On the Tractability of Un/Satisfiability

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Abstract

The Petri net approach proves to be effective to tackle the \( \mathcal{P} \) vs \( \mathcal{NP} \) problem. A safe acyclic Petri net (PN) is associated with an Exactly-1 3SAT formula, in which a clause is an exactly-1 disjunction \( \hat{\vee} \) of literals. A clause also corresponds to a set of conflicting transitions in the PN. Some 2SAT/XOR-SAT formula arisen in the \textit{inversed} PN checks if the truth assignment of a literal (a transition firing) \( z_v \) is “incompatible” for the satisfiability of the 3SAT formula (the reachability of the target state in the \textit{inversed} PN). If \( z_v \) is incompatible, then \( z_v \) is discarded and \( x_v \) becomes true. Therefore, a clause \( (\tau_u \hat{\vee} z_i \hat{\vee} z_j) \) reduces to the conjunction \( (\tau_u \land \tau_i \land \tau_j) \), and a 3-literal clause \( (z_v \hat{\vee} z_u \hat{\vee} z_x) \) reduces to the 2-literal clause \( (z_u \oplus z_x) \). This reduction facilitates checking un/satisfiability; the 3SAT formula is un/satisfiable iff the target state of the \textit{inversed} PN is un/reachable. The solution complexity is \( O(n^5) \). Therefore, it is the case that \( \mathcal{P} = \mathcal{NP} = \text{coNP} \).

1 Introduction

It is well known that if an \( \mathcal{NP} \)-complete problem is tractable, then all \( \mathcal{NP} \)-complete problems are tractable, i.e., \( \mathcal{P} = \mathcal{NP} \), which is called the \( \mathcal{P} \) vs \( \mathcal{NP} \) problem. It is also well known that there are various formulations to specify an \( \mathcal{NP} \)-complete problem, e.g., the traveling salesman formulation. On the other hand, even if every formulation has the same solution efficiency, some formulation can be more effective, i.e., facilitate finding the solution.

This paper shows that reachability in safe acyclic Petri nets (PNs) brings about effective formulation to attack the \( \mathcal{P} \) vs \( \mathcal{NP} \) problem. A safe acyclic PN is associated with some Exactly-1 3SAT formula, and its reachability problem is \( \mathcal{NP} \)-complete \cite{1}\cite{2}\cite{4}. This effective formulation takes place due to the \textit{inverse} of the PN, and due to the set of \textit{conflicts}, which specifies exactly-1 disjunction. In other words, un/reachability in the \textit{inversed} safe acyclic PN is \textit{tractable}, compared to un/reachability in the safe acyclic PN, because some 2SAT/XOR-SAT formula arises in the \textit{inversed} PN that efficiently checks “incompatibility” of a truth assignment, and because some clauses reduce to conjunctions due to the incompatible assignments to be discarded.
The truth assignment of a literal \( z_i \) in the 3SAT formula corresponds to its associated transition occurrence (or firing) in the safe acyclic PN. In the inversed PN, a three-literal clause \( c_k \) satisfied is denoted by a marked place, \( c_k \in M^0 \), with three output transitions, each of which is a variable \( x_{ik} \) or its negation \( \overline{x}_{ik} \) in the clause \( c_k \), while a literal \( z_i \) is denoted by a place \( \ell_i \) with two input transitions, one for \( x_i \) and the other for \( \overline{x}_i \), i.e., \( z_i \in \ell_i = \{ x_i, \overline{x}_i \} \). A clause \( c_k = (z_i \lor z_j \lor z_u) \) is an exactly-1 disjunction \( \lor \), rather than disjunction \( \lor \), of three literals to specify some Exactly-1 3SAT formula \( \phi = c_1 \land c_2 \land \cdots \land c_m \). Then, \( c_k \) is true iff either \( z_i \) or \( z_j \) or \( z_u \) is true. The initial marking (state) \( M^0 \) in the inversed PN denotes that every clause-place \( c_k \) is marked, i.e., \( c_k \in M^0 \forall k \), while the target marking denotes that each literal-place \( \ell_i \) is to be marked by \( x_i \) or \( \overline{x}_i \). That is, the inversed PN assumes the formula \( \phi \) is satisfiable. This assumption is then checked by the “PN scan” algorithm proposed. If the truth assignment of a literal \( z_v \) in \( \phi \) (the occurrence of a transition \( z_v \) in the “PN scan”) is incompatible, then \( z_v \) is discarded, and its negation \( \overline{z}_v \) becomes true. Because \( z_v \) is true, \( z_i \) and \( z_j \) become false to satisfy \( (\overline{z}_v \lor z_i \lor z_j) \), i.e., \( \overline{z}_v \) and \( \overline{z}_j \) are true. Therefore, if \( z_v \) is incompatible, a clause \( (\overline{z}_v \lor z_i \lor z_j) \) reduces to the conjunction \( (\overline{z}_v \land \overline{z}_i \land \overline{z}_j) \), and a 3-literal clause \( (z_v \lor z_u \lor \overline{z_x}) \) reduces to the 2-literal clause \( (z_u \lor \overline{z}_x) \). This reduction, due to the PN approach, facilitates checking un/satisfiability.

### 2 Basic Definitions

This section introduces underlying tools to attack the \( \mathcal{P} \) vs \( \mathcal{NP} \) problem.

**Definition 2.1.** A safe PN is a tuple \( PN = (P,T,F,M^0) \), in which:

- \( P = \{ p_1, p_2, \ldots, p_m \} \) is a set of places,
- \( T = \{ t_1, t_2, \ldots, t_n \} \) is a set of transitions such that \( P \cap T = \emptyset \),
- \( F \subseteq (P \times T) \cup (T \times P) \) is a flow relation,
- \( M^0 \subseteq P \) is a set of places marked initially (the initial marking/state).
  - \( x^* (\ast x) \) denotes the post-set (pre-set) of \( x = \{ p_i, t_j \} \).

**Definition 2.2.** \( \mathcal{C} = \{ k : |p^t_{ik}| > 1 \} \) is a set of the indices of conflicts in \( PN \), and \( \mathcal{C} = \{ C_k : k \in \mathcal{C} \} \) is a family of sets of conflicts \( C_k \), where \( C_k = p^t_{ik} \).

**Definition 2.3.** A leveled-acyclic net \( \mathcal{N} = (L, P_l, T_l, F, M^0) \) is a safe PN (Defn. 2.1), in which:

- \( L = \{ 0, 1, \ldots, d \} \) is a set of levels, and \( l(x) \) denotes the level of \( x \),
- \( P_l \) is a set of places in \( l \in (L \cup \{ S \}) \), where \( P_S \) denotes sink places,
- \( T_l \) is a set of transitions in \( l \in L \),
- \( M^0 = \{ p \in P_l | p = \emptyset \} \), and \( M_l \subseteq P_l \) is a set of places marked.
  - \( i. \ \forall t \in T_l | \ast t \subseteq P_l \) (levelled \( \mathcal{N} \)) and \( t^* \subseteq P_{l' > l} \) (acyclic \( \mathcal{N} \)).
Fig. 1 depicts a net $\mathcal{N}$; $P = \{p_1, p_2, \ldots, p_{17}\}$, $T = \{t_1, t_2, \ldots, t_{14}\}$, $F = \{(p_1, t_1), (p_1, t_2), \ldots, (t_{14}, p_{17})\}$, and $M^0 = \{p_1, p_2, p_3, p_{11}, p_{12}, p_{13}\}$. Then, $C = \{C_1, C_2, C_3\}$ denotes the sets of conflicts (conflicting transitions), where $C_1 = p_3^0 = \{t_1, t_2\}$, $C_2 = p_3^0$, and $C_3 = p_5^0$. Further, $L = \{0, 1, 2\}$, $M_0 = P_0 = \{p_1, p_2, p_3\}$, $P_1 = \{p_4, p_5, \ldots, p_{13}\}$, $P_2 = \{p_{17}\}$, and $T_2 = \{t_{14}\}$. Note by (i) in Defn. 2.3 that $t_8 \in T_1$ and $\bullet t_8 = \{p_5, p_{11}\} \subset P_1$, i.e., $l(t_8) = l(p_5) = l(p_{11}) = 1$ (levelled $\mathcal{N}$), and that $t_2 \in T_0$ and $t_2^* = \{p_4, p_6\} \subset P_1$ (acyclic $\mathcal{N}$).

**Definition 2.4** (The enabling and firing rule). $t \in T$ is enabled in $M$ if $\bullet t \subseteq M$, i.e., if each of its input places is marked in $M$. If $t$ is enabled, it can fire (occur). Then, the tokens (balls in circles) are removed from $\bullet t$ and the new ones are created in $t$. This firing results in a 1-step transition, and yields the consequent marking/state $M'$ in $\mathcal{N}$, $M \xrightarrow{t} M'$; $M' = (M \cup \bullet t) - \bullet t$.

**Definition 2.5** (Token game). A safe acyclic $PN$ is executed by a token game, $M^0 \xrightarrow{\sigma} M$, played from $M^0$ by the enabling-firing rule until no $t_j \in T$ is enabled. $M^0 \xrightarrow{\sigma} M$ denotes a $k$-step transition. That is, $M$ is reached from $M^0$ by (firing) $\sigma$, where $\sigma = (t_{j_1}, t_{j_2}, \ldots, t_{j_k})$ is a $k$-step transition firing sequence from $M^0$.

For example, $T_0 = \{t_1, t_2, \ldots, t_6\}$ is enabled in $M^0 = \{p_1, p_2, p_3, p_{11}, p_{12}, p_{13}\}$ in Fig. 1. If $t_5$ fires, then $M^0 \xrightarrow{t_5} M$, where $M = (M^0 \cup \bullet t_5) - \bullet t_5 = \{p_1, p_2, p_5, p_{10}, p_{11}, p_{12}, p_{13}\}$. Hence, $t_6$ never fires due to $C_3$. Recall that $C_1 = \{t_1, t_2\}$, $C_2 = \{t_3, t_4\}$, and $C_3 = \{t_5, t_6\}$. $M^0 \xrightarrow{\sigma} M$ is a token game by $\sigma = (t_1, t_3, t_{10}, t_5, t_8, t_{13}, t_{14})$. No $t_j \in T$ is enabled in the final marking $M = \{p_{17}\}$. $M^0 \xrightarrow{\sigma} M$ is another token game by $\sigma = (t_2, t_7, t_3, t_{10}, t_6)$. No $t_j \in T$ is enabled in the final marking $M = \{p_{14}, p_{15}, p_6, p_8, p_{13}\}$. 

![Figure 1: A (leveled-acyclic) net $\mathcal{N}$](image)
3 The Net \( N \) of Exactly-1 3SAT: \( N^\phi \) vs \( N^\varphi \)

Recall that reachability in safe acyclic PNs is \( N^P \)-complete. The proof is due to Esparza [2] based on a polynomial time construction that associates a safe acyclic PN to a boolean formula in conjunctive normal form. The net nondeterministically selects a truth assignment for the variables of the formula, and checks if the formula is true under the assignment.

Fig. 2 depicts this construction based on Esparza [2]. The net \( N \) of the formula \( \phi \) is denoted by \( N^\phi = (L, P, T, I, F, M^0) \). Note that this construction is in fact based on the reduction of Exactly-1 3SAT due to places \( c_k, k = 1, 2, \ldots, m \), which ensures there can exist exactly one true literal in the clause \( c_k \) (the place \( \ell_k \) is marked by at most one token). In \( N^\phi \), \( P_0 = \{ \ell_1, \ell_2, \ldots, \ell_n \} \), i.e., the (source) places in \( l_0 \), specifies each literal \( \ell_i \) that is either a variable \( x_i \) or its negation \( \bar{x}_i \), i.e., \( \ell_i = \{ x_i, \bar{x}_i \} \), and \( T_0 = \{ x_1, \bar{x}_1, \ldots, x_n, \bar{x}_n \} \) specifies the variables. Furthermore, \( T_1 = \{ z_{i_1}, z_{i_2}, \ldots, z_{u_3} \} \), where \( z_{vi,k} \in \{ x_{vi,k}, \bar{x}_{vi,k} \} \) denotes the variable \( x_{vi} \) or \( \bar{x}_{vi} \) in \( c_k \), and \( P_2 = \{ c_1, c_2, \ldots, c_m \} \) specifies each clause \( c_k = ( z_{vi,k} \lor \bar{z}_{vi,k} ) \). Recall that \( \lor \) denotes exactly one transition \( z_{i,v} \) fires to mark \( c_i \), and that \( C = \{ c_1, c_2, \ldots, c_n \} \), where \( c_i = \ell^*_i \). Then, \( \phi \) is (un)satisfiable iff \( M^\tau \) is (un)reachable, i.e., if \( (M^\tau \neq M^i) \), \( M^\tau = M^i \), where \( M^i = \{ T^p \} \) is the target marking and \( M^i \) is a final marking in which no \( t_j \in T \) is enabled. Note that \( M^\tau = M^i \) iff \( M^2 = \{ T^p \} \) and \( M^1 = \emptyset \) for all \( i \in L \).

![Figure 2: \( N^\phi; \phi = c_1 \land c_2 \land \ldots \land c_m, c_k = z_{v_1,k} \lor \bar{z}_{v_2,k} \lor \bar{z}_{v_3,k}, z_{v_i,k} \in \{ x_{v_i,k}, \bar{x}_{v_i,k} \} \)](image)

On the other hand, the inverse of \( N^\phi \), denoted by \( N^\varphi \) in Fig. 3, brings about effective formulation for the \( P \) vs \( N^P \) problem, because some 2SAT/XOR-SAT formula arisen in the \( N^\varphi \) scan efficiently checks incompatibility of a truth assignment (of a transition firing). In \( N^\varphi \), \( P_0 = \{ c_1, c_2, \ldots, c_m \} \) corresponds to the clauses \( c_k \), and to the sets of the conflicts \( C = \{ C_1, C_2, \ldots, C_m \} \).
That is, $c^*_k = C_k = \{z_{v_1 k}, z_{v_2 k}, z_{v_3 k}\}$, where $z_{v_1 k} \in \{x_{v_1 k}, \overline{x_{v_1 k}}\}$, $v_t \in \Sigma$, and $\Sigma = \{1, 2, \ldots, n\}$, which is the set of the indices of the literals $\ell_t \in P_2 = \{\ell_1, \ell_2, \ldots, \ell_n\}$. Further, $d = 2$ is the depth of $\mathcal{N}^\varphi$, and $|C_k| = 2, 3$ because the clauses involve 2 or 3 literals in 3SAT. That is, $C_k \notin \mathcal{C}$ iff $|c^*_k| = 1$, and $c_k$ is called a conjunct rather than a clause if $|c^*_k| = 1$. Also, $\ell_t = \{x_i, \overline{x_i}\} \forall\ell_t \in P_2$.

Note that $\mathcal{N}^\varphi$ is constructed directly over $\phi$, i.e., there is no need to invert $\mathcal{N}^\phi$. Note also that $\mathcal{N}^\varphi$ assumes that $\phi$ is satisfiable (due to $P_0 \subset M^0$), and that the $\mathcal{N}^\varphi$ scan checks this assumption by the reachability of $M^\gamma = \{T^p\}$, i.e., of $M_2 = P_2 = \{\ell_1, \ell_2, \ldots, \ell_n\}$.

Figure 3: $\mathcal{N}^\varphi$, the inverse of $\mathcal{N}^\phi$; $\varphi = \phi = c_1 \land c_2 \land \ldots \land c_m \land M^\gamma \models M^\gamma$

The next section introduces the $\mathcal{N}^\varphi$ scan, which checks incompatibility of every transition firing (truth assignment of a literal) for the reachability of the marking $M_2$ (satisfiability of the formula $\phi$). This scan is executed on the current $\mathcal{N}^\varphi$ structure (formula $\phi$). The current net $\mathcal{N}^\varphi$ (formula $\phi$) is obtained by discarding incompatible transitions (literals) from $\mathcal{N}^\varphi$ ($\phi$).

### 3.1 The $\mathcal{N}^\varphi$ scan

Recall that a literal $\ell_i$ denotes a variable $x_i$ or its negation $\overline{x_i}$ in $\phi$, and that $\ell_i \in P_2 = \{\ell_1, \ldots, \ell_n\}$ is a place also in $\mathcal{N}^\varphi$. Recall also that $z_{ik}$ is a variable (output transition) $x_{ik}$ or $\overline{x_{ik}}$ of the clause (place) $c_k$. In this respect, the following definitions regard both $z_i$ the transition and $\ell_i$ the place as a literal.

**Definition 3.1.** $z_i \in \ell_i'$ and $\ell_i \in z_i$, $\ell_i = \{x_i, \overline{x_i}\}$.

**Definition 3.2.** $z_i = x_i$ iff $z_i = \overline{x_i}$, and $z_i = \overline{x_i}$ iff $z_i = x_i$.

**Definition 3.3.** $C_z = \{k \in C \mid z_{ik} \Rightarrow z_i\} = \{k_1, k_2, \ldots, k_m\}$ is the set of the indices of conflicts $C_z$ involved in the $z_i$ firing in the $\mathcal{N}^\varphi$ scan (of the indices of clauses $c_k$ in the formula $\phi$ that the literal $z_i$ contributes to).
Definition 3.4 (Non/necessary $z_i$). $z_i$ is said to be necessary if it must fire to unmark some places $c_k$ (must assume true to satisfy some clauses $c_k$).

Remark 3.1. $x_i$ and $\overline{x}_i$ are alternatives/nonnecessary; either $x_i$ or $\overline{x}_i$ can fire.

Remark 3.2 (Truth assignment vs transition firing). The literal $z_i$ is assigned true in $\phi$, $z_i = T$, iff the transition $z_i$ fires in the $N^\phi$ scan. Then, $z_i = F$ iff $z_i$ does not fire, denoted by $\neg z_i$, i.e., $z_i = F$ iff $\neg z_i$. Therefore, $\overline{z}_i = F$ iff $\neg \overline{z}_i$.

Lemma 3.1 (Necessary/Conjunct $z_i$). $z_i$ is necessary iff $\phi^*_i = \{z_i\}$.

Definition 3.5. $\phi(z_i) = \phi \land z_i$, where $z_i \in \epsilon_i$ is a literal in the formula $\phi$.

Lemma 3.2 (Reduction of clauses). $\phi(z_i)$ entails $(z_i \lor z_v \lor z_y) \rightarrow (z_v \lor z_y)$ and $(z_i \lor z_v \lor z_u) \rightarrow (z_i \land z_j \land z_u)$.

Proof: Because $z_i = T$ to satisfy $\phi(z_i) = \phi \land z_i$, it is the case that $\overline{z}_i = F$. Further, because $z_i = T$, it is the case that $z_j = \overline{u}_u = F$ to satisfy $(z_i \lor z_j \lor \overline{z}_u)$ due to exactly-1 disjunction $\lor$. Therefore, $\overline{z}_j = z_u = T$. Consequently, $\phi(z_i)$ entails that every clause $(z_j \lor z_j \lor \overline{z}_u)$ reduce to the conjunction $(z_i \land \overline{z}_j \land z_u)$, i.e., $(z_i \lor z_j \lor \overline{z}_u) \lor (z_i \land z_j \land z_u)$, and that every 3-literal clause $(z_i \lor z_v \lor z_y)$ reduce to the 2-literal clause $(z_i \lor z_v)$, i.e., $(z_i \lor z_v \lor z_y) \rightarrow (z_v \lor z_y)$. Note that $(z_i \lor z_j \lor \overline{z}_u) \not\equiv (z_i \lor z_j \lor \overline{u}_u)$, and that $(z_v \lor z_y) \equiv (z_v \lor z_y)$. □

Definition 3.6 (General vs special nets). $\{x_{ik}, \overline{x}_{ik}\} \not\subseteq C_k \forall k \in \mathcal{C} \forall i \in \mathcal{I}$ in a general $N^\phi$ (structure), while $\{x_{ik}, \overline{x}_{ik}\} \subseteq C_k$ for some $k$ in a special $N^\phi$.

Lemma 3.3 (Conversion of a special formula/net). $\phi(\overline{z}_j)$ converts a special formula $\phi$ to the general formula $\phi \equiv \phi(\overline{z}_j)$, where $C_k \in \mathcal{C} = \{z_{jk}, x_{ik}, \overline{x}_{ik}\}$.

Proof: If $z_j$ marks $\ell_j$, then neither $x_i$ nor $\overline{x}_i$ marks $\ell$, i.e., $\ell_i \not\in M_2$ in $N^\phi$. Therefore, $z_j$ cannot fire, and to be discarded. Because $z_j$ is discarded (false), $\overline{z}_j$ becomes necessary (true). Because $\overline{z}_j$ is necessary, $\phi^*_j = \{\overline{z}_{jk}\}$ for some $k$ (Lemma 3.1), which is the case in the formula $\phi(\overline{z}_j) = \phi = \overline{z}_j$ (Defn. 3.5).

Consequently, a special $\phi$ is converted to the general $\phi$ by means of $\phi(\overline{z}_j)$. □

For example, $\phi = \lor = (x_1 \lor \overline{x}_3) \land (x_1 \lor \overline{x}_2 \lor x_3) \land (x_2 \lor \overline{x}_3) \land x_4$ is a general formula, while $\phi' = \phi' = (x_1 \lor \overline{x}_3 \lor x_4) \land (x_1 \lor \overline{x}_2 \lor x_2) \land (x_2 \lor \overline{x}_3)$ is a special formula due to $C_2 = \phi^*_2 = \{x_{12}, \overline{x}_{22}, x_{22}\}$. Note that, in $N^\phi$, $\mathcal{C}^z = \{1, 2\}$ and $\mathcal{C}^{x_4} = \emptyset$ (Defn. 3.3) due to $\phi^* = \{x_{44}\}$ (Lemma 3.1), while $\mathcal{C}^{x_4} = \{1\}$ in $N^\phi'$. Then, neither $\overline{x}_2$ nor $x_2$ marks $\ell_2$, i.e., $\ell_2$ is not marked ($\ell_2 \not\in M_2$), if $x_1$ marks $\ell_1$ in $N^\phi'$. Therefore, $\phi'$ is converted to the general formula (Lemma 3.3) by discarding $x_1$, i.e., by means of $\phi'(\overline{z}_1) = \phi' \land \overline{z}_1 \equiv (\overline{x}_3 \lor x_4) \land (\overline{x}_2 + x_2) \land (x_2 \lor \overline{x}_3) \land \overline{x}_1 \equiv (\overline{x}_3 \lor x_4) \land (\overline{x}_2 + x_2) \land (x_2 \lor \overline{x}_3) \land \overline{x}_1$.

Recall that some 2SAT/XOR-SAT formula arises in the $N^\phi$ scan to check incompatibility of a firing. For example, consider incompatibility of $\overline{x}_1$, for the formula $\phi$ above. If $\overline{x}_1$ fires, then $\neg x_1$, i.e., $x_{31}$ and $(\overline{x}_{22} \lor x_{32})$ unmark the places $c_1$ and $c_2$. Thus, $\overline{x}_1 \Rightarrow \phi(\overline{x}_1)$, where $\phi(\overline{x}_1) = \overline{x}_1 \land \overline{x}_3 \land (\overline{x}_2 \lor x_3) \land (x_2 \lor \overline{x}_3) \land x_4$. Then, $\overline{x}_1$ is incompatible if $\phi(\overline{x}_1)$ is unsatisfiable. Theorems in the sequel incorporate this feature, which arises due to the PN approach.
Definition 3.7. $\phi(z_v)$ denotes the partial effect of the $z_v$ firing.

Definition 3.8. $\phi(\neg z_v)$ denotes the effect of the $z_v$ unfiring.

Definition 3.9. Let $c^v = \{k_1, k_2, \ldots, k_r\}$ for $\phi(z_v)$ and $\phi(\neg z_v)$, and let

$c_{k_1} = (z_v \lor z_{i_1} \lor z_{i_2})$, $c_{k_2} = (z_v \lor z_{j_1} \lor z_{j_2}) \cdots c_{k_r} = (z_v \lor z_{u_1} \lor z_{u_2})$.

Lemma 3.4. $\phi(z_v) \Rightarrow \phi(z_v) = \odot_{i_1} \land \odot_{i_2} \land \odot_{j_1} \land \odot_{j_2} \cdots \land \odot_{u_1} \land \odot_{u_2}$.

Proof: $z_v \Rightarrow (z_v \land \odot_{i_1} \land \odot_{i_2}) \land (z_v \land \odot_{j_1} \land \odot_{j_2}) \cdots \land (z_v \land \odot_{u_1} \land \odot_{u_2})$ due to the reduction of clauses (Lemma 3.2). Therefore, $z_v \Rightarrow \phi(z_v)$. □

Lemma 3.5. $\neg z_v \Rightarrow \phi(\neg z_v) = (z_{i_1} \lor z_{i_2}) \land (z_{j_1} \lor z_{j_2}) \cdots \land (z_{u_1} \lor z_{u_2})$.

Proof: If $\neg z_v$, then $(z_{i_1} \lor z_{i_2}) \land (z_{j_1} \lor z_{j_2}) \cdots \land (z_{u_1} \lor z_{u_2})$ must fire. □

Lemma 3.6 (The overall effect of the $z_v$ firing). $\psi(z_v) = v \lor \phi(z_v) \lor \phi(\neg z_v)$.

Proof: $z_v$ iff $\neg z_v$ (of $z_v$ fires iff $\neg z_v$ does not fire). Then, $v \Rightarrow \neg z_v$. Also, $z_v \Rightarrow z_v$, $z_v \Rightarrow \phi(z_v)$, and $\neg z_v \Rightarrow \phi(\neg z_v)$. Therefore, $z_v \Rightarrow z_v \land \phi(z_v) \land \phi(\neg z_v)$. □

Remark 3.3. $c^v \cap c^{\neg v} = \emptyset$ because $\{x_{vk}, \neg x_{vk}\} \not\subseteq C_k$ due to Lemma 3.3.

Definition 3.10. $z_v$ is incompatible to assume true iff $\phi(z_v) = z_v \land \phi = \bot$.

Lemma 3.7. $\phi(z_v) \equiv \psi(z_v) \land \phi'$, where $\phi'$ is a sub-formula of $\phi$.

Lemma 3.8 (A sufficient condition for incompatibility of $z_v$). If $\psi(z_v) = \bot$, then $\phi(z_v) = \bot$, i.e., $z_v$ is incompatible.

Lemma 3.9 (The scope $\Psi(z_v)$ over $\phi$). $\phi(z_v) \equiv \Psi(z_v) \land \phi^*$.

Proof: $\phi(z_v)$ entails reduction of clauses to conjunctions (Lemma 3.2). Each conjunction entails further reductions, i.e., $z_v \Rightarrow z_v \land z_v \land \cdots z_v$. Therefore, $\Psi(z_v) = \psi(z_v) \land \psi(z_v) \land \cdots \land \psi(z_v)$ and $\phi(z_v) \equiv \Psi(z_v) \land \phi^*$ (Lemma 3.7). □

Remark 3.4. Lemma 3.9 generalizes Lemma 3.7. $\phi^*$ is a sub-formula of $\phi$, and can be empty. If $\phi^*$ is empty, the scope $\Psi(z_v)$ is said to cover the formula $\phi$, i.e., $\phi(z_v) \equiv \Psi(z_v)$. Otherwise, $\phi^*$ is said to be beyond the scope $\Psi(z_v)$.

For example, consider $\phi(x_1) \lor \phi (x_1 \lor x_2 \lor x_3 \lor x_4 \lor x_5\lor x_6)$.

Then, $\phi(x_1) = (x_1 \land (x_2 \land x_3))$ and $\phi(\neg x_1) = (\neg x_1 \land x_2 \land x_3)$, while $\phi(\neg x_2)$ and $\phi(\neg x_3)$ are empty as $c_{x_1} = \emptyset$. Thus, $\phi(x_1) = \phi \land x_1$ entails $x_1 \Rightarrow \psi(x_1)$, in which $\psi(x_1) = x_1 \land \phi(x_1) \land \phi(\neg x_1) = x_1 \land x_2 \land x_3 \land x_2 \land x_3$ (Lemma 3.6). Further, $\phi(x_1) \equiv \psi(x_1) \land \phi'$ (Lemma 3.7), where $\phi' = c_5 = (x_2 \lor x_3)$. Therefore, $x_1$ is incompatible as $\psi(x_1) \Rightarrow \bot$ (Lemma 3.8) due to $x_3 \land \neg x_3$, i.e., if $x_1 = T$, then $\phi$ is unsatisfiable ($M_2$ is unreachable as $\ell_3 \not\subseteq M_2$). Similarly, $\phi(\neg x_1)$ entails $\neg x_1 \Rightarrow \phi(\neg x_1)$, in which $\psi(\neg x_1) = \neg x_1 \land \phi(\neg x_1) \land \phi(\neg x_1) = \neg x_1 \land \neg x_1$. Then, $\phi(x_1) \equiv \psi(x_1) \land \phi' = \neg x_1 \land x_2 \land x_3 \land x_2 \land x_3$, and $\phi(x_2) = c_4 = (x_2 \lor x_3)$. Since $x_3$ is a conjunct for $\phi(\neg x_1)$, i.e., $c_{x_3} = \{x_2\}$, it is the case that $\neg x_3 \Rightarrow \psi(\neg x_3)$, in which $\psi(\neg x_3) = \neg x_3 \land \phi(\neg x_3) \land \phi(\neg x_3) = \neg x_3 \land \neg x_3$, where $\phi(x_3) = (x_2 \lor x_3)$ by $c_{x_3} = \{4\}$ and $\phi(\neg x_3) = (\neg x_3) \lor \phi^*$ by $c_{\neg x_3} = \{3\}$ in $\phi(x_1)$. Therefore, the scope of $\neg x_1$ over $\phi$ is $\Psi(\neg x_1) = \psi(\neg x_1) \land \psi(x_3) = \neg x_1 \land x_2 \land x_3$, and $\phi(\neg x_1) \equiv \Psi(\neg x_1) \land \phi^*$, where $\phi^*$ is empty. Consequently, $\Psi(\neg x_1)$ covers $\phi$, i.e., $\phi(x_1) \equiv \Psi(\neg x_1) \equiv \phi$.
Definition 3.11. \(X_s\) denotes the current net/formula for the \(s^{th}\) \(N^\varphi\) scan.

Theorem 3.10. \(z_v\) becomes incompatible for \(\phi_s\) iff \(\Psi_s(z_v) = \perp\).

Proof: The validity of the theorem follows from \(\phi_s(z_v) = z_v \land \phi_s = \perp\) (Defn. 3.10) and from \(\phi_s(z_v) \equiv \Psi_s(z_v) \land \phi^*_s\) (Lemma 3.8/3.9), which depends on the soundness of the \(\Psi_s(z_v)\) construction over \(\phi_s(z_v)\), on the satisfiability of \(\Psi_s(z_v)\) and \(\phi^*_s\), and on the monotonicity of \(\Psi_s(z_v)\). Related to the soundness, the \(\Psi_s(z_v)\) construction is a deterministic chain of reductions of some clauses to conjunctions, and of some 3-clausal clauses to 2-clausal clauses, specified in the algorithm Incompatible \((z_v)\) below. Related to the satisfiability of \(\Psi_s(z_v)\) and \(\phi^*_s\), if \(\Psi_s(z_v) = T\) and \(\phi^*_s = \perp\), then not only \(z_v\) is incompatible, but also \(\phi_s = \perp\), i.e., \(\phi\) is unsatisfiable. Because unsatisfiability of \(\phi^*_s\) is already checked by the algorithm Scan, introduced in the sequel, it is irrelevant to check if \(\phi^*_s = \perp\) for the incompatibility of \(z_v\). Related to the monotonicity of \(\Psi_s(z_v)\), because \(\Psi_s(z_v)\) is the formula arisen from \(\Psi_s(z_v)\) by reducing some 3/2-clausal clauses to 2/1-clausal clauses, if \(\Psi_s(z_v) = \perp\), then \(\Psi_s(z_v) = \perp\) for all \(s > s\), i.e., incompatible \(z_v\) never becomes compatible again. 

Corollary 3.11. \(\phi_s(z_v) \equiv \phi\) is satisfiable if \(\Psi_s(z_v) = T\) and \(\phi^*_s\) is empty.

Algorithm 1 Incompatible \((z_v)\)

1: \(E \leftarrow \{z_v\}\) \(\triangleright\) \(E\) is a set of conjuncts emerged during the reductions in \(\phi_s(z_v)\)
2: while \(\phi^*_s(z_v)\) is not empty and \(E \neq E'\) do \(\triangleright\) Reductions start off
3: for all \(z_j \in (E - E')\) do \(\triangleright\) \(E'\) denotes conjuncts already reduced clauses
4: for all \(k \in c^z_j\) do \(\triangleright\) \(z_j = T\); Determine \(\varphi_s(z_j)\) (Lemma 3.4)
5: for all \(z_{ik} \in (C_k - \{z_{jk}\})\) do \(\triangleright\) Reduce \(c_k\) to conjunction
6: \(E_k \leftarrow (E_k \cup \{\overline{z}_i\})\) \(\triangleright\) \(E_k\) is a set of conjuncts reduced from \(c_k\)
7: end for
8: \(E \leftarrow (E \cup E_k\), \(c^*_k \leftarrow \emptyset\) \(\triangleright\) \(\phi_s(z_v)\) is updated (16); \(c^*_k = \emptyset\)
9: end for
10: if \(\{\overline{x}, \overline{i}\} \subseteq E\) then return \(z_v\) is incompatible \(\triangleright\) \(x_i \land \overline{i} = \perp\)
11: for all \(k \in c^{\overline{z}_j}\) do \(\triangleright\) \(\overline{z}_j = F\); Determine \(\varphi_s(\overline{z}_j)\) (Lemma 3.5)
12: \(c^*_k \leftarrow (c^*_k - \{\overline{z}_j\})\) \(\triangleright\) Reduce 3/2-clausal \(c_k\) to 2/1-clausal \(c_k\)
13: if \(c^*_k = \{z_{uk}\}\) then \(E \leftarrow (E \cup \{z_{uk}\}\), \(c^*_k \leftarrow \emptyset\)
14: end for
15: \(E' \leftarrow (E' \cup \{z_{jk}\}\) \(\triangleright\) \(z_j\), and \(\overline{z}_j\), has reduced clauses; \(E'\) is updated
16: Determine the current \(\varphi_s(z_v) \equiv \phi_s(z_v)\), and Update \(c\) for \(\varphi_s(z_v)\)
17: end for
18: end while \(\triangleright\) Reductions in \(\phi_s(z_v)\) terminate; \(\phi^*_s(z_v)\) is empty or \(E = E'\)
19: if \(\Psi_s(z_v) = (\bigwedge_{z_v \in E}) \land \varphi^*_s(z_v) = \perp\) then return \(z_v\) is incompatible
20: else return \(z_v\) is not yet incompatible \(\triangleright\) It can be incompatible for \(\phi_{s > s}\)
21: end if
22: Because reductions in \(\phi_s(z_v)\) terminate, \(\Psi_s(z_v)\) can be determined
In the algorithm Incompatible \((z_v)\), \(\Psi_s(z_v) = (\bigwedge_{z_u \in E} \phi_s(z_u)) \land \phi_s^2(z_v)\) becomes the scope over \(\phi_s(z_v)\) (L:19), when the reductions started off (L:2) terminate (L:18), i.e., if the formula of 3-literal clauses over \(\phi_s(z_v)\), \(\phi_s^2(z_v)\), is empty or \(E = E'\), \(\Psi_s(z_v)\) incorporates two formulas based on \(\varphi_s(z_j)\) and \(\varphi_s(-z_j)\) due to every conjunct \(z_j\) emerged during the reductions. The former, \(\bigwedge_{z_u \in E} = z_u_1 \land \varphi_s(z_u_1) \land \cdots \land z_u_n \land \varphi_s(z_u_n) \land R\), is the conjunction due to \(\varphi_s(z_j)\) (L:4-9) and \(R\) (L:13), while the latter, \(\phi_s^2(z_v)\), is the formula of 2-literal clauses due to \(\varphi_s(-z_j)\) (L:11-14), as well as due to the 2-literal clauses not reduced, i.e., remained in \(\phi_s(z_v)\). Consequently, \(\phi_s(z_v) \equiv \Psi_s(z_v) \land \phi_s^2(z_v)\), where \(\phi_s^2(z_v)\) is beyond the scope \(\Psi_s(z_v)\). If \(\Psi_s(z_v) = \bot\), a 2SAT/XOR-SAT formula, then \(z_v\) becomes incompatible for \(\phi_s\) (L:19), else \(z_v\) is not yet incompatible for \(\phi_s\).

The \(\mathcal{N}_s^\phi\) scan, the algorithm Scan \(\mathcal{N}_s^\phi\), is introduced below. Recall that \(\{x_{ik}, \overline{x}_{ik}\} \not\subseteq C_k\) by the conversion of a special net to a general net (Lemma 3.3). Note that there is no need to check incompatibility of \(z_v\) if \(|\ell_v| = 2\), i.e., \(|\ell_v| = 1\) (L:4 in Scan \(\mathcal{N}_s^\phi\)), because its incompatibility has already been checked in the algorithm Discard \((z_v)\). More precisely, if \(z_v\) is incompatible, then it is discarded (L:6 in Scan \(\mathcal{N}_s^\phi\)), i.e., the \(z_v\) firing initiates a chain of reductions (L:2-12 in Discard \((z_v)\)). Then, \(z_v\) is incompatible, i.e., \(\phi = \bot\) (L:8 in Discard \((z_v)\)) or \(z_v\) is not incompatible, i.e., \(\mathcal{N}_s^\phi\) is updated (L:14).

**Algorithm 2 Scan \(\mathcal{N}_s^\phi\)** \(\triangleright\) Incompatibility of all \(z_v \in \ell_v \forall v \in \mathcal{L}\) are checked

1: for all \(\ell_v \in \mathcal{L}_v\) & \(|\ell_v| = 2\) do \(\triangleright\) \(z_v\) is necessary (Lemma 3.1), i.e., \(\overline{x}_v = T\)
2: Discard \((z_v)\) \(\triangleright\) \(z_v\) is incompatible as \(z_v\) is necessary, i.e., \(z_v = F\)
3: end for \(\triangleright\) If \(|\ell_v| = 1\), then \(z_v\) has already been discarded
4: for all \(\ell_v \in P_2\) & \(|\ell_v| = 2\) do
5: for all \(z_v \in \ell_v = \{x_v, \overline{x}_v\}\) do
6: \(\text{if Incompatible } (z_v) \text{ then Discard } (z_v) \triangleright\) Theorem 3.10
7: end for \(\triangleright\) If \(|\ell_v| = 1\), then incompatibility of \(z_v\) has already been checked
9: return \(|\ell_v| = 1\) indicates \(z_v \in \mathcal{N}_s\) has already been discarded.

The algorithm Discard \((z_v)\) is introduced below, in which the \(z_v\) firing initiates the same chain of reductions as the one in Incompatible \((z_v)\). That is, if \(z_v\) is incompatible, then it is false and discarded, i.e., \(z_v\) is a conjunct. Therefore, \(z_v = T\) and \(z_{i1} = z_{i2} = F\), i.e., \(z_{i1} = z_{i2} = T\), to satisfy each clause \((z_v \lor z_{i1} \lor z_{i2})\). As a result, each clause \(c_k = (z_v \lor z_{i1} \lor z_{i2})\) reduces to the conjunction \((z_v \land z_{i1} \land z_{i2})\) (L:2-7). Then, \(N_k = \{z_{i1}, z_{i2}\}\) is a set of conjuncts over \((z_{i1} \land z_{i2})\) reduced from \(c_k\) (L:3-5), while \(N\) is the set of all conjuncts over the formula \(\phi\) (L:1). If \(\{x_i, \overline{x}_i\} \subseteq N\), i.e., if \(x_i \land \overline{x}_i = \bot\), then \(\phi_s \equiv \phi\) is unsatisfiable (L:8). Otherwise, \(z_{i1}\) is discarded from each clause \(c_k\) (L:9-12), and \(\phi/N_s\) is updated (L:14). Note that some clauses reduce to conjunctions iff \(N\) is updated (L:6/11). In this case, Lines (1-3) are executed in Scan \(\mathcal{N}_s^\phi\), where \(|\ell_v| = 1\) indicates \(z_v \in \mathcal{N}_s\) has already been discarded.
Algorithm 3 Discard \((z_v)\) ▷ Incompatible/nonnecessary \(z_v\) from \( \mathcal{N}^\varphi/\phi_s \)

1: \(N \leftarrow (N \cup \{z_v\})\) ▷ \(z_v\) is necessary/conjunct, \(N\) is their set over \(\mathcal{N}^\varphi/\phi\)
2: for all \(k \in \mathcal{C}^\varphi\) do ▷ \(z_v\) is necessary; Determine \(\varphi_s(z_v)\) (Lemma 3.4)
3: for all \(z_{ik} \in (C_k - \{z_{ek}\})\) do ▷ Reduce \(\mathcal{C}_k\) to conjunction due to \(z_v\)
4: \(N_k \leftarrow (N_k \cup \{z_i\})\) ▷ \(N_k\) is a set of conjuncts \(z_i\) reduced from \(\mathcal{C}_k\)
5: end for
6: \(N \leftarrow (N \cup N_k), \mathcal{C}_k^* \leftarrow \emptyset\) ▷ \(\mathcal{C}_k\) is reduced/\(\phi_s\) is updated; \(\mathcal{C}_k^*\) is empty
7: end for
8: if \(\{x_i, \pi_i\} \subseteq N\) then return \(\phi\) is unsatisfiable ▷ \(\phi_s \equiv \phi = \bot\)
9: for all \(k \in \mathcal{C}^\varphi\) do ▷ \(z_v\) is incompatible; Determine \(\varphi_s(-z_v)\) (Lemma 3.5)
10: \(c_k^* \leftarrow (c_k^* - \{z_{vk}\})\) ▷ Reduce 3/2-literal \(\mathcal{C}_k\) to 2/1-literal \(\mathcal{C}_k\)
11: if \(c^*_k = \{z_{uk}\}\) then \(N \leftarrow (N \cup \{z_{uk}\})\), \(c_k^* \leftarrow \emptyset\)
12: end for
13: \(\ell_{v} \leftarrow \{\pi_i\}\) ▷ Discard \(z_v\), i.e., \(\ell_{v} = \{\pi_i\}\)
14: Determine \(\phi_{s+1}\) and \(\mathcal{N}_{s+1}^\varphi\) ▷ \(N\) is updated or 3-literal \(\mathcal{C}_k\) is reduced
15: Scan \(\mathcal{N}_{s+1}^\varphi\) ▷ Re-scan \(\mathcal{N}_{s}\) due to \(\mathcal{N}_{s}\) re-structured

Let \(\phi = \varphi = (x_1 \lor \pi_3) \land (x_1 \lor \pi_2 \lor x_3) \land (x_2 \lor \pi_3)\), and check if \(\phi(x_1) = \varphi(x_1) = \phi \land x_1 = \bot\), i.e., consider incompatible \((x_1)\). Then, the \(x_1\) firing initiates a chain of reductions in \(\mathcal{N}_{s}^\varphi\). That is, \(x_1 \Rightarrow \psi(x_1)\), where \(\psi(x_1) = x_1 \land \varphi(x_1) \land \varphi(-\pi_1)\) (Lemma 3.6). Therefore, \(E \leftarrow \{x_1\}\) (L:1), \(\mathcal{C}^{\varphi_1} = \{1, 2\}\) (L:4), i.e., \(C_1\) and \(C_2\) participate in the \(x_1\) firing \((x_1)\) contributes to the clauses \(c_1\) and \(c_2\), \(E_1 \leftarrow \{x_3\}\) (L:6), \(E \leftarrow \{x_1, x_3\}\), \(c_1^* \leftarrow \emptyset\) (L:8), and \(E_2 \leftarrow \{x_2, \pi_3\}\) (L:6), \(E \leftarrow \{x_1, x_3, x_2, \pi_3\}\), \(c_2^* \leftarrow \emptyset\) (L:8). Because \(\{x_3, \pi_3\} \subset E\) (L:10), i.e., \(x_3 \land \pi_3 = \bot\), \(\phi(x_1) = \bot\) (\(x_1\) becomes incompatible for \(\phi\)). Note that \(\varphi(-\pi_1)\) is empty. Consequently, \(\bigwedge_{z_v \in E} x_1 \lor \varphi(x_1)\), where \(\varphi(x_1) = x_3 \land x_2 \land \pi_3\) (Lemma 3.4) is the formula over \(E_1 \cup \mathcal{E}_2\), i.e., \(x_1 \Rightarrow \{x_1 \land x_3\} \land \{x_1 \land x_2 \land \pi_3\}\). That is, \((x_1 \lor \pi_3) \Rightarrow (x_1 \lor \pi_3)\), and \((x_1 \lor \pi_2 \lor x_3) \Rightarrow (x_1 \lor \pi_2 \lor x_3)\).

Because \(x_1\) is incompatible, it is discarded \((\pi_1)\) becomes a conjunct. Then, \(\pi_1\) initiates a chain of reductions. That is, \(\pi_1 \Rightarrow \psi(x_1)\), in which \(\psi(x_1) = \pi_1 \land \varphi(\pi_1) \land \varphi(-\pi_1) = \pi_1 \land (\pi_2 \lor x_3)\). Therefore, \(\phi_2 = \psi(\pi_1) \land (x_2 \lor \pi_3)\), in which \(c_1^* = \{\pi_{11}\}\), \(c_2^* = \{\pi_{32}\}\), i.e., \(\pi_3\) is a conjunct, \(c_3^* = \{\pi_{23}, \pi_{33}\}\), and \(c_4^* = \{x_{24}, \pi_{34}\}\). Subsequently, \(\pi_3 \Rightarrow \psi_2(\pi_3)\), in which \(\psi_2(\pi_3) = \pi_3 \land \varphi(\pi_3) \land \varphi(-\pi_3)\), where \(\varphi(\pi_3) = \pi_2\) by \(\mathcal{C}^{\pi_3} = \{4\}\) and \(\varphi(-\pi_3) = \pi_2\) by \(\mathcal{C}^{\pi_3} = \{3\}\). Therefore, \(\phi_3 = \psi(\pi_1) \land \psi_2(\pi_3) = \pi_1 \land \pi_3 \land \pi_2\). Note that \(\phi_3 \equiv \phi\). Note also that \(\pi \Rightarrow \phi\) if \((x_1 \lor \pi_2 \lor x_3) \Rightarrow (\pi_2 \lor x_3)\), and \(\phi \Rightarrow \phi\) if \((x_2 \lor \pi_3) \Rightarrow (\pi_2 \land \pi_3)\).

3.2 An Illustrative Example

Fig. 4b depicts \(\mathcal{N}^\varphi\) for \(\phi = \varphi = (x_1 \lor \pi_3) \land (x_1 \lor \pi_2 \lor x_3) \land (x_2 \lor \pi_3)\). In \(\mathcal{N}^\varphi\), \(P_0 = M_0 = \{c_1, c_2, c_3\}\), \(P_1 = \{p_1, \ldots, p_{32}\} \cup \{\tilde{p}_1, \tilde{p}_2, \tilde{p}_3\}\), \(P_2 = \{\ell_1, \ell_2, \ell_3\}\), \(M_0 = P_0 \cup \{\tilde{p}_1, \tilde{p}_2, \tilde{p}_3\}\), and \(C_k = c^*_k \forall k \in \mathcal{C} = \{1, 2, 3\}\), e.g., \(C_1 = \{x_{11}, \pi_{31}\}\).
Scan $\mathcal{N}^\varphi$; Because $|c^*_1| \neq 1 \forall k$ (L:1-3), i.e., all $z_i \in \mathbb{E}$ are nonnecessary, Incompatible $\{x_1, \overline{x}_1, x_2, \overline{x}_2, x_3, \overline{x}_3\}$ are executed (L:4-8). Recall that $\mathcal{N}^\varphi$ is the net for the $s^{th}$ scan, $\mathcal{N}^\varphi = \mathcal{N}_s^\varphi$, and that the order of incompatibility check is insignificant due to the monotonicity of $\Psi_\varphi(z)$ (Theorem 3.10).

(a) $\mathcal{N}^\varphi$, based on Esparza [2]

(b) $\mathcal{N}^\varphi$

Figure 4: $\mathcal{N}^\varphi$ and $\mathcal{N}^\varphi$; $\phi = \varphi = (x_1 \lor \overline{x}_3) \land (x_1 \lor \overline{x}_2) \lor (x_2 \lor \overline{x}_3)$

Incompatible $(\pi_2)$ in Scan $\mathcal{N}^\varphi$; $\phi(\pi_2) = \phi \land \pi_2$; $E \leftarrow \{\pi_2\}$ (L:1), $\mathfrak{C}^{\pi_2} = \{2\}$ (L:4), $E_2 \leftarrow \{\overline{x}_1, \overline{x}_3\}$ (L:6), and $E \leftarrow \{\overline{x}_2, \overline{x}_1, \overline{x}_3\}$, $c^*_2 \leftarrow \emptyset$ (L:8). $\mathfrak{C}^{\pi_2} = \{3\}$ (L:11), $c^*_3 \leftarrow \{(x_{23}, \overline{x}_{33}) - \{x_{23}\}\}$ (L:12). Because $c^*_3 = \{\overline{x}_{33}\}$, $E \leftarrow (E \cup \overline{x}_{33})$ and $c^*_3 \leftarrow \emptyset$ (L:13). Then, $\phi(\overline{x}_{33}) = \overline{x}_2 \land \overline{x}_1 \land \overline{x}_3 \land (x_1 \lor \overline{x}_3)$ (L:16), where $c^*_3 = \{\overline{x}_{21}\}, c^*_5 = \{\overline{x}_{12}\}, c^*_6 = \{\overline{x}_{33}\}, c^*_7 = \{x_{14}, \overline{x}_{34}\}$, and $\mathfrak{C} = \{4\}$. Because $\phi(\overline{x}_{33})$ is empty, the reductions terminate (L:18). Therefore, $\Psi(\overline{x}_{33}) = (\bigwedge z_i \in E) \land \phi(\overline{x}_{33})$, where $\phi(\overline{x}_{33}) = c_4$. Note that $(x_1 \lor \overline{x}_3) = c_4$ for $\phi(\overline{x}_{33})$, while $(x_1 \lor \overline{x}_3) = c_1$ for $\phi(\overline{x}_{33})$. Note also that $\phi(\overline{x}_{33})$ is a sub-formula of $\phi(\pi_2)$ that is not reduced from some 3-literal clause in $\phi(\pi_2)$. Because $\phi(\overline{x}_{33})$ is empty, $\Psi(\overline{x}_{33})$ covers $\phi(\pi_2)$. Also, because $\Psi(\overline{x}_{33})$ is satisfiable, $\phi$ is satisfiable (Corollary 3.11), which has arisen without any cycle between the execution of Scan and Discard. This verification arises in this cycle in general, as demonstrated in the sequel.

On the other hand, if $\phi(\overline{x}_{33})$ were not empty and the reductions terminated due to $E = E'$ (L:18), then either $\overline{x}_2$ would become incompatible in Scan $\mathcal{N}^\varphi$ due to $\Psi(\pi_2) = \perp$ (L:19), or $\overline{x}_2$ would not yet become incompatible due to $\Psi(\pi_2) = T$. In the former, Discard $(\pi_2)$ would be executed. In the latter, incompatibility of another $z_i \in \mathbb{E}$ would be checked (L:4-8 in Scan).

Recall that the order of incompatibility check is insignificant. Assume Incompatible $(x_1)$ is executed first in Scan $\mathcal{N}^\varphi$. Recall also that $\mathcal{N}^\varphi = \mathcal{N}_s^\varphi$.

Incompatible $(x_1)$ in Scan $\mathcal{N}^\varphi$; $\phi(x_1) = \phi \land x_1$; $E \leftarrow \{x_1\}$ (L:1), $\mathfrak{C}^{x_1} = \{1, 2\}$ (L:4), $E_1 \leftarrow \{x_3\}$ (L:6), $E \leftarrow \{x_1, x_3\}$, $c^*_1 \leftarrow \emptyset$ (L:8), and $E_2 \leftarrow \{x_2, \overline{x}_3\}$ (L:6), $E \leftarrow \{x_1, x_3, x_2, \overline{x}_3\}$, $c^*_2 \leftarrow \emptyset$ (L:8). Because $\{x_3, \overline{x}_3\} \subset E$ (L:10), $x_1$ becomes incompatible for $N^\varphi$, and Discard $(x_1)$ is executed in Scan $\mathcal{N}^\varphi$ (L:6).
Discard \((x_1)\) in \(\text{Scan} \mathcal{N}^\phi\): \(N ← (N ∪ \{\overline{x}_1\})\) (L:1), i.e., \(N = \{\overline{x}_1\}\). \(c^{x_1} = \emptyset\) (L:2) and \(\phi\) is not unsatisfiable (L:8). Then, \(c^{x_2} = \{1, 2\}\) (L:9) over \(\phi = (x_1 ∨ x_3) ∧ (x_1 ∨ \overline{x}_2 ∨ x_3) ∧ (x_2 ∨ \overline{x}_3), c_1^* ← (c_1^* - \{x_{11}\})\) (L:10), and because \(c_1^* = \{\overline{x}_{31}\}, N ← (N ∪ \{\overline{x}_3\})\) and \(c_1^* ← \emptyset\) (L:11), i.e., \(N = \{\overline{x}_1, \overline{x}_3\}\). Also, \(c_2^* ← (c_2^* - \{x_{12}\})\) (L:10), i.e., \(c_2^* = \{\overline{x}_{22}, x_{32}\}\). Therefore, \(\ell_1 ← \{\overline{x}_1\}\) (L:13), and \(\phi_2 = (\overline{x}_3) ∧ (\overline{x}_2 ∨ x_3) ∧ (x_2 ∨ \overline{x}_3) ∧ \overline{x}_1\) (L:14). As a result, \(c = \{2, 3\}\), and \(C_2 = \{\overline{x}_{22}, x_{32}\}\) and \(C_3 = \{x_{23}, \overline{x}_3\}\) in \(\mathcal{N}^{c_2}_3\). Note that \(c_3^* = \{\overline{x}_{31}\}, |\ell_3| = 2\) and \(c_3^* = \{\overline{x}_{14}\}, |\ell_4| = 1\) in \(\mathcal{N}^{c_2}_2/\phi_2\). Consequently, \(\text{Scan} \mathcal{N}^{c_2}_2\) is executed.

\(\text{Scan} \mathcal{N}^{c}_3\): Because \(c_1^* = \{\overline{x}_{31}\}\) and \(|\ell_3| = 2\) (L:1), i.e., \(\overline{x}_3\) is necessary for \(\phi_2 = \overline{x}_3 ∧ (\overline{x}_2 ∨ x_3) ∧ (x_2 ∨ \overline{x}_3) ∧ \overline{x}_1\). \(\text{Discard} \ (x_3)\) is executed (L:2).

\(\text{Scan} \mathcal{N}^{c}_3\): \(N ← (N ∪ \{\overline{x}_3\})\) (L:1); \(N ← (\{\overline{x}_1, \overline{x}_3\} ∪ \{\overline{x}_3\})\). \(c^{x_3} = \{3\}\) over \(\phi_2\) (L:2), \(N ← \{\overline{x}_3\}\) (L:3-5), \(N ← (\{\overline{x}_1, \overline{x}_3\} ∪ N_3), c_1^* ← \emptyset\) (L:6), and \(ϕ_2 ≡ ϕ\) is not unsatisfiable over \(N = \{\overline{x}_1, \overline{x}_3, \overline{x}_2\}\) (L:8). \(c^{x_2} = \{2\}\) over \(ϕ_2\) (L:9), \(c_2^* ← (c_2^* - \{x_{32}\})\) (L:10), i.e., \(c_2^* ← \{\overline{x}_{22}\}\), and \(N ← (N ∪ \{\overline{x}_2\})\), \(c_2^* ← \emptyset\) (L:11). As a result, \(N = \{\overline{x}_1, \overline{x}_3, \overline{x}_2\}\). Therefore, \(\ell_3 ← \{\overline{x}_3\}\) (L:13), and \(ϕ_3 = \overline{x}_1 ∧ \overline{x}_3 ∧ \overline{x}_2\) (L:14). Consequently, \(\text{Scan} \mathcal{N}^{c}_3\) is executed.

\(\text{Scan} \mathcal{N}^{c}_1\): Because \(c_3^* = \{\overline{x}_{23}\}\) and \(|\ell_2| = 2\) in \(\mathcal{N}^{c}_3/\phi_3\) (L:1), i.e., \(\overline{x}_2\) is necessary for \(ϕ_3 = \overline{x}_3 ∧ \overline{x}_3 ∧ \overline{x}_2\). \(\text{Discard} \ (x_2)\) is executed (L:2). Note that \(c_1^* = \{\overline{x}_{11}\}, |\ell_1| = 1\) and \(c_2^* = \{\overline{x}_{32}\}, |\ell_3| = 1\) in \(\mathcal{N}^{c}_3/ϕ_3\).

\(\text{Discard} \ (x_2)\) in \(\mathcal{N}^{c}_1\): \(c = \{k : |c_k^*| > 1\} = \emptyset\) over \(ϕ_3\). Then, \(c^{x_2} = \emptyset\) and \(c^{x_3} = \emptyset\). Note that \(c^{x_2} = \{k ∈ c : x_{2k} ⇒ \overline{x}_2\}\). Therefore, \(\ell_2 ← \{\overline{x}_2\}\), and \(ϕ_4 = ϕ_3\). Consequently, \(\text{Scan} \mathcal{N}^{c}_1\) is executed.

\(\text{Scan} \mathcal{N}^{c}_1\): Because \(|\ell_i| = 1\) ∀\(i \in c\) (L:1/4 in \(\text{Scan}\)), \(\{\ell_1, \ell_2, \ell_3\}\) satisfies \(ϕ ≡ ϕ_4 = \overline{x}_1 ∧ \overline{x}_3 ∧ \overline{x}_2\), where \(\ell_i = \{\overline{x}_i\} \forall i \in \{1, 2, 3\}\).

4 Efficiency of the \(\mathcal{N}^\phi\) scan

Complexity of the algorithm \(\text{Scan}\) is determined as follows. \(z_v\) is incompatible if \(Ψ_v(z_v) = L\) (Theorem 3.10). Because \(Ψ_v(z_v)\) is of 2SAT/XOR-SAT formula, its complexity is assumed to be \(n^3\), where \(n = |P_2|\) is the number of the literals. \(Ψ_v(z_v)\) is determined when the reductions of the clauses terminate (L:18 in \(\text{Incompatible} (z_v)\)). Then, the number of the reductions is the number of the clauses, i.e., \(|P_0| = m\), where \(P_0 = \{c_1, c_2, \ldots, c_m\}\) denotes the clauses. Thus, the complexity of \(\text{Incompatible} (z_v)\) is \(m + n^3\). The number of the incompatibility checks is \(|T_2| = 2|P_2| = 2n\) (L:4-8 in \(\text{Scan}\)), where \(P_2 = \{ℓ_1, ℓ_2, \ldots, ℓ_n\}\) denotes the literals. Also, \(\mathcal{N}^\phi\) is re-scanned each time some \(z_v\) is discarded, and the number of the re-scans is \(|P_2| = n\). Therefore, the complexity of \(\text{Scan}\) is \((m + n^3) × 2n × n = 2mn^2 + 2n^5\), i.e., \(O(n^5)\).

On the other hand, the complexity of \(\text{Scan}\) becomes the complexity of \(\text{Incompatible} (z_v)\), i.e., \(O(n^3)\), if Corollary 3.11 holds. Also, efficiency of \(\text{Scan}\) is improved by finding incompatible literals promptly, because incompatible literals facilitate checking un/satisfiability via the reductions. Thus, checking incompatibility of \(z_1, z_2, \ldots, z_n\) in parallel improves the efficiency.
5 Conclusion

Reachability in safe acyclic PNs proves to be effective to attack the $\mathcal{P}$ vs $\mathcal{NP}$ problem, because some 2SAT/XOR-SAT formula arisen in the inverted PN checks if the truth assignment of a literal (transition firing) $z_v$ is incompatible for the satisfiability of the 3SAT formula (the reachability of the target state in the inverted PN). If $z_v$ is incompatible, then $z_v$ is false, i.e., $z_v$ is discarded and $\overline{z}_v$ becomes true. This incompatibility reduces, by means of exactly-1 disjunction $\lor$, a clause $(z_v \lor z_i \lor z_j)$ to the conjunction $(z_v \land z_i \land z_j)$, and a 3-literal clause $(z_v \lor z_i \lor z_j)$ to the 2-literal clause $(z_u \oplus z_x)$. The Exactly-1 3SAT formulation is facilitated by sets of conflicts in the PN, i.e., a clause corresponds to a set of conflicts. Checking incompatibility in parallel is possible also, which further improves the efficiency. Because the complexity of checking un/satisfiability is $O(n^5)$, it is the case that $\mathcal{P} = \mathcal{NP} = \text{coNP}$.

References


