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Convergence analysis of a finite volume method for the Stokes system using non-conforming arguments

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We consider finite volume methods for the Stokes system in a polyhedral domain of $\mathbb{R}^d$, $d = 2$ or 3. We prove different error estimates using non-conforming tools, namely by regarding the finite volume scheme as a non-conforming approximation of the continous variational problem. This point of view allows us to extend recent error estimates obtained by Blanc *et al*. (2004, *Numer. Meth. PDE*, **20**, 907–918.) for equilateral triangulations to a larger class of 2D meshes (incompletely proved by Alami-Idrissi & Atounti (2002) *JIPAM*, **3**, for meshes made of triangles) and to obtain its 3D version. Some numerical tests confirm our theoretical considerations.

Keywords: finite volume method; cell-center method; non-conforming approximation.

1. Introduction

These days, finite volume methods are widely used to approximate many problems of Physics or Mechanics. The convergence analysis of such schemes started in the 1980s and is still in progress, see for instance Manteufel & White (1986), Weiser & Wheeler (1988), Heinrich (1987), Forsyth & Sammon (1988), Baranger *et al*. (1996) and Eymard *et al*. (2000) and the references cited therein. We may distinguish two main techniques:

1. Define a mesh depending norm and prove error estimates using the principle of conservation of flux and Taylor's expansions. In this case, error estimates are usually obtained under some geometrical assumptions on the mesh.
2. Associate with the finite volume scheme a mixed finite element method with an appropriate quadrature rule and use the finite element error estimates to get the error estimates for the finite volume scheme.

On the other hand, the finite volume method is sometimes called a "discontinuous finite element method" because the trial and test functions are piecewise constant, but not continuous. Accordingly, the finite volume method may be considered as a non-conforming approximation of the (continuous) problem. Therefore, our goal is to prove error estimates for a finite volume scheme for the Stokes

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problem using non-conforming tools. By a variant of the second Strang Lemma (Ciarlet, 1978; Brezzi & Fortin, 1991), the main steps of the proof of the error estimates are then:

1. define an appropriate mesh depending norm,
2. verify the coerciveness and the continuity of the bilinear forms,
3. check the discrete inf–sup condition,
4. estimate the consistency terms.

Note that this point of view was applied in Coudière & Villedieu (2000) for a convection–diffusion equation on locally refined meshes made of rectangles.

For the Stokes system, different discretizations by finite volume methods have been proposed and analyzed. The main difficulties are the coupling between the velocity and the pressure and the stability of the scheme (an inf–sup condition should be satisfied as for finite element methods). For structured grids (rectangles) the Marker And Cell scheme uses different overlapped control volume grids for the different unknowns, and its convergence analysis is performed in Nicolaides (1992). Since this scheme is only applicable on structured grids, many efforts have been made to obtain schemes on unstructured grids. The first attempt consists in using finite volume element methods (Emonot, 1992; Chou, 1997): these methods are close to the finite element methods but the flux of the velocity is no longer discretized (by finite differences), which is a fundamental principle of finite volume methods. The second attempt is based on a discretization of the problem in terms of fluxes, more precisely a (standard cell-centered) finite volume scheme is used to approximate the velocity and a Galerkin method is used for the approximation of the pressure (Eymard *et al.*, 2000). Unfortunately, the convergence of that scheme, in its general form, is difficult to establish (see Eymard *et al.*, 2000, and Alami-Idrissi & Atounti, 2002, the proof of the convergence in the last paper containing a gap (Blanc *et al.*, 2004)). Two solutions have then been supplied:

1. slightly modify the scheme (Eymard & Herbin, 2003a,b),
2. prescribe some geometrical constraints on the meshes, see Blanc *et al.* (2004) for equilateral triangulations.

We adopt the second solution and establish error estimates, using a non-standard approach, namely non-conforming arguments. This furnishes a general framework that can be used for other schemes. Note further that we consider 2D problems as well as 3D ones, approximated by meshes made of triangles or rectangles in 2D and made of tetrahedra or hexahedra in 3D. For meshes made of rectangles or hexahedra, the standard scheme is not well-posed. For such meshes, we then introduce a new scheme, but which is relatively close to the standard one. To our knowledge, no proof of convergence exists for rectangular meshes or for 3D meshes. As mentioned before, error estimates are obtained under some geometrical constraints on the meshes (relatively strong for triangles or tetrahedra, and not present for rectangles or hexahedra). The necessity of these constraints is checked numerically, since for some examples where the constraints are not satisfied the scheme does not converge. These examples further show that the scheme proposed in Eymard *et al.* (2000) cannot converge for arbitrary regular meshes and that its convergence may require some additional geometrical conditions. Moreover, they underline the limitation of this scheme. From our results, we may conclude that this limitation comes from the so-called orthogonality condition on the meshes (see below). If we relax this condition (using for instance the diamond path technique, leading to a scheme which slightly differs from the one studied here), then the geometrical constraints on the meshes may be avoided, but some difficulties in the estimation of the error between the numerical and continuous fluxes appear (see Lemma 2.5).

The plan of the paper is as follows: in Section 2 we describe the discretization of the Stokes system proposed in Eymard *et al*. (2000) and transform it into a non-conforming approximation of the variational formulation of the problem. Using a variant of the second Strang Lemma we deduce an optimal error estimate under some geometrical constraints on the mesh. Section 3 is devoted to the extension of the previous scheme to meshes made of rectangles or hexahedra. As before the previous approach allows us to show error estimates. In Section 4 we present some illustrative numerical tests for our finite volume schemes obtained by adapting the standard Uzawa algorithm (Temam, 1984).

As usual, we denote by $L^p(\cdot)$ ($1 \leqslant p \leqslant \infty$) the Lebesgue spaces and by $W^{s,p}(\cdot)$ ($s \geqslant 0$, $1 \leqslant p \leqslant \infty$) the standard Sobolev spaces. Sometimes we write $W^{0,p}(\cdot)$ for $L^p(\cdot)$ and $H^s(\cdot)$ for $W^{s,2}(\cdot)$. The usual norm and seminorm of $W^{s,p}(D)$ are denoted by $\|\cdot\|_{s,p,D}$ and $|\cdot|_{s,p,D}$. For brevity, the $L^2(D)$-norm will be denoted by $\|\cdot\|_D$. In the case $p = 2$, we will drop the index p, similarly in the case $D = \Omega$, we will drop the index Ω. The space $H_0^1(\Omega)$ is defined, as usual, by $H_0^1(\Omega) := \{v \in H^1(\Omega)/v = 0$ on $\Gamma\}$. In the sequel, the symbol $|\cdot|$ will denote the Euclidean norm in $\mathbb{R}^d$, the length of a line segment or the area of a plane region. Finally, the notation $a \lesssim b$ means here and below that there exists a positive constant C independent of a and b (and of the mesh size of the triangulation) such that $a \leqslant Cb$.

2. Discretization of the Stokes system

Let Ω be an open bounded subset of $\mathbb{R}^d$, $d = 2$ or 3, with a polygonal boundary Γ ($d = 2$) or a polyhedral boundary ($d = 3$) (see Fig. 1).

Over the domain Ω, we consider the stationary Stokes problem with Dirichlet boundary conditions: given a vector function $f = (f_1, \ldots, f_d)$, find a vector function $u = (u_1, \ldots, u_d)$ representing the velocity of the fluid and a scalar function p representing the pressure and satisfying

$$\begin{cases} -\nu \Delta u + \nabla p = f, & \text{in } \Omega, \\ \operatorname{div} u = 0, & \text{in } \Omega, \\ u = 0, & \text{on } \partial\Omega, \end{cases} \tag{2.1}$$

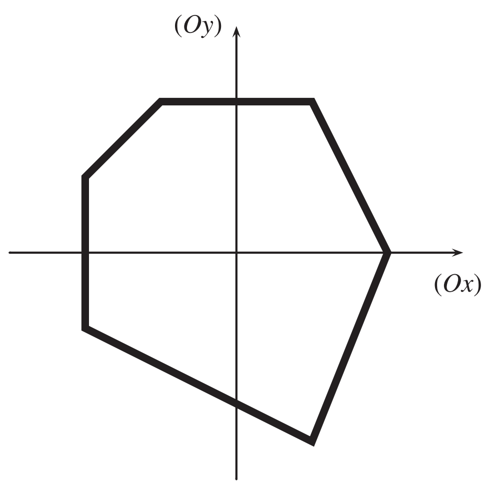

FIG. 1. The domain Ω.

where $\nu > 0$ represents the kinematic viscosity. Here we use the weak formulation which has a unique solution $(u, p) \in V \times Q$,

$$V := (H_0^1(\Omega))^d, \quad Q := \left\{ q \in L^2(\Omega) : \int_\Omega q = 0 \right\},$$

for $f \in L^2(\Omega)^d$ as shown in Girault & Raviart (1986, Theorem I.5.1), namely

$$\begin{cases} \nu a(u, v) + b(v, p) = (f, v), & \forall\, v \in V, \\ b(u, q) = 0, & \forall\, q \in Q, \end{cases} \tag{2.2}$$

where

$$a(v, w) = \sum_{i=1}^{d} \int_\Omega \nabla v_i(x) \cdot \nabla w_i(x), \quad b(v, q) = -\int_\Omega q \operatorname{div} v.$$

In the whole section we assume that this solution (u, p) belongs to $(H^2(\Omega))^d H^1(\Omega)$. This regularity holds either if Ω is convex or if f satisfies some orthogonality relations (Dauge, 1989; Bernardi & Raugel, 1981, Theorem II.1).

2.1 *The numerical scheme*

Let us fix a conforming mesh $\mathcal{T}$ of Ω made on triangles ($d = 2$) or of tetrahedra ($d = 3$) (Ciarlet, 1978). We further assume that $\mathcal{T}$ is a restricted admissible mesh in the sense of Eymard *et al.* (2000, Definition 9.4), i.e. a mesh satisfying standard orthogonality conditions (see Fig. 2), and the regularity assumption $d(x_K, \sigma) \sim h_K$, h_K being the diameter of K, the constants of equivalence being independent of the mesh size $h = \max_{K \in \mathcal{T}} h_K$ of $\mathcal{T}$, x_K being the "center" of the box K (in 2D it is not necessarily the intersection of the orthogonal bisectors). Note that such a mesh is regular in Ciarlet's sense (Ciarlet, 1978), i.e. it satisfies

$$\max_{K \in \mathcal{T}} \frac{h_K}{\rho_K} \leqslant C,$$

where the positive constant C is independent of $\mathcal{T}$ and h (we recall that ρ_K is the maximum of the diameters of the balls included in K). In the whole paper we use the notation from Eymard *et al.* (2000), except that the elements K in $\mathcal{T}$ are supposed to be closed.

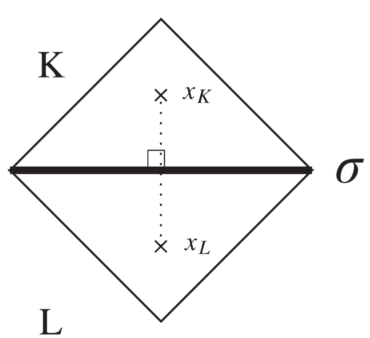

FIG. 2. Illustration of the orthogonality condition.

The finite volume scheme considered in Eymard *et al.* (2000) uses a cell-centered method for the velocity and a Galerkin one for the pressure. In other words, we consider $\{u_K = (u_{iK})_{i=1,\dots,d}\}_{K\in\mathcal{T}}$ (u_K being the approximation of $u(x_K)$, for $K \in \mathcal{T}$) and $\{p_S\}_{S\in\mathcal{S}}$ (p_S being the approximation of $p(S)$, for $S \in \mathcal{S}$, where $\mathcal{S}$ is the set of vertices of the triangulation $\mathcal{T}$) the unique solution (see Eymard *et al.*, 2000, Section 11) of

$$-\nu \sum_{\sigma\in\mathcal{E}_K} F_{K,\sigma} + \sum_{S\in\mathcal{S}_K} p_S \int_K \nabla\phi_S \,\mathrm{d}x = \int_K f(x)\,\mathrm{d}x, \quad \forall K \in \mathcal{T}, \tag{2.3}$$

$$\sum_{K\in\mathcal{T}} \int_K u_K \cdot \nabla\phi_S \,\mathrm{d}x = 0, \quad \forall S \in \mathcal{S}, \tag{2.4}$$

$$\int_\Omega \sum_{S\in\mathcal{S}} p_S \phi_S \,\mathrm{d}x = 0, \tag{2.5}$$

where $\mathcal{S}_K$ is the set of vertices of K, $\mathcal{E}_K$ is the set of edges of K and $F_{K,\sigma}$ is defined by

$$F_{K,\sigma} := \begin{cases} \frac{|\sigma|}{d_\sigma}(u_L - u_K), & \text{if } \sigma = K \cap L, \\ -\frac{|\sigma|}{d_\sigma} u_K, & \text{if } \sigma \subset K \cap \partial\Omega, \end{cases}$$

when $d_\sigma = d(x_K, x_L)$ if $\sigma = K \cap L$, with $K, L \in \mathcal{T}$ and $d_\sigma = d(x_K, \sigma)$ if $\sigma \subset \partial K \cap \Gamma$. Finally, ϕ_S is the shape function associated with the vertex S, i.e. ϕ_S is piecewise linear on $\mathcal{T}$ and satisfies $\phi_S(S') = \delta_{S,S'}$ for all $S' \in \mathcal{S}$.

Let us now transform the above problem into a non-conforming approximation of (2.2). For this purpose we introduce the space V_h made of piecewise constant vector-valued functions in $\mathcal{T}$

$$V_h := \{v_h \in (L^2(\Omega))^d : v_{h|K} \in (\mathbb{P}_0(K))^d, \forall K \in \mathcal{T}\},$$

equipped with the mesh depending norm (Eymard *et al.*, 2000)

$$\|v_h\|_{\mathcal{T}}^2 := \sum_{\sigma\in\mathcal{E}} \frac{|\sigma|}{d_\sigma} |D_\sigma v_h|^2,$$

where $\mathcal{E}$ is the set of edges of the mesh $\mathcal{T}$, $D_\sigma v_h := v_L - v_K$ if $\sigma = K \cap L$, with $K, L \in \mathcal{T}$ and $D_\sigma v_h := -v_K$, if $\sigma \subset \partial K \cap \Gamma$. This space V_h is considered as a non-conforming approximation of $(H_0^1(\Omega))^d$.

Let us further introduce

$$Q_h = \{q_h \in Q \cap C(\bar{\Omega}) : q_{h|K} \in \mathbb{P}_1(K), \forall K \in \mathcal{T}\},$$

endowed with the $L^2(\Omega)$-norm.

For $v_h, w_h \in V_h$ and $q_h \in Q_h$ we define

$$a_h(v_h, w_h) := -\sum_{K\in\mathcal{T}} \sum_{\sigma\in\mathcal{E}_K} F_{K,\sigma}(v_h) \cdot w_K,$$

$$b_h(v_h, q_h) := \int_\Omega v_h \cdot \nabla q_h \,\mathrm{d}x,$$

$$(f, w_h) := \int_\Omega f \cdot w_h \,\mathrm{d}x,$$

where $F_{K,\sigma}(v_h)$ is defined by

$$F_{K,\sigma}(v_h) := \begin{cases} \frac{|\sigma|}{d_\sigma}(v_L - v_K), & \text{if } \sigma = K \cap L, \\ -\frac{|\sigma|}{d_\sigma} v_K, & \text{if } \sigma \subset K \cap \partial\Omega, \end{cases}$$

and v_K (resp. w_K) is the restriction of v_h (resp. w_h) to K.

With these notations we readily obtain the following result.

LEMMA 2.1 $\{u_K\}_{K\in\mathcal{T}}$ and $\{p_S\}_{S\in\mathcal{S}}$ are solutions of (2.3)–(2.5) if and only if $u_h = \sum_{K\in\mathcal{T}} u_K \chi_K \in V_h$ (χ_K being the characteristic function of the control volume K) and $p_h = \sum_{S\in\mathcal{S}} p_S \phi_S \in Q_h$ are solutions of

$$\begin{cases} \nu a_h(u_h, v_h) + b_h(v_h, p_h) = (f, v_h), & \forall v_h \in V_h, \\ b_h(u_h, q_h) = 0, & \forall q_h \in Q_h. \end{cases} \tag{2.6}$$

At this stage we look at (2.6) as a non-conforming approximation of problem (2.2). In order to apply Proposition II.2.16 of Brezzi & Fortin (1991), we first need that a_h is continuous and coercive on V_h, properties that follow from the principle of conservation of flux. We secondly need that b_h is continuous and satisfies the uniform inf–sup condition. These properties are now checked. First introduce the inner product associated with $\|\cdot\|_{\mathcal{T}}$:

$$(v_h, w_h)_{\mathcal{T}} := \sum_{\sigma\in\mathcal{E}} \frac{|\sigma|}{d_\sigma} D_\sigma v_h D_\sigma w_h, \quad \forall\, v_h, w_h \in V_h.$$

LEMMA 2.2 For all $v_h, w_h \in V_h$ one has

$$a_h(v_h, w_h) = (v_h, w_h)_{\mathcal{T}}, \tag{2.7}$$

and consequently a_h satisfies

$$a_h(v_h, v_h) = \|v_h\|^2_{\mathcal{T}}, \tag{2.8}$$

$$|a_h(v_h, w_h)| \leqslant \|v_h\|_{\mathcal{T}} \|w_h\|_{\mathcal{T}}. \tag{2.9}$$

Proof. By the definition of a_h we may write

$$a_h(v_h, w_h) = \sum_{K\in\mathcal{T}} \left(\sum_{\sigma=K\cap L} F_{K,\sigma}(v_h) \cdot (w_L - w_K) - 2 \sum_{\sigma\subset\partial K\cap\partial\Omega} F_{K,\sigma}(v_h) \cdot w_K \right)$$
$$+ \sum_{K\in\mathcal{T}} \left(- \sum_{\sigma=K\cap L} F_{K,\sigma}(v_h) \cdot w_L + \sum_{\sigma\subset\partial K\cap\partial\Omega} F_{K,\sigma}(v_h) \cdot w_K \right).$$

Since for $\sigma = K \cap L$, $F_{L,\sigma}(v_h) = -F_{K,\sigma}(v_h)$, the above identity becomes

$$a_h(v_h, w_h) = \sum_{K\in\mathcal{T}} \left(\sum_{\sigma=K\cap L} \frac{|\sigma|}{d_\sigma} D_\sigma v_h \cdot D_\sigma w_h + 2 \sum_{\sigma\subset\partial K\cap\partial\Omega} \frac{|\sigma|}{d_\sigma} D_\sigma v_h \cdot D_\sigma w_h \right)$$
$$+ \sum_{K\in\mathcal{T}} \left(\sum_{\sigma=K\cap L} F_{L,\sigma}(v_h) \cdot w_L + \sum_{\sigma\subset\partial K\cap\partial\Omega} F_{K,\sigma}(v_h) \cdot w_K \right).$$

This is equivalent to

$$a_h(v_h, w_h) = 2(v_h, w_h)_{\mathcal{T}} + \sum_{L\in\mathcal{T}} \sum_{\sigma\in\mathcal{E}_L} F_{L,\sigma}(v_h)\cdot w_L$$
$$= 2(v_h, w_h)_{\mathcal{T}} - a_h(v_h, w_h),$$

which is nothing else but (2.7). □

LEMMA 2.3 b_h satisfies

$$\sup_{v_h\in V_h} \frac{b_h(v_h, q_h)}{\|v_h\|_{\mathcal{T}}} \gtrsim \|q_h\|, \quad \forall\, q_h \in Q_h. \tag{2.10}$$

Proof. Fix $q_h \in Q_h$. Let $v \in (H_0^1(\Omega))^d$ be such that (see e.g. Corollary I.2.4 of Girault & Raviart, 1986)

$$\operatorname{div} v = -q_h, \quad \text{in } \Omega, \tag{2.11}$$

$$\|v\|_{1,\Omega} \lesssim \|q_h\|. \tag{2.12}$$

Green's formula yields

$$\|q_h\|^2 = -\int_\Omega \operatorname{div} v\, q_h = \int_\Omega v\cdot\nabla q_h.$$

Since ∇q_h is piecewise constant, the above identity may be transformed as follows:

$$\|q_h\|^2 = \sum_{K\in\mathcal{T}} \left(\int_K v\right)\cdot\nabla q_{h|K} = \sum_{K\in\mathcal{T}} \left(\frac{1}{|K|}\int_K v\right)\cdot\int_K \nabla q_h.$$

Therefore, the (vector-valued) function $v_h \in V_h$ defined by

$$v_K = \frac{1}{|K|}\int_K v, \quad \forall\, K \in \mathcal{T},$$

satisfies

$$b_h(v_h, q_h) = \|q_h\|^2. \tag{2.13}$$

Moreover, a scaling argument yields

$$|D_\sigma v_h| = |v_K - v_L| \lesssim |v|_{1,K\cup L}, \quad \text{for } \sigma = K\cap L,$$

$$|D_\sigma v_h| = |v_K| \lesssim |v|_{1,K}, \quad \text{for } \sigma \subset \partial K\cap\Gamma.$$

By the regularity of the mesh we get

$$\|v_h\|_{\mathcal{T}}^2 \lesssim \sum_{\sigma\in\mathcal{E}} |D_\sigma v_h|^2$$
$$\lesssim \sum_{\sigma\in\mathcal{E}} \sum_{K\in\mathcal{T}:\sigma\in\mathcal{E}_K} |v|_{1,K}^2 \lesssim |v|_{1,\Omega}^2.$$

Using the estimate (2.12) we conclude that

$$\|v_h\|_{\mathcal{T}} \lesssim \|q_h\|. \tag{2.14}$$

This estimate and (2.13) lead to

$$\frac{b_h(v_h, q_h)}{\|v_h\|_{\mathcal{T}}} \gtrsim \|q_h\|,$$

which proves (2.10). □

LEMMA 2.4 b_h satisfies

$$|b_h(v_h, q_h)| \lesssim \|v_h\|_{\mathcal{T}} \|q_h\|, \quad \forall\ v_h \in V_h,\ \ q_h \in Q_h. \tag{2.15}$$

Proof. Using Green's formula on each control volume K we have

$$b_h(v_h, q_h) = \sum_{K \in \mathcal{T}} \sum_{\sigma \in \mathcal{E}_K} \int_\sigma q_h v_K \cdot n_{K,\sigma},$$

where $n_{K,\sigma}$ is the outward normal vector to K along σ. The continuity of q_h through the edges leads to

$$b_h(v_h, q_h) = \frac{1}{2} \sum_{K \in \mathcal{T}} \sum_{\sigma = K \cap L} \int_\sigma q_h (v_K - v_L) \cdot n_{K,\sigma} + \sum_{K \in \mathcal{T}} \sum_{\sigma \subset \partial K \cap \partial \Omega} \int_\sigma q_h v_K \cdot n_K.$$

A discrete Cauchy–Schwarz's inequality yields

$$|b_h(v_h, q_h)| \leqslant \|v_h\|_{\mathcal{T}} \left(\sum_{\sigma \in \mathcal{E}} \frac{d_\sigma}{|\sigma|} \left(\int_\sigma q_h \right)^2 \right)^{1/2}. \tag{2.16}$$

A scaling argument and the fact that all norms are equivalent in a finite-dimensional space give

$$\left| \int_\sigma q_h \right| \lesssim |\sigma| |K|^{-1/2} \|q_h\|_K.$$

This estimate and the property $|K| \sim |L|$ if $\sigma = K \cap L$ allow us to obtain

$$\begin{aligned} \sum_{\sigma \in \mathcal{E}} \frac{d_\sigma}{|\sigma|} \left(\int_\sigma q_h \right)^2 &\lesssim \sum_{\sigma \in \mathcal{E}} d_\sigma |\sigma| |K|^{-1} \|q_h\|_K^2 \\ &\lesssim \sum_{K \in \mathcal{T}} (|K| + |L|) |K|^{-1} \|q_h\|_K^2 \\ &\lesssim \|q_h\|^2. \end{aligned}$$

This estimate in (2.16) leads to the continuity property of b_h. □

2.2 *The error estimate*

For $u, v \in (H^2(\Omega))^d + V_h$, we define

$$a_h(u, v) := a_h(I_h u, I_h v),$$

where $I_h u \in V_h$ is the "interpolant" of u defined by

$$I_h u := \sum_{K \in \mathcal{T}} u(x_K)\chi_K.$$

Now regarding (2.6) as a non-conforming approximation of (2.2) (since $V_h \not\subset (H_0^1(\Omega))^d$), we deduce by a variant of the second Strang Lemma (Brezzi & Fortin, 1991, Proposition II.2.16) that

$$|u - u_h|_{1,h} + \|p - p_h\| \lesssim \inf_{v_h \in V_h} |u - v_h|_{1,h} + \inf_{q_h \in Q_h} \|p - q_h\| + \sup_{w_h \in V_h} \frac{|\nu a_h(u, w_h) + b_h(w_h, p) - (f, w_h)|}{|w_h|_{1,h}} + \sup_{q_h \in Q_h} \frac{|b_h(u, q_h)|}{\|q_h\|}, \tag{2.17}$$

where $|u - u_h|_{1,h}^2 = a_h(u - u_h, u - u_h)$ (by Lemma 2.2), $b_h(u, q_h) = b_h(I_h u, q_h)$ and $b_h(w_h, p) = b_h(w_h, I_C p)$, $I_C p$ being the Clément interpolant of p, defined by

$$I_C p = \sum_{S \in \mathcal{S}} \frac{1}{|\omega_S|} \left(\int_{\omega_S} p(x)\, dx \right) \phi_S,$$

where the patch ω_S is defined by $\omega_S = \bigcup_{K \in \mathcal{T}: S \in K} K$ (one readily checks that $\int_\Omega I_C p\, dx = \int_\Omega p\, dx = 0$).

The first term, called the approximation error term on the velocity, is equal to 0 in our case, since for $v_h = I_h u$, one has $|u - v_h|_{1,h} = 0$. Moreover a scaling argument yields (Clément, 1975)

$$\|p - I_C p\| \lesssim h|p|_{1,\Omega}.$$

Therefore, it remains to estimate the consistency terms.

We start with the first consistency term.

LEMMA 2.5 For all $w_h \in V_h$ we have

$$|\nu a_h(u, w_h) + b_h(w_h, p) - (f, w_h)| \lesssim h|w_h|_{1,h}(|u|_{2,\Omega} + |p|_{1,\Omega}). \tag{2.18}$$

Proof. Using Green's formula we may write

$$\nu a_h(u, w_h) + b_h(w_h, p) - (f, w_h) = \nu I_1 - I_2, \tag{2.19}$$

where we set

$$I_1 = \sum_{K \in \mathcal{T}} \sum_{\sigma \in \mathcal{E}_K} R_{K,\sigma} \cdot w_K |\sigma|$$

$$I_2 = \sum_{K \in \mathcal{T}} \int_{\partial K} (p - I_C p) w_K \cdot n_K\, d\sigma,$$

recalling that

$$R_{K,\sigma} := \begin{cases} \frac{u(x_L) - u(x_K)}{d_\sigma} - \frac{1}{|\sigma|} \int_\sigma \nabla u \cdot n_{K,\sigma}\, ds, & \text{if } \sigma = K \cap L, \\ \frac{-u(x_K)}{d_\sigma} - \frac{1}{|\sigma|} \int_\sigma \nabla u \cdot n_{K,\sigma}\, ds, & \text{if } \sigma \subset \partial K \cap \partial\Omega. \end{cases}$$

Using the property of conservation of flux $R_{K,\sigma} = -R_{L,\sigma}$ if $\sigma = K \cap L$, the first term may be written as

$$I_1 = \frac{1}{2} \sum_{K \in \mathcal{T}} \sum_{\sigma = K \cap L} (w_K - w_L) \cdot R_{K,\sigma} |\sigma| + \sum_{K \in \mathcal{T}} w_K \cdot \sum_{\sigma \subset \partial K \cap \partial \Omega} R_{K,\sigma} |\sigma|,$$

and by the discrete Cauchy–Schwarz inequality and Lemma 2.2 we get

$$\begin{aligned} |I_1| &\leqslant \left(\sum_{K \in \mathcal{T}} \sum_{\sigma \in \mathcal{E}_K} \frac{|\sigma|}{d_\sigma} |w_K - w_L|^2 \right)^{1/2} \left(\sum_{K \in \mathcal{T}} \sum_{\sigma \in \mathcal{E}_K} d_\sigma |\sigma| |R_{K,\sigma}|^2 \right)^{1/2} \\ &\leqslant \sqrt{2} \|w_h\|_{\mathcal{T}} \left(\sum_{K \in \mathcal{T}} \sum_{\sigma \in \mathcal{E}_K} d_\sigma |\sigma| |R_{K,\sigma}|^2 \right)^{1/2} \\ &\leqslant \sqrt{2} |w_h|_{1,h} \left(\sum_{K \in \mathcal{T}} \sum_{\sigma \in \mathcal{E}_K} d_\sigma |\sigma| |R_{K,\sigma}|^2 \right)^{1/2}. \end{aligned}$$

According to the estimate (9.63) of Eymard *et al.* (2000), we conclude that

$$|I_1| \lesssim h |u|_{2,\Omega} |w_h|_{1,h}. \tag{2.20}$$

For the second term using the continuity of $p - I_C p$ through the edges and the Cauchy–Schwarz inequality we get

$$\begin{aligned} |I_2| &= \Bigg| \frac{1}{2} \sum_{K \in \mathcal{T}} \sum_{\sigma = K \cap L} \int_\sigma (p - I_C p)(w_K - w_L) \cdot n_{K,\sigma} \, d\sigma \\ &\quad + \sum_{K \in \mathcal{T}} \sum_{\sigma \subset \partial K \cap \partial \Omega} \int_\sigma (p - I_C p) w_K \cdot n_{K,\sigma} \, d\sigma \Bigg| \\ &\lesssim \|w_h\|_{\mathcal{T}} \left(\sum_{\sigma \in \mathcal{E}} d_\sigma \|p - I_C p\|_\sigma^2 \right)^{1/2}. \end{aligned}$$

As a trace theorem and scaling arguments yield (Clément, 1975)

$$\|p - I_C p\|_\sigma \lesssim h_K^{1/2} |p|_{1,K}, \quad \text{if } \sigma \in \mathcal{E}_K, \ \ K \in \mathcal{T},$$

we deduce that

$$|I_2| \lesssim h \|w_h\|_{\mathcal{T}} |p|_{1,\Omega}.$$

With the help of Lemma 2.2, we arrive at

$$|I_2| \lesssim h |w_h|_{1,h} |p|_{1,\Omega}.$$

This estimate and (2.20) in the identity (2.19) lead to the conclusion. □

To estimate the second consistency term we need the following assumption: if g_K denotes the centre of gravity of $K \in \mathcal{T}$, then

$$\left|\sum_{i=1}^{d} \sum_{K \in \mathcal{T}} \int_K \nabla u_i(x) \cdot (x_K - g_K) \partial_i q_h \, \mathrm{d}x\right| \lesssim h \|u\|_{2,\Omega} \|q_h\|, \quad \forall\, q_h \in Q_h. \tag{2.21}$$

At the end of this section we shall give a sufficient condition which guarantees (2.21). This condition means that x_K should not be too far from g_K (the best being $x_K = g_K$, which holds for instance for equilateral triangles as considered in Blanc *et al.* (2004) or for regular tetrahedra). We further give a class of triangulations satisfying this condition.

LEMMA 2.6 Assume that (2.21) holds. Then for all $q_h \in Q_h$ we have

$$|b_h(u, q_h)| \lesssim h \|u\|_{2,\Omega} \|q_h\|. \tag{2.22}$$

Proof. Since $C^\infty(\bar{\Omega})$ is dense in $H^2(\Omega)$, it suffices to show that

$$|b_h(u, q_h)| \lesssim (h\|u\|_{2,\Omega} + \|\operatorname{div} u\|_{0,\Omega})\|q_h\|_{0,\Omega} + \|u \cdot n\|_{0,\Gamma} \|q_h\|_{0,\Gamma}, \tag{2.23}$$

for all $u \in C^\infty(\bar{\Omega})$ and $q_h \in Q_h$. Indeed by density and the properties $\operatorname{div} u = 0$ in Ω and $u = 0$ on Γ, estimate (2.23) leads to (2.22).

In the remainder of the proof u is now fixed in $C^\infty(\bar{\Omega})$ and q_h in Q_h. Using a Taylor expansion with integral remainder, for $i = 1, \ldots, d$ we may write

$$\begin{aligned} u_i(x_K) - u_i(x) &= \nabla u_i(x) \cdot (x_K - x) \\ &\quad + \int_0^1 H(u_i)(tx + (1-t)x_K)(x_K - x) \cdot (x_K - x) t \, \mathrm{d}t, \quad \forall\, x \in K, \ K \in \mathcal{T}, \end{aligned} \tag{2.24}$$

where $H(u_i)(z)$ denotes the Hessian matrix of u_i at the point z. Multiplying this identity by $\partial_i q_h$, integrating the resulting identity on K and summing through i we get

$$b_h(u, q_h) = \sum_{K \in \mathcal{T}} \int_K u \cdot \nabla q_h \, \mathrm{d}x + I_2 + I_3, \tag{2.25}$$

where

$$I_2 = \sum_{i=1}^{d} \sum_{K \in \mathcal{T}} \int_K \nabla u_i(x) \cdot (x_K - x) \partial_i q_h \, \mathrm{d}x$$

$$I_3 = \sum_{i=1}^{d} \sum_{K \in \mathcal{T}} \int_K \int_0^1 H(u_i)(tx + (1-t)x_K)(x_K - x) \cdot (x_K - x) t \, \mathrm{d}t \partial_i q_h \, \mathrm{d}x.$$

It then remains to estimate both terms on the right-hand side. By Green's formula the first term becomes

$$\sum_{K \in \mathcal{T}} \int_K u \cdot \nabla q_h \, \mathrm{d}x = \int_\Omega u \cdot \nabla q_h \, \mathrm{d}x = -\int_\Omega \operatorname{div} u q_h \, \mathrm{d}x + \int_\Gamma u \cdot n q_h \, \mathrm{d}\sigma.$$

By the Cauchy–Schwarz inequality we obtain

$$\left|\sum_{K\in\mathcal{T}}\int_K u\cdot\nabla q_h\,\mathrm{d}x\right| \leqslant \|\operatorname{div} u\|_{0,\Omega}\|q_h\|_{0,\Omega} + \|u\cdot n\|_{0,\Gamma}\|q_h\|_{0,\Gamma}. \tag{2.26}$$

The second term on the right-hand side of (2.25) is transformed as follows:

$$I_2 = \sum_{i=1}^{d}\sum_{K\in\mathcal{T}}\int_K \nabla u_i(x)\cdot(x_K - g_K)\partial_i q_h\,\mathrm{d}x + \sum_{i=1}^{d}\sum_{K\in\mathcal{T}}\int_K \nabla u_i(x)\cdot(g_K - x)\partial_i q_h\,\mathrm{d}x.$$

The first term is estimated via assumption (2.21). For the second one, since g_K is the center of gravity of K, we have $\int_K (g_K - x)\,\mathrm{d}x = 0$. Therefore, for all $r \in (\mathbb{P}_1(\mathcal{T}))^d$, where $\mathbb{P}_1(\mathcal{T}) = \{r \in C(\bar{\Omega}) : r_{|K} \in \mathbb{P}_1(K), \forall K \in \mathcal{T}\}$, we get

$$\sum_{i=1}^{d}\sum_{K\in\mathcal{T}}\int_K \nabla r_i(x)\cdot(g_K - x)\partial_i q_h\,\mathrm{d}x = \sum_{i=1}^{d}\sum_{K\in\mathcal{T}}\nabla r_{i|K}\cdot\int_K (g_K - x)\,\mathrm{d}x\,\partial_i q_{h|K} = 0.$$

This identity allows us to write

$$\sum_{i=1}^{d}\sum_{K\in\mathcal{T}}\int_K \nabla u_i(x)\cdot(g_K - x)\partial_i q_h\,\mathrm{d}x = \sum_{i=1}^{d}\sum_{K\in\mathcal{T}}\int_K \nabla(u_i - r_i)(x)\cdot(g_K - x)\partial_i q_h\,\mathrm{d}x,$$

for all $r \in (\mathbb{P}_1(\mathcal{T}))^d$, and by the Cauchy–Schwarz inequality

$$\left|\sum_{i=1}^{d}\sum_{K\in\mathcal{T}}\int_K \nabla u_i(x)\cdot(g_K - x)\partial_i q_h\,\mathrm{d}x\right| \leqslant \sum_{K\in\mathcal{T}} h_K|u - r|_{1,K}|q_h|_{1,K}, \quad \forall\, r \in (\mathbb{P}_1(\mathcal{T}))^d.$$

A standard a priori error estimate and an inverse inequality (Ciarlet, 1978) lead to

$$\left|\sum_{i=1}^{d}\sum_{K\in\mathcal{T}}\int_K \nabla u_i(x)\cdot(g_K - x)\partial_i q_h\,\mathrm{d}x\right| \lesssim \sum_{K\in\mathcal{T}} h_K|u|_{2,K}\|q_h\|_K.$$

By the discrete Cauchy–Schwarz inequality we arrive at

$$\left|\sum_{i=1}^{d}\sum_{K\in\mathcal{T}}\int_K \nabla u_i(x)\cdot(g_K - x)\partial_i q_h\,\mathrm{d}x\right| \lesssim h|u|_{2,\Omega}\|q_h\|.$$

This estimate and the assumption (2.21) lead to

$$|I_2| \lesssim h\|u\|_{2,\Omega}\|q_h\|. \tag{2.27}$$

Let us now estimate the third term on the right-hand side of (2.25). For $K \in \mathcal{T}$ and $i = 1, \ldots, d$, let

$$I_i(K) = \int_K\int_0^1 H(u_i)(tx + (1-t)x_K)(x_K - x)\cdot(x_K - x)t\,\mathrm{d}t\,\mathrm{d}x.$$

Then we directly get

$$|I_i(K)| \leqslant h_K^2 \int_K \int_0^1 \|H(u_i)(tx + (1-t)x_K)\|_2 t \, \mathrm{d}t \, \mathrm{d}x.$$

We now estimate this last integral. For $t \in (0, 1)$, we denote

$$K_t = \{y = tx + (1-t)x_K : x \in K\}.$$

Since for $y \in K_t$, we have $y - x_K = t(x - x_K)$ for some $x \in K$, we directly deduce that $K_t \subset B(x_K, \rho_t)$, with $\rho_t = th_K$. Consequently, we may estimate

$$|K_t| \lesssim t^d h_K^d.$$

Now making the change of variables $y = tx + (1-t)x_K$ we obtain

$$|I_i(K)| \leqslant h_K^2 \int_0^1 \int_{K_t} \|H(u_i)(y)\|_2 \, \mathrm{d}y t^{1-d} \, \mathrm{d}t.$$

By the Cauchy–Schwarz inequality and the above estimate on $|K_t|$ we get

$$|I_i(K)| \lesssim h_K^{\frac{4+d}{2}} \int_0^1 \left(\int_{K_t} \|H(u_i)(y)\|_2^2 \, \mathrm{d}y \right)^{1/2} t^{1-\frac{d}{2}} \, \mathrm{d}t.$$

As K_t is included in K and $\int_0^1 t^{1-\frac{d}{2}} \, \mathrm{d}t = d - 1$, we can conclude that

$$|I_i(K)| \lesssim h_K^{\frac{4+d}{2}} |u|_{2,K}.$$

As $\partial_i q_h$ is constant on each control volume K, we deduce that

$$|I_3| \lesssim \sum_{K \in \mathcal{T}} h_K^{\frac{4+d}{2}} |u|_{2,K} |\partial_i q_{h|K}|.$$

The standard inverse inequality

$$|\partial_i q_{h|K}| \lesssim h_K^{-\frac{2+d}{2}} \|q_h\|_K \tag{2.28}$$

finally leads to

$$|I_3| \lesssim h|u|_{2,\Omega} \|q_h\|. \tag{2.29}$$

The estimates (2.26), (2.27) and (2.29) in the identity (2.25) yield the conclusion. □

Lemmas 2.5 and 2.6 allow us to obtain the following error estimate.

THEOREM 2.7 Assume that (2.21) holds. Then

$$|u - u_h|_{1,h} + \|p - p_h\| \lesssim h(\|u\|_{2,\Omega} + \|p\|_{1,\Omega}). \tag{2.30}$$

We now give a sufficient condition guaranteeing (2.21).

LEMMA 2.8 Assume that

$$|x_K - g_K| \lesssim h_K^2, \quad \forall K \in \mathcal{T}. \tag{2.31}$$

Then (2.21) holds.

Proof. By the Cauchy–Schwarz inequality we simply estimate

$$\left|\sum_{i=1}^{d}\sum_{K\in\mathcal{T}}\int_K \nabla u_i(x)\cdot(x_K - g_K)\partial_i q_h \,\mathrm{d}x\right| \lesssim |u|_{1,\Omega}\left(\sum_{K\in\mathcal{T}}|x_K - g_K|^2|K||\partial_i q_h|^2\right)^{1/2}.$$

Also by the inverse inequality (2.28) we get

$$\left|\sum_{i=1}^{d}\sum_{K\in\mathcal{T}}\int_K \nabla u_i(x)\cdot(x_K - g_K)\partial_i q_h \,\mathrm{d}x\right| \lesssim |u|_{1,\Omega}\left(\sum_{K\in\mathcal{T}}|x_K - g_K|^2 h_K^{-2}\|q_h\|_K^2\right)^{1/2}.$$

The assumption (2.31) allows us to conclude the proof. □

Even though the above lemma is based on a rough technique, we do not investigate finer sufficient conditions since our numerical experiments show that condition (2.31) is necessary for the convergence of our scheme.

Let us illustrate our assumption (2.31) by giving a class of 2D triangulations satisfying it.

LEMMA 2.9 Assume that $d = 2$. For any $K \in \mathcal{T}$, assume that the interior angles of K are smaller than $\pi/2$ and take x_K as the intersection of the orthogonal bisectors (which then belongs to K). Denote by $h_{3,K} \leqslant h_{2,K} \leqslant h_{1,K} = h_K$ the length of the three edges of K. If there exists $C \geqslant 0$ such that

$$\frac{h_K}{h_{3,K}} \leqslant 1 + Ch_K, \quad \forall K \in \mathcal{T}, \tag{2.32}$$

then the assumption (2.31) holds.

Proof. Denote by α_K^i, $i = 1, 2, 3$, the three interior angles of K. We recall that the regularity assumption on the mesh is equivalent to

$$\alpha_0 \leqslant \alpha_K^i, \quad \forall\, i = 1, 2, 3, \quad K \in \mathcal{T},$$

for some $\alpha_0 > 0$ (this is the so-called Zlámal condition (Zlámal, 1968)). For a fixed $K \in \mathcal{T}$, with the notation from Fig. 3 (for the sake of simplicity we now drop the index K), we have

$$g_K = \left(\frac{2h_2\cos\alpha_1 - h_1}{6}, \frac{h_2\sin\alpha_1}{3}\right) \text{ and } x_K = \left(0, \frac{h_2 - h_1\cos\alpha_1}{2\sin\alpha_1}\right).$$

Therefore, the assumption (2.31) is equivalent to

$$|2h_2\cos\alpha_1 - h_1| \lesssim h_1^2 \text{ and } |3h_2 - 3h_1\cos\alpha_1 - 2h_2\sin^2\alpha_1| \lesssim h_1^2,$$

or dividing by h_1:

$$\begin{cases} 1 - Ch_1 \leqslant \frac{2h_2\cos\alpha_1}{h_1} \leqslant 1 + Ch_1, \\ 3\cos\alpha_1 - Ch_1 \leqslant \frac{h_2}{h_1}(3 - 2\sin^2\alpha_1) \leqslant 3\cos\alpha_1 + Ch_1, \end{cases}$$

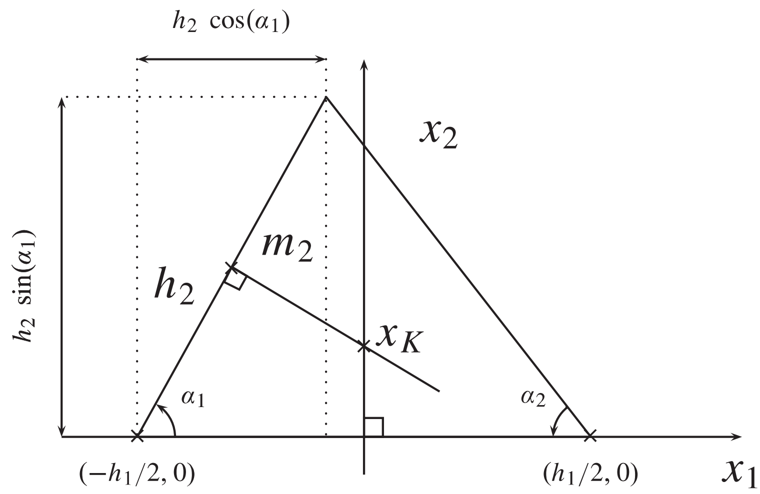

FIG. 3. Some notation.

for some $C \geqslant 0$. By simple calculations this is still equivalent to

$$\begin{cases} \cos\alpha_1 = \frac{h_1}{2h_2} + r_c, \\ \sin^2\alpha_1 = \frac{3}{2} - \frac{3h_1\cos\alpha_1}{2h_2} + r_s, \end{cases} \tag{2.33}$$

with $|r_c| + |r_s| \leqslant Ch_K$.

Using the identity $h_3^2 = h_1^2 + h_2^2 - 2h_1h_2\cos\alpha_1$, it is easy to check that (2.32) implies that

$$\cos\alpha_1 = \frac{1}{2} + \tilde{r}_c,$$

with $|\tilde{r}_c| \leqslant Ch_K$. This condition combined with (2.32) leads to (2.33). □

REMARK 2.10

1. Using the trigonometric identity $\cos^2\alpha_1 + \sin^2\alpha_1 = 1$, we see that the condition (2.33) implies that

$$\frac{h_K}{h_{2,K}} \leqslant 1 + C\sqrt{h_K},$$

for some $C \geqslant 0$. Since this condition is not equivalent to (2.33), we have chosen the stronger assumption (2.32).

2. From the proof of the above lemma we see that triangles satisfying (2.32) are “almost” equilateral but this constraint comes from the particular choice of x_K. As shown in Section 4 for some triangulations made of rectangular triangles, we may take $x_K = g_K$ and therefore (2.31) automatically holds.
3. The assumption (2.31) holds if $x_K = g_K$. In particular, it holds for equilateral triangles or for regular tetrahedra. As a consequence, our results extend the error estimates obtained in Blanc *et al.* (2004) to a larger class of 2D triangulations and further give its 3D counterpart.

In view of Theorem 2.1 of Blanc *et al.* (2004) we end up with an error estimate between u_h and $\mathcal{M}_h u \in V_h$ defined by

$$(\mathcal{M}_h u)_{|K} = \frac{1}{|K|}\int_K u(x)\,\mathrm{d}x.$$

THEOREM 2.11 Assume that (2.31) holds. Then we have

$$\|\mathcal{M}_h u - u_h\|_{\mathcal{T}} \lesssim h(\|u\|_{2,\Omega} + \|p\|_{1,\Omega}). \tag{2.34}$$

Proof. Integrating the identity (2.24) on $K \in \mathcal{T}$ we get (with the notation from Lemma 2.6):

$$u_i(x_K) - \frac{1}{|K|}\int_K u_i(x)\,\mathrm{d}x = |K|^{-1}\left(\int_K \nabla u_i(x)\cdot(x_K - x)\,\mathrm{d}x + I_i(K)\right).$$

The arguments of Lemmas 2.6 and 2.8 lead to

$$|K|^{1/2}\left|u_i(x_K) - \frac{1}{|K|}\int_K u_i(x)\,\mathrm{d}x\right| \lesssim h_K\|u\|_{2,K}.$$

Summing over the squares of this estimate we arrive at

$$\|I_h u - \mathcal{M}_h u\|_{\mathcal{T}} \lesssim h\|u\|_{2,\Omega}.$$

We conclude using Lemma 2.2, the triangle inequality and the estimate (2.30). □

3. Extension to rectangular or hexahedral meshes

3.1 *The scheme*

In this section, we extend the previous results to meshes $\mathcal{T}$ made of rectangles ($d = 2$) or of hexahedra ($d = 3$). We assume that the mesh is regular in Ciarlet's sense and we further take $x_K = g_K$, the center of gravity of K, so that the orthogonality condition is automatically satisfied (see Fig. 4). Without loss of generality we may assume that the edges of the rectangles or of the hexahedra are parallel to the x_i-axis.

Since the gradient of a function in $\mathbb{Q}_1$ is not constant, if V_h is defined as before and Q_h is given by

$$Q_h = \{q_h \in Q \cap C(\bar{\Omega}) : q_{h|K} \in \mathbb{Q}_1(K), \forall K \in \mathcal{T}\}, \tag{3.1}$$

then the bilinear form b_h, defined as before, does not satisfy the inf–sup condition on $V_h \times Q_h$,

We therefore need to modify the previous finite volume scheme by approximating the velocity on each element K by a function in $V_K := \nabla\mathbb{Q}_1(K)$. In other words, if $d = 2$, then

$$\begin{aligned} V_K &= (\mathbb{P}_0(K))^2 \oplus \operatorname{Span}\begin{pmatrix} x_2 - x_{K2} \\ x_1 - x_{K1}\end{pmatrix} \\ &= \left\{v_h : v_h(x) = v_K + \alpha_K\begin{pmatrix} x_2 - x_{K2} \\ x_1 - x_{K1}\end{pmatrix}, \forall x \in K; v_K \in (\mathbb{P}_0(K))^2, \alpha_K \in \mathbb{P}_0(K)\right\},\end{aligned}$$

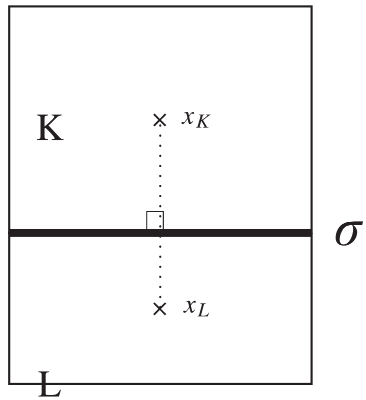

FIG. 4. The orthogonality condition for rectangles.

when $x_K = (x_{K1}, x_{K2})$. On the other hand, if $d = 3$, then writing $x_K = (x_{K1}, x_{K2}, x_{K3})$, we have

$$\begin{aligned} V_K &= (\mathbb{P}_0(K))^3 \oplus \operatorname{Span}\begin{pmatrix} x_2 - x_{K2} \\ x_1 - x_{K1} \\ 0 \end{pmatrix} \oplus \operatorname{Span}\begin{pmatrix} x_3 - x_{K3} \\ 0 \\ x_1 - x_{K1} \end{pmatrix} \\ &\oplus \operatorname{Span}\begin{pmatrix} 0 \\ x_3 - x_{K3} \\ x_1 - x_{K1} \end{pmatrix} \oplus \operatorname{Span}\begin{pmatrix} (x_2 - x_{K2})(x_3 - x_{K3}) \\ (x_1 - x_{K1})(x_3 - x_{K3}) \\ (x_1 - x_{K1})(x_2 - x_{K2}) \end{pmatrix} \\ &= \{v : v(x) = v_K + \alpha_{1K}\begin{pmatrix} x_2 - x_{K2} \\ x_1 - x_{K1} \\ 0 \end{pmatrix} + \alpha_{2K}\begin{pmatrix} x_3 - x_{K3} \\ 0 \\ x_1 - x_{K1} \end{pmatrix} \\ &+ \alpha_{3K}\begin{pmatrix} 0 \\ x_3 - x_{K3} \\ x_1 - x_{K1} \end{pmatrix} + \alpha_{4K}\begin{pmatrix} (x_2 - x_{K2})(x_3 - x_{K3}) \\ (x_1 - x_{K1})(x_3 - x_{K3}) \\ (x_1 - x_{K1})(x_2 - x_{K2}) \end{pmatrix}, \forall x \in K; \\ &\forall v_K \in (\mathbb{P}_0(K))^3, \alpha_{iK} \in \mathbb{P}_0(K), i = 1, \dots, 4\}. \end{aligned}$$

In the sequel for $v_h \in V_K$, we always denote by v_K its constant part, or equivalently $v_K = v_h(x_K)$.

As before we now introduce the space V_h made of functions being piecewise in V_K

$$V_h := \{v_h \in (L^2(\Omega))^d : v_{h|K} \in V_K, \forall K \in \mathcal{T}\},$$

equipped with the mesh depending norm

$$\|v_h\|_{\mathcal{T}}^2 := \sum_{\sigma \in \mathcal{E}} \frac{|\sigma|}{d_\sigma} |D_\sigma v_h|^2 + \sum_{K \in \mathcal{T}} \int_K |\nabla v_h|^2 \, dx,$$

where $D_\sigma v_h$ is defined as in the previous section (recalling that $v_K = v_h(x_K)$). The space Q_h defined by (3.1) is further equipped with the $L^2(\Omega)$-norm.

Now multiplying the equation $-\nu \Delta u + \nabla p = f$ by $v_h \in V_h$ and integrating the result on $K \in \mathcal{T}$, we get after integration by parts

$$-\nu \int_{\partial K} \frac{\partial u}{\partial n} \cdot v_h \, d\sigma + \nu \int_K \nabla u \cdot \nabla v_h \, dx + \int_K v_h \cdot \nabla p \, dx = \int_K f \cdot v_h \, dx.$$

Approximating u by $u_h \in V_h$, p by $p_h \in Q_h$ and $\frac{\partial u}{\partial n}$ by the numerical flux $F_{K,\sigma}(u_h)$ (defined as in the previous section) on each edge/face σ of K, we arrive at

$$-\nu \sum_{\sigma \in \mathcal{E}_K} F_{K,\sigma}(u_h) v_K + \nu \int_K \nabla u_h \cdot \nabla v_h \, \mathrm{d}x + \int_K v_h \cdot \nabla p_h \, \mathrm{d}x = \int_K f \cdot v_h \, \mathrm{d}x, \quad \forall v_h \in V_K, \; K \in \mathcal{T}.$$

These equations are completed with (2.4) and (2.5) to get the approximation scheme of the Stokes system (2.1).

For $v_h, w_h \in V_h$ and $q_h \in Q_h$ we define $b_h(v_h, q_h)$ as before and take here

$$a_h(v_h, w_h) := \sum_{K \in \mathcal{T}} \left(- \sum_{\sigma \in \mathcal{E}_K} F_{K,\sigma}(v_h) \cdot w_K + \int_K \nabla v_h \cdot \nabla w_h \, \mathrm{d}x \right).$$

With these notations the scheme proposed above may be formulated as in Section 2: find $u_h \in V_h$ and $p_h \in Q_h$ to solve (2.6).

REMARK 3.1 The proposed scheme means that we approximate the pressure by a Galerkin method and the velocity by a kind of discontinous Galerkin method (Cockburn *et al.*, 2000), but the simple structure of V_h implies that this method is close to a cell-centered method. □

As before we shall check the requested properties on a_h and b_h. Introducing the natural inner product associated with $\| \cdot \|_{\mathcal{T}}$

$$(v_h, w_h)_{\mathcal{T}} := \sum_{\sigma \in \mathcal{E}} \frac{|\sigma|}{d_\sigma} D_\sigma v_h D_\sigma w_h + \sum_{K \in \mathcal{T}} \int_K \nabla v_h \cdot \nabla w_h \, \mathrm{d}x, \quad \forall \, v_h, w_h \in V_h,$$

we immediately see that the statements of Lemma 2.2 hold. Therefore, a_h is coercive and continuous on V_h.

Let us pass to the inf–sup condition for b_h.

LEMMA 3.2 b_h satisfies the inf–sup condition (2.10).

Proof. For a fixed $q_h \in Q_h$, we consider $v \in (H_0^1(\Omega))^d$ satisfying (2.11) and (2.12). As in Lemma 3.2, Green's formula yields

$$\|q_h\|^2 = -\int_\Omega \operatorname{div} v q_h = \int_\Omega v \cdot \nabla q_h.$$

Since $\nabla q_{h|K}$ belongs to V_K, this identity may be transformed as follows:

$$\|q_h\|^2 = \sum_{K \in \mathcal{T}} \int_K P_K v \cdot \nabla q_h,$$

where $P_K v$ is the orthogonal projection of v on V_K (for the $L^2(K)^d$-inner product). Therefore, the (vector-valued) function $v_h \in V_h$ defined by $v_K = P_K v$ satisfies (2.13).

The conclusion holds if the estimate (2.14) is valid. We check this estimate in 2D, the 3D-case being treated similarly. Direct calculations yield

$$P_K v = \mathcal{M}_K v + \alpha_K \begin{pmatrix} x_2 - x_{K2} \\ x_1 - x_{K1} \end{pmatrix},$$

where $\mathcal{M}_K v = \frac{1}{|K|}\int_K v$ is the mean of v on K and

$$\alpha_K = \left(\int_K \|x - x_K\|^2 \,\mathrm{d}x\right)^{-1} \int_K v(x) \cdot \begin{pmatrix} x_2 - x_{K2} \\ x_1 - x_{K1} \end{pmatrix} \mathrm{d}x.$$

Since $x_i - x_{Ki}$ has a zero mean on K, we may write

$$\alpha_K = \left(\int_K \|x - x_K\|^2 \,\mathrm{d}x\right)^{-1} \int_K (v - \mathcal{M}_K v) \cdot \begin{pmatrix} x_2 - x_{K2} \\ x_1 - x_{K1} \end{pmatrix} \mathrm{d}x.$$

Therefore, by Poincaré's inequality, we get

$$|\alpha_K| \lesssim h^{-1} |v|_{1,K}.$$

This estimate implies that

$$\int_K |\nabla v_h|^2 \,\mathrm{d}x \lesssim h^2 |\alpha_K|^2 \lesssim |v|_{1,K}^2.$$

On the other hand, as in Lemma 2.3, we have

$$\sum_{\sigma \in \mathcal{E}} |D_\sigma v_h|^2 \lesssim |v|_{1,\Omega}^2.$$

These two estimates combined with the estimate (2.12) lead to (2.14). □

LEMMA 3.3 b_h is continuous, i.e. it satisfies (2.15).

Proof. Using Green's formula on each control volume K we may write

$$b_h(v_h, q_h) = b_{1h}(v_h, q_h) + b_{2h}(v_h, q_h),$$

where we have set

$$b_{1h}(v_h, q_h) = \sum_{K \in \mathcal{T}} \sum_{\sigma \in \mathcal{E}_K} \int_\sigma q_h v_K \cdot n_K$$

$$b_{2h}(v_h, q_h) = \sum_{K \in \mathcal{T}} \int_K (v_h - v_K) \cdot \nabla q_h.$$

By the proof of Lemma 2.4, we have

$$|b_{1h}(v_h, q_h)| \lesssim \|v_h\|_{\mathcal{T}} \|q_h\|. \tag{3.2}$$

On the other hand, by inverse inequalities we have

$$\|v_h - v_K\|_K \lesssim h_K \|\nabla v_h\|_K,$$

$$\|\nabla q_h\|_K \lesssim h_K^{-1} \|q_h\|_K.$$

These estimates and the Cauchy–Schwarz inequality lead to

$$|b_{2h}(v_h, q_h)| \lesssim \sum_{K \in \mathcal{T}} \|\nabla v_h\|_K \|q_h\|_K \lesssim \|v_h\|_{\mathcal{T}} \|q_h\|.$$

This estimate and (3.2) yield the conclusion. □

3.2 *The error estimate*

As in the previous section regarding (2.6) as a non-conforming approximation of (2.2) we deduce that the error estimate (2.17) holds, with the same notation, except for $I_h u$, here defined on each element K by

$$(I_h u)_{|K} = u(x_K) + r_K,$$

where r_K is uniquely determined by the condition

$$\int_K \nabla r_K \cdot \nabla v_h = \int_K \nabla u \cdot \nabla v_h, \quad \forall\, v_h \in V_K.$$

We then need to estimate each term on the right-hand side of (2.17). The first two terms are treated as before, while the estimate of the consistency terms slightly differs.

LEMMA 3.4 For all $w_h \in V_h$, the estimate (2.18) holds.

Proof. Using Green's formula we write

$$\nu a_h(u, w_h) + b_h(w_h, p) - (f, w_h) = \nu I_1 - I_2 + I_3,$$

where I_1, I_2 have the same meaning as in Lemma 2.5, with the notation $w_K = w_h(x_K)$, while I_3 is defined by

$$I_3 = \sum_{K\in\mathcal{T}} \int_K \nabla(I_{\mathrm{C}} p - p) \cdot (w_h - w_K)\, \mathrm{d}x.$$

By the Cauchy–Schwarz inequality we may write

$$|I_3| \leqslant \sum_{K\in\mathcal{T}} |I_{\mathrm{C}} p - p|_{1,K} \|w_h - w_K\|_K.$$

An inverse inequality and the stability of the Clément interpolant in H^1 (i.e. $|I_{\mathrm{C}} p|_{1,\Omega} \lesssim |I_{\mathrm{C}} p|_{1,\Omega}$) yields

$$|I_3| \lesssim h|p|_{1,\Omega} \left(\sum_{K\in\mathcal{T}} \|\nabla w_h\|_K^2 \right)^{1/2}.$$

This estimate and the estimate of the terms I_1 and I_2 obtained as in Lemma 2.5 lead to the required estimate. □

LEMMA 3.5 For all $q_h \in Q_h$, (2.22) holds.

Proof. As before it suffices to show (2.23) for $u \in C^\infty(\bar{\Omega})$ and $q_h \in Q_h$. As in Lemma 2.6 using a Taylor expansion, we obtain the identity (2.25), with the same definition for I_2 and I_3. The first term of the right-hand side of (2.25) as well as I_3 are estimated in the same manner as in Lemma 2.6. It then remains to estimate I_2. For this purpose, we write equivalently

$$I_2 = I_{21} + I_{22},$$

where

$$I_{21} = \sum_{i=1}^{d} \sum_{K \in \mathcal{T}} \int_K \partial_i u_i(x)(x_{Ki} - x_i)\partial_i q_h \, \mathrm{d}x,$$
$$I_{22} = \sum_{\substack{i,j=1,\dots,d \\ i \neq j}} \sum_{K \in \mathcal{T}} \int_K \partial_j u_i(x)(x_{Kj} - x_j)\partial_i q_h \, \mathrm{d}x.$$

For the first term, recalling that x_K is the center of gravity of each element K, we remark that

$$\sum_{i=1}^{d} \sum_{K \in \mathcal{T}} \int_K \partial_i r_i(x)(x_{Ki} - x_i)\partial_i q_h \, \mathrm{d}x = 0,$$

for all $r \in (\mathbb{Q}_1(\mathcal{T}))^d$, where $\mathbb{Q}_1(\mathcal{T}) = \{r \in C(\bar{\Omega}) : r_{|K} \in \mathbb{Q}_1(K), \forall K \in \mathcal{T}\}$. Taking $r = Iu$, the Lagrange interpolant of u, we get

$$I_{21} = \sum_{i=1}^{d} \sum_{K \in \mathcal{T}} \int_K \partial_i (u_i - r_i)(x)(x_{Ki} - x_i)\partial_i q_h(x) \, \mathrm{d}x,$$

and by the Cauchy–Schwarz inequality

$$|I_{21}| \leqslant \sum_{K \in \mathcal{T}} h_K |u - r|_{1,K} |q_h|_{1,K}.$$

A standard a priori error estimate and an inverse inequality (Ciarlet, 1978) lead to

$$|I_{21}| \lesssim \sum_{K \in \mathcal{T}} h_K |u|_{2,K} \|q_h\|_K.$$

By the discrete Cauchy–Schwarz inequality we arrive at

$$|I_{21}| \lesssim h\|u\|_{2,\Omega}\|q_h\|. \tag{3.3}$$

For the second term, Green's formula on each element K yields

$$I_{22} = -\sum_{\substack{i,j=1,\dots,d \\ i \neq j}} \sum_{K \in \mathcal{T}} \int_K \partial_{ij}^2 u_i(x)(x_{Kj} - x_j) q_h \, \mathrm{d}x + \sum_{\substack{i,j=1,\dots,d \\ i \neq j}} \sum_{K \in \mathcal{T}} \int_{\partial K} \partial_j u_i(x)(x_{Kj} - x_j) n_i q_h \, \mathrm{d}\sigma(x).$$

We remark that the second term of this right-hand side is equal to zero, since for a fixed edge/face $\sigma = K \cap L$ with $K, L \in \mathcal{T}$ for which $n_i \neq 0$, $(x_{Kj} - x_j) = (x_{Lj} - x_j)$ on σ, since $j \neq i$. Consequently, by the Cauchy–Schwarz inequality we obtain

$$|I_{22}| \lesssim h\|u\|_{2,\Omega}\|q_h\|. \tag{3.4}$$

By the estimates (3.3) and (3.4) and the splitting of I_2, we conclude that

$$|I_2| \lesssim h\|u\|_{2,\Omega}\|q_h\|.$$

□

4. Some numerical tests

We start with some tests for meshes made of triangles. To implement the system (2.3)–(2.5) or equivalently (2.6) for a triangular mesh, we use a variant of the standard Uzawa algorithm (see the System (5.4) and (5.5) of Temam, 1984). The only problem is that for u_h in V_h, $\operatorname{div} u_h$ does not belong to $L^2(\Omega)$, so we consider $\operatorname{div} u_h$ as an element of Q'_h (see below). Consequently, for a fixed parameter $\delta \in\ ]0,1[$ our algorithm is the following one:

- start with arbitrary $u_h^0 \in V_h$ and $p_h^0 \in Q_h$,
- for $k = 0, 1, \ldots$, find $u_h^{k+1} \in V_h$ satisfying

$$-\nu \sum_{\sigma \in \mathcal{E}_K} F_{K,\sigma}(u_h^{k+1}) = \int_K f \,\mathrm{d}x - \int_K \nabla p_h^k \,\mathrm{d}x, \quad \forall K \in \mathcal{T}, \tag{4.1}$$

  and find $p_h^{k+1} \in Q_h$ satisfying

$$\int_K p_h^{k+1} q \,\mathrm{d}x = \int_K p_h^k \, q \,\mathrm{d}x - \delta \int_\Omega u_h^{k+1} \cdot \nabla q \,\mathrm{d}x, \quad \forall q \in Q_h, \tag{4.2}$$

- stop the algorithm if $\| p_h^k - p_h^{k+1} \|_{0,\Omega}$ is small enough.

REMARK 4.1 Note that at each step of the proposed algorithm, (4.1) requires the resolution of two linear systems as for the standard Uzawa algorithm, while (4.2) is directly solved using a mass lumping process.

The above algorithm is now applied in the domain $\Omega :=]0,1[^2$ and for a uniform family of triangulations $\mathcal{T} = T_n, n = 4, 8, \ldots, 128$, as described in Fig. 5 for $n = 8$. The parameter δ is fixed at 0.8 and we stop the algorithm once $\| p_\tau^k - p_\tau^{k+1} \|_{0,\Omega} \leqslant 10^{-8}$. The tests will be performed with the exact solutions u and p of the Stokes system with $\nu = 1$ given by

$$\begin{aligned} u_1(x,y) &:= -2(x-1)^2 x^2 (y-1) y (2y-1)\,, \\ u_2(x,y) &:= 2(y-1)^2 y^2 (x-1) x (2x-1)\,, \\ p(x,y) &:= x - 0.5 \end{aligned}$$

Note that (u, p) belongs to $(H^2(\Omega))^2 \times H^1(\Omega)$ so that the regularity assumption of Section 2 is satisfied.

For the first test, we chose the points x_K as the center of gravity of K (so that the assumption (2.21) holds and first-order of convergence is expected for $\| I_h u - u_h \|_{\mathcal{T}} + \| p - p_h \|_{0,\Omega}$). Table 1 presents different errors with respect to n. Note that in the last column iter(n) denotes the number of iterations of the algorithm. Figure 6 illustrates the rates of convergence for the natural norm $\| I_h u - u_h \|_{\mathcal{T}} + \| p - p_h \|$ and for the discrete L^2-norm $\| I_h u_h - u_h \|$ in a double logarithmic scale, so that the slope of the curves corresponds to the order of convergence. From these results we may conclude that the theoretical order of convergence 1 is satisfied for the natural norm and is of order 2 in the discrete L^2-norm (a standard phenomenom).

In the following tests we want to underline the necessity of the assumption (2.31) in the case of a triangular mesh. For this purpose we have made the following choices for x_K: the first one for which (2.31) still holds, namely by taking $d(x_K, g_K) \sim h_K^2$ as illustrated by Fig. 7(left) and the second one for which (2.31) no longer holds, i.e. by taking $d(x_K, g_K) \sim h_K$ as illustrated by Fig. 7(right).

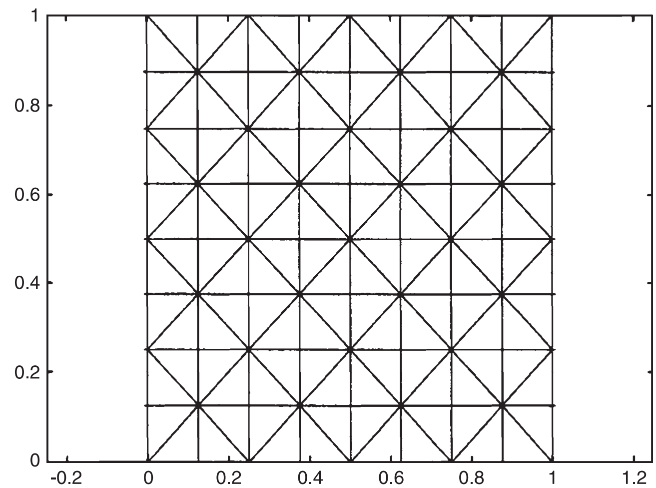

FIG. 5. The triangulation T_n for $n = 8$.

TABLE 1 *Errors w.r.t. n*

n	$\|I_h u - u_h\|_{0,\Omega}$	$\|I_h u - u_h\|_{\mathcal{T}}$	$\|u - u_h\|_{0,\Omega}$	$\|p - p_h\|_{0,\Omega}$	iter(n)
4	1.27×10^{-3}	1.88×10^{-2}	3.42×10^{-3}	1.38×10^{-2}	49
8	3.04×10^{-4}	8.79×10^{-3}	1.71×10^{-3}	4.59×10^{-3}	60
16	8.45×10^{-5}	4.79×10^{-3}	8.46×10^{-3}	1.80×10^{-3}	68
32	2.26×10^{-5}	2.53×10^{-3}	4.22×10^{-4}	6.60×10^{-4}	73
64	5.85×10^{-6}	1.30×10^{-3}	2.11×10^{-4}	2.36×10^{-4}	76
128	1.49×10^{-6}	6.63×10^{-4}	1.05×10^{-4}	8.42×10^{-5}	79

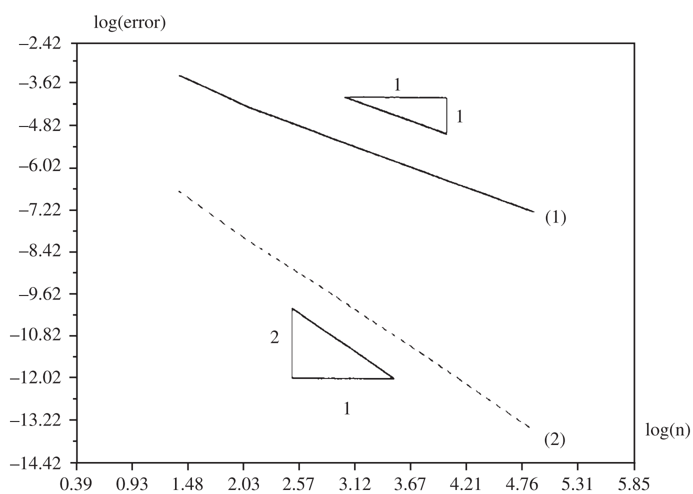

FIG. 6. Rate of convergence of $\|I_h u - u_h\|_{\mathcal{T}} + \|p - p_h\|$ (line (1)) and $\|I_h u_h - u_h\|$ (line (2)) w.r.t. n.

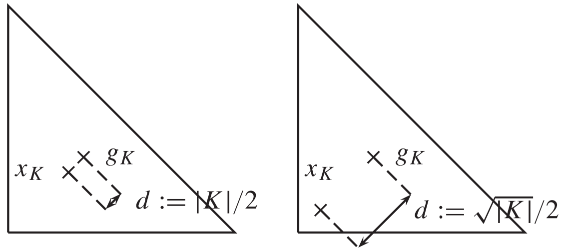

FIG. 7. Illustration of the choice of x_K, $K \in \mathcal{T}$.

TABLE 2 *Errors w.r.t. n as* $d(x_K, g_K) \sim h_K^2$

n	$\|I_h u - u_h\|_{0,\Omega}$	$\|I_h u - u_h\|_{\mathcal{T}}$	$\|u - u_h\|_{0,\Omega}$	$\|p - p_h\|_{0,\Omega}$
4	1.26×10^{-3}	1.85×10^{-3}	3.41×10^{-3}	1.38×10^{-2}
8	3.02×10^{-4}	8.73×10^{-3}	1.71×10^{-3}	4.59×10^{-3}
16	8.44×10^{-5}	4.78×10^{-3}	8.46×10^{-4}	1.80×10^{-3}
32	2.26×10^{-5}	2.53×10^{-3}	4.22×10^{-4}	6.60×10^{-4}
64	5.85×10^{-6}	1.30×10^{-3}	2.11×10^{-4}	2.36×10^{-4}
128	1.49×10^{-6}	6.63×10^{-4}	1.05×10^{-4}	8.42×10^{-5}

TABLE 3 *Errors w.r.t. n as* $d(x_K, g_K) \sim h_K$

n	$\|I_h u - u_h\|_{0,\Omega}$	$\|I_h u - u_h\|_{\mathcal{T}}$	$\|u - u_h\|_{0,\Omega}$	$\|p - p_h\|_{0,\Omega}$
4	1.36×10^{-3}	2.14×10^{-2}	4.17×10^{-2}	1.51×10^{-2}
8	4.13×10^{-4}	1.28×10^{-2}	2.06×10^{-3}	7.36×10^{-3}
16		No	convergence	
32		No	convergence	
64		No	convergence	
128		No	convergence	

TABLE 4 *Different error norms w.r.t. n for rectangular meshes*

n	$\|I_h u - u_h\|_{0,\Omega}$	$\|I_h u - u_h\|_{\mathcal{T}}$	$\|u - u_h\|_{0,\Omega}$	$\|p - p_h\|_{0,\Omega}$	iter(n)
4	2.15×10^{-3}	1.62×10^{-2}	4.72×10^{-3}	1.41×10^{-2}	64
8	6.98×10^{-4}	7.12×10^{-3}	2.20×10^{-3}	7.12×10^{-3}	89
16	1.98×10^{-4}	2.77×10^{-3}	1.05×10^{-3}	2.59×10^{-3}	87
32	5.26×10^{-5}	1.03×10^{-3}	5.19×10^{-4}	9.19×10^{-4}	94
64	1.35×10^{-5}	3.72×10^{-4}	2.58×10^{-4}	3.21×10^{-4}	98
128	3.43×10^{-6}	1.33×10^{-4}	1.29×10^{-4}	1.12×10^{-4}	104

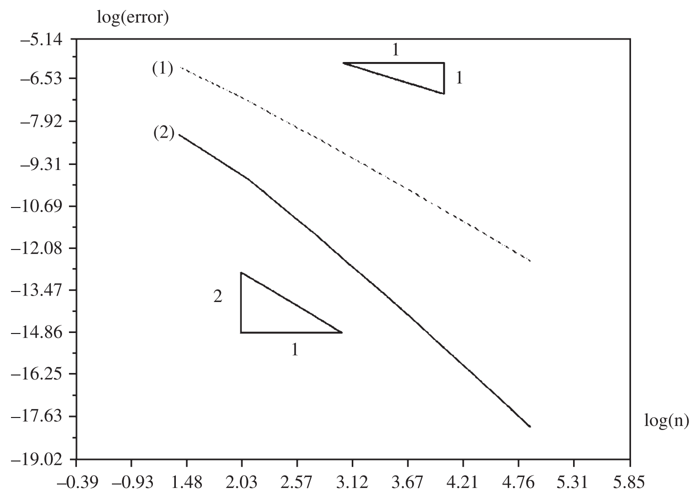

FIG. 8. Rate of convergence of $\|I_h u - u_h\|_{\mathcal{T}} + \|p - p_h\|$ (line (1)) and $\|I_h u_h - u_h\|$ (line (2)) w.r.t. n for rectangular meshes.

Tables 2 and 3 present the same errors with respect to n as before for both choices. In Table 2, we see in the case $d(x_K, g_K) \sim h_K^2$ similar rates of convergence as in the case $d(x_K, g_K) = 0$. In the case $d(x_K, g_K) \sim h_K$, from Table 3 we observe lower rates of convergence and even for n large enough no convergence of the algorithm. From these two tests we may conclude the necessity of the condition (2.31) to ensure convergence of (u_h, p_h) to (u, p).

We finish this section by a test concerning the scheme proposed in Section 3 for meshes made of squares. As before we consider the Stokes system in the unit square with the same solution (u, p). The numerical results are obtained using a similar algorithm as before. We give different error norms in Table 4 and Fig. 8. They confirm the order 1 for the norm $\|I_h u - u_h\|_{\mathcal{T}} + \|p - p_h\|$ as theoretically expected and an order 2 for the discrete L^2-norm.

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