A Uniform Distribution Question Related
to Numerical Analysis

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Abstract. Using the theory of uniform distribution modulo one, it is shown that
under certain conditions on the real-valued functions $\alpha(x)$ and $g(x)$ on $[0, 1]$,

$$h \sum_{\gamma=1}^{[h^{-1}]} \{h^{-1}\alpha(\gamma h)\}^m g(\gamma h) = (m + 1)^{-1} \int_0^1 g(x)\,dx + O\left(h^{1/3}\log\frac{1}{h}\right) \quad \text{as } h \to 0+,$$

where $m > 0$ and $\{x\}$ denotes the fractional part of x. The conditions are as fol-
lows: $\alpha''(x)$ exists for all but finitely many points in $[0, 1]$, changes sign at most
finitely often, and is bounded away in absolute value from both 0 and ∞, whereas
$g(x)$ is of bounded variation on $[0, 1]$. Also, under these conditions on $\alpha(x)$,

$$h \sum_{\gamma=1}^{[h^{-1}]} \{h^{-1}\alpha(\gamma h)\}^m = (m + 1)^{-1} + O(h^{1/3}) \quad \text{as } h \to 0+.$$

These results, which are, in fact, deduced from somewhat more general proposi-
tions, answer questions of Feldstein connected with discretization methods for
differential equations.

In connection with algorithms for solving retarded ordinary differential equations
by discretization methods, Feldstein [1] was led to search for conditions on the real-
valued functions $\alpha(x)$ and $g(x)$ defined on $[0, 1]$ which guarantee that

$$\lim_{h\to 0+} \int_0^1 \{h^{-1}\alpha(x)\}^m g(x)\,dx = \frac{1}{m+1} \int_0^1 g(x)\,dx$$

for $m > 0$, where $\{x\} = x - [x]$ is the fractional part function. Sufficient conditions
of a quite general nature were given in [1]. The question was raised as to what hap-
pens when the integral on the left-hand side is replaced by a Riemann sum of mesh
size h. We address ourselves to this problem. Thus, we inquire about conditions on
$\alpha(x)$ and $g(x)$ in order for

(1)
$$\lim_{h\to 0+} h \sum_{\gamma=1}^{[h^{-1}]} \{h^{-1}\alpha(\gamma h)\}^m g(\gamma h)$$

to exist and equal $(m + 1)^{-1}\int_0^1 g(x)\,dx$, and furthermore, what could be said about
the rate of convergence.

Since it is straightforward to check that for $m > 0$, $g(x) \equiv 1$, and $\alpha(x) = x + a$
with a constant $a \neq 0$, the limit in (1) fails to exist, it is clear that some restrictions
on the derivatives of $\alpha(x)$ are to be expected. Actually, the simple condition on the
second derivative of $\alpha(x)$ enunciated in Theorem 1 below will suffice (compare with

Corollary 2). The regularity conditions on $g(x)$ are rather mild, since $g(x)$ need only be of bounded variation on $[0, 1]$. If these requirements are met, an estimate for the remainder term can also be established. The condition on $\alpha''(x)$ can even be relaxed somewhat (see the remark following Corollary 3).

The reader may check that, using the proofs presented below, the implied constant in the error estimate in Corollary 2 may be effectively computed in terms of the upper and lower bounds on $|\alpha''(x)|$, the total variation of $g(x)$ on $[0, 1]$, and $\sup_{0 \leqslant x \leqslant 1}|g(x)|$, and in Corollary 3 in terms of the upper and lower bounds on $|\alpha''(x)|$ alone. It is interesting to note that the constants turn out to be independent of m. This is due to the fact that the function x^m has total variation 1 on $[0, 1]$, no matter which $m > 0$ we consider.

The results concerning (1) are deduced from somewhat more general statements contained in Theorem 1 and, for $g(x) \equiv 1$, in Theorem 2. In the proofs of these theorems, we need the notions of discrepancy of a finite sequence of points in $[0, 1]^2$ and in $[0, 1]$, respectively, for which we refer to [3, Chapter 2, Section 1]. We recall also that a function $F(x, y)$ on $[0, 1]^2$ is said to be of bounded variation in the sense of Hardy and Krause if it is of bounded variation on $[0, 1]^2$ in the sense of Vitali (see [2] and [3, Definition 5.1, p. 147]), and if the four functions $F(x, 0)$, $F(x, 1)$, $F(0, y)$, and $F(1, y)$ are of bounded variation on $[0, 1]$. We shall use Vinogradov's notation $\ll$ and Landau symbols interchangeably.

THEOREM 1. *Let $F(x, y)$ be of bounded variation on $[0, 1]^2$ in the sense of Hardy and Krause, and let $\alpha(x)$ be twice differentiable on $[0, 1]$ with $A \leqslant \alpha''(x) \leqslant B$ for $0 \leqslant x \leqslant 1$ or $-B \leqslant \alpha''(x) \leqslant -A$ for $0 \leqslant x \leqslant 1$ and positive constants A and B. Then,*

$$h \sum_{\gamma=1}^{[h^{-1}]} F(\{h^{-1}\alpha(\gamma h)\}, \gamma h) = \int_0^1 \int_0^1 F(x, y)\,dx\,dy + O\left(h^{1/3}\log\frac{1}{h}\right) \quad as\ h \to 0+.$$

PROOF. Let h be sufficiently small, say $0 < h \leqslant 1/8$, and let $M = [h^{-1/3}]$. For integers j and k we set

(2)
$$S_{j,k} = \left| \frac{1}{[h^{-1}]} \sum_{\gamma=1}^{[h^{-1}]} \exp[2\pi i(jh^{-1}\alpha(\gamma h) + k\gamma h)] \right|.$$

If $1 \leqslant |j| \leqslant M$, we write

(3)
$$S_{j,k} = \left| \frac{1}{[h^{-1}]} \sum_{\gamma=1}^{[h^{-1}]} \exp[2\pi i q(\gamma)] \right|,$$

where the function q is defined by $q(t) = jh^{-1}\alpha(ht) + kht$ for $1 \leqslant t \leqslant [h^{-1}]$. Of course, q depends also on h, j, and k, but we suppress this dependence in the notation. We observe that $q''(t) = jh\alpha''(ht)$, so that $q''(t)$ does not change sign and $|q''(t)| \geqslant |j|hA > 0$. Therefore, by (3) and an inequality of van der Corput (see [3, Chapter 1, Theorem 2.7] and [5]) we have

$$S_{j,k} \leqslant [h^{-1}]^{-1}(|q'([h^{-1}]) - q'(1)| + 2)(4(|j|hA)^{-1/2} + 3)$$

$$= [h^{-1}]^{-1}(|j\alpha'(h[h^{-1}]) - j\alpha'(h)| + 2)(4(|j|hA)^{-1/2} + 3).$$

From $|\alpha''(x)| \leqslant B$ and the mean-value theorem we infer

$$|j\alpha'(h[h^{-1}]) - j\alpha'(h)| \leqslant |j|B.$$

Since $|j|h \leqslant Mh \leqslant h^{2/3} \leqslant 1/4$, we have

$$4(|j|hA)^{-1/2} + 3 \leqslant (4A^{-1/2} + 3/2)(|j|h)^{-1/2};$$

and so,

(4) $$S_{j,k} \ll (|j|h)^{1/2} \quad \text{for} \quad 1 \leqslant |j| \leqslant M \text{ and all } k,$$

where the implied constant is independent of h, j, and k. For $j = 0$ and $1 \leqslant |k| \leqslant M$, we have

$$S_{0,k} = \left| \frac{1}{[h^{-1}]} \sum_{\gamma=1}^{[h^{-1}]} e^{2\pi ik\gamma h} \right| = \frac{|e^{2\pi ikh[h^{-1}]} - 1|}{[h^{-1}]|e^{2\pi ikh} - 1|}.$$

Now $|k|h \leqslant Mh \leqslant h^{2/3} \leqslant 1/4$ and $|k| - |k|h < |k|h[h^{-1}] \leqslant |k|$, so that for certain δ and θ with $0 \leqslant \delta < \theta \leqslant 1/4$ we have

$$S_{0,k} = |e^{2\pi i\delta} - 1|/[h^{-1}]|e^{2\pi i\theta} - 1|;$$

and hence,

(5) $$S_{0,k} < [h^{-1}]^{-1} \ll h \quad \text{for } 1 \leqslant |k| \leqslant M.$$

Let $r(a) = \max(1, |a|)$ for $a \in \mathbf{Z}$. By a result on quantitative Fourier inversion (see [4]), the discrepancy $D_{[h^{-1}]}$ of the points $(\{h^{-1}\alpha(\gamma h)\}, \gamma h)$, $\gamma = 1, 2, \ldots, [h^{-1}]$, in $[0, 1]^2$ satisfies

$$D_{[h^{-1}]} \ll \frac{1}{M} + \sum_{|j| \leqslant M; |k| \leqslant M}^{*} \frac{1}{r(j)r(k)} S_{j,k},$$

where the asterisk indicates that $(j, k) = (0, 0)$ is deleted from the range of summation and where the implied constant is absolute. It follows now from (4) and (5) that

$$D_{[h^{-1}]} \ll \frac{1}{M} + \sum_{1 \leqslant |j| \leqslant M; |k| \leqslant M} \frac{1}{|j|r(k)} S_{j,k} + \sum_{1 \leqslant |k| \leqslant M} \frac{1}{|k|} S_{0,k}$$

$$\ll \frac{1}{M} + h^{1/2} \sum_{1 \leqslant |j| \leqslant M} |j|^{-1/2} \sum_{|k| \leqslant M} \frac{1}{r(k)} + h \sum_{1 \leqslant |k| \leqslant M} \frac{1}{|k|}$$

$$\ll \frac{1}{M} + h^{1/2}M^{1/2}\log M + h \log M \ll h^{1/3} + h^{1/3} \log \frac{1}{h} \ll h^{1/3} \log \frac{1}{h},$$

by the definition of M. According to [3, Chapter 2, Theorem 5.5], we get

$$\frac{1}{[h^{-1}]} \sum_{\gamma=1}^{[h^{-1}]} F(\{h^{-1}\alpha(\gamma h)\}, \gamma h) = \int_0^1 \int_0^1 F(x, y)\,dx\,dy + O(h^{1/3}\log 1/h).$$

Since F is bounded, we have

$$h \sum_{\gamma=1}^{[h^{-1}]} F(\{h^{-1}\alpha(\gamma h)\}, \gamma h) = \frac{1}{[h^{-1}]} \sum_{\gamma=1}^{[h^{-1}]} F(\{h^{-1}\alpha(\gamma h)\}, \gamma h) + O(h),$$

and the desired result follows.

COROLLARY 1. *Let $f(x)$ and $g(x)$ be of bounded variation on $[0, 1]$, and let $\alpha(x)$ satisfy the conditions of Theorem 1. Then,*

$$h \sum_{\gamma=1}^{[h^{-1}]} f(\{h^{-1}\alpha(\gamma h)\})g(\gamma h)$$
$$= \left(\int_0^1 f(x)\,dx\right)\left(\int_0^1 g(x)\,dx\right) + O(h^{1/3}\log 1/h) \quad as \ h \to 0+.$$

PROOF. It suffices to note that $F(x, y) = f(x)g(y)$ is of bounded variation on $[0, 1]^2$ in the sense of Hardy and Krause. The rest follows from Theorem 1.

COROLLARY 2. *Let $m > 0$ be real, and let $g(x)$ and $\alpha(x)$ be as in Corollary 1. Then,*

$$h \sum_{\gamma=1}^{[h^{-1}]} \{h^{-1}\alpha(\gamma h)\}^m g(\gamma h) = \frac{1}{m+1}\int_0^1 g(x)\,dx + O(h^{1/3}\log 1/h) \quad as \ h \to 0+.$$

PROOF. Take $f(x) = x^m$ in Corollary 1.

In the special case where $g(x)$ is constant, say $g(x) \equiv 1$, slight improvements on the above results can be obtained.

THEOREM 2. *Let $f(x)$ be of bounded variation on $[0, 1]$, and let $\alpha(x)$ be as in Theorem 1. Then,*

$$h \sum_{\gamma=1}^{[h^{-1}]} f(\{h^{-1}\alpha(\gamma h)\}) = \int_0^1 f(x)\,dx + O(h^{1/3}) \quad as \ h \to 0+.$$

PROOF. Let h with $0 < h \le 1/8$ be given. With $S_{j,k}$ defined by (2) and $M = [h^{-1/3}]$, the discrepancy $D_{[h^{-1}]}$ of the points $\{h^{-1}\alpha(\gamma h)\}$, $\gamma = 1, 2, \ldots, [h^{-1}]$, in $[0, 1]$ satisfies

$$D_{[h^{-1}]} \ll \frac{1}{M} + \sum_{j=1}^{M} \frac{1}{j} S_{j,0}$$

according to an inequality in [4], where the implied constant is absolute. It follows from (4) that

$$D_{[h^{-1}]} \ll \frac{1}{M} + h^{1/2} \sum_{j=1}^{M} j^{-1/2} \ll \frac{1}{M} + h^{1/2}M^{1/2} \ll h^{1/3}.$$

The proof is now completed as in Theorem 1, by using [3, Chapter 2, Theorem 5.1].

COROLLARY 3. *Let $m > 0$ be real, and let $\alpha(x)$ be as in Theorem 1. Then,*

$$h \sum_{\gamma=1}^{[h^{-1}]} \{h^{-1}\alpha(\gamma h)\}^m = \frac{1}{m+1} + O(h^{1/3}) \quad as \ h \to 0+.$$

In all of the above results, the conditions on $\alpha(x)$ may be relaxed somewhat by assuming only that $\alpha''(x)$ exists for all but finitely many points in $[0, 1]$, that $0 < A \le |\alpha''(x)| \le B$ whenever $\alpha''(x)$ exists, and that $\alpha''(x)$ changes sign at most finitely often in $[0, 1]$. This will only affect the estimation of (3), where one now splits up the sum into a fixed number of subsums in such a way that van der Corput's inequality can be.

applied to each subsum containing more than one term.

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