)A(x, y) \text{ for } A \text{ positive} \quad (\text{UP}) $$

$\text{UP}$ entails the following bounded uniformity axiom:

$$ \forall x(\exists y \leq t)A[x, y] \rightarrow (\exists y \leq t)(\forall x)A[x, y] \text{ for } A \text{ positive} \quad (\text{UP}') $$

In the presence of $\text{UP}'$ we can add an universal type to PET:

$$ \mathbb{R}(a) \rightarrow \mathbb{R}(\text{all } a) \land \forall x(x \in \text{ all } a \leftrightarrow \forall y(\langle x, y \rangle \in a)) \quad (\text{all}) $$

**Lemma**

$\text{PT} + (\text{UP})$ is contained in $\text{PET} + (\text{all})$ and $\text{PET} + (\text{all})$ is a conservative extension of $\text{PT} + (\text{UP})$ for closed $\mathcal{L}$ formulae.
Other Extensions

Choice \((\forall x \in W)(\exists y \in W)A(x, y) \rightarrow (\exists f : W \rightarrow W)(\forall x \in W)A(x, fx)\)

for \(A\) positive and containing type variables only in the form \(t \in X\)

Totality \(\forall x \forall y (xy \downarrow)\)

Extensionality \(\forall f \forall g (\forall x (fx \simeq gx) \rightarrow f = g)\)

Theorem

The provably total functions of PET augmented by any combination of the principles (all), Choice, Totality, and Extensionality coincide with the polynomial time computable functions.
Further Work

We are currently studying the addition of disjoint join:

\[
\text{Join: } \mathcal{R}(a) \land f \mapsto \mathcal{R} \rightarrow \mathcal{R}(j(a, f)) \land \\
\forall x (x \in j(a, f) \leftrightarrow (x)_0 \in a \land (x)_1 \in f(x)_0)
\]

Furthermore, we plan to study weak theories of partial (self referential) truth.