A PRIMAL-DUAL ACTIVE SET METHOD FOR BILATERALLY CONTROL CONSTRAINED OPTIMAL CONTROL OF THE NAVIER-STOKES EQUATIONS

J. C. DE LOS REYES

Abstract. This paper is devoted to the analysis and numerical solution of distributed optimal control of the Navier-Stokes equations in presence of bilateral pointwise control constraints. The analysis of the problem involves the proof of existence of an optimal solution, as well as the presentation of necessary and sufficient conditions for optimality. For the numerical solution of the problem we apply a primal-dual active set strategy and show global and local convergence properties of the method. Finally, some numerical experiments, which illustrate the performance of the method, are discussed.

1. Introduction

In this paper we propose a primal-dual active set strategy for the numerical solution of the following bilaterally control-constrained flow control problem:

\[
\begin{array}{l}
\min \ J(y, u) = \frac{1}{2} \int_\Omega |y - z_d|^2 \, dx + \frac{\alpha}{2} \int_\Omega |u|^2 \, dx \\
\quad \text{subject to} \\
\quad \nu \Delta y + (y \cdot \nabla) y + \nabla p = Bu \\
\quad \text{div} \ y = 0 \\
\quad y|_\Gamma = g \\
\quad a \leq u \leq b, \ a.e.
\end{array}
\]

(1.1)

The method proposed is based on a reformulation of the complementarity problem, present in the resulting optimality system for (1.1), as an operator equation involving the max and min functions. The strategy exhibits important features like global and local superlinear convergence and termination after a finite number of steps. After the original presentation in the context of control-constrained optimal control of the Laplace equation [1], the method was investigated for optimal control of ODE’s [2], general control-constrained optimal control problems [3], state-constrained optimal control problems [4, 5, 6] and numerical solution of variational inequalities [7]. The efficiency of the method is tested for control- and state-constrained control problems in [5], where the method is compared with two interior point algorithms. The equivalence of the primal-dual strategy and the semi-smooth Newton method in the linear case is presented in [8], together with local superlinear convergence results.

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In [9] the authors extend the strategy to a general class of nonlinear optimal control problems and show global and local convergence properties. The method is also investigated for flow problems in [10], where the strategy is applied to the boundary optimal control of the Navier-Stokes equations.

Previous contributions on the numerical treatment of distributed control-constrained optimal control of the Navier-Stokes equations are [11] and [12]. In [11] the authors apply a SQP method for the numerical solution of the control problem and, using the efficiency of the primal-dual active set strategy for linear problems, solve the constrained linear-quadratic subproblems with the mentioned method. In [12], a semi-smooth Newton method is used for the numerical solution of the problem and local superlinear convergence of the method is verified.

Differently from [11] and [12], we propose the direct application of the primal-dual active set method to the nonlinear problem. Based on the comparison of algorithms carried out in [13] and [9], we have strong evidence to expect a better behavior of our method with respect to the Navier-Stokes optimal control problem. Additionally, the nonlinear primal-dual framework allows us to obtain not only local, but also global convergence results.

The outline of the paper is as follows. In Section 1 we present existence and uniqueness results for the state equations, as well as some extra regularity results for the solution. Section 2 deals with the distributed control problem and presents existence results, first order necessary and second order sufficient optimality conditions. In Section 3 the primal-dual active set method is applied to the resulting optimality system and the convergence analysis is carried out. Finally in Section 4, some selected numerical experiments are presented.

2. State equations

Firstly, we introduce the notation to be used. We denote by \( (\cdot, \cdot)_X \) the inner product in the Hilbert space \( X \) and by \( \| \cdot \|_X \) the associated norm. The topological dual of \( X \) is denoted by \( X' \) and the duality pairing is written as \( \langle \cdot, \cdot \rangle_{X',X} \). If the \( L^2 \)-inner product or norm are meant, the subindex is suppressed. The space of infinitely differentiable functions with compact support on \( \Omega \) is denoted by \( \mathcal{D}(\Omega) \) and its dual, the distributions space, by \( \mathcal{D}'(\Omega) \). The Sobolev space \( W^{m,p}(\Omega) \) is the space of \( L^p(\Omega) \) functions whose distributional derivatives up to order \( m \) are also in \( L^p(\Omega) \). For these spaces a norm is introduced in the following way:

\[
\| u \|_{W^{m,p}} = \left( \sum_{|j| \leq m} \| D^j u \|_{L^p}^p \right)^{1/p},
\]

where \( D^j \) denotes the differentiation operator with respect to the multi-index \( j = (j_1, \ldots, j_n) \), i.e. \( D^j = \frac{\partial^{j_1} \partial^{j_n}}{\partial x_1 \cdots \partial x_n} \), with \( |j| = \sum_{i=1}^n j_i \). If \( p = 2 \) we use for \( W^{m,2}(\Omega) \) the standard notation \( H^m(\Omega) \), which constitute Hilbert spaces with the scalar product

\[
(u, v)_{H^m} = \sum_{|j| \leq m} \frac{1}{2} (D^j u, D^j v).
\]
We denote the closure of \( D(\Omega) \) in the \( W^{m,p}(\Omega) \) norm by \( W_0^{m,p}(\Omega) \). If the domain \( \Omega \) is smooth enough, \( H_0^1(\Omega) = \{ v \in H^1(\Omega) : v|_{\partial\Omega} = 0 \} \) and the Poincaré inequality holds, i.e.,

\[
\|u\| \leq \kappa \|\nabla u\|, \text{ for all } u \in H_0^1(\Omega),
\]

where \( \kappa \) is a constant dependent on \( \Omega \). Consequently, in \( H_0^1(\Omega) \) the \( H^1 \)-norm is equivalent to the norm

\[
\|u\|_{H_0^1} = \|\nabla u\|
\]

and \( H_0^1(\Omega) \) is a Hilbert space with the inner product

\[(u,v)_{H_0^1} = (\nabla u, \nabla v).\]

The dual of \( H_0^1(\Omega) \) is denoted by \( H^{-1}(\Omega) \). We introduce the bold notation for the product of spaces, i.e., for example \( L^2(\Omega) = \prod_{i=1}^m L^2(\Omega) \), and provide them with the Euclidean product norm. The divergence free distribution space is denoted by \( \mathcal{V} \) and its closure in \( H_0^1(\Omega) \) by \( \mathbf{V} \), which is characterized as \( \mathbf{V} = \{ v \in H_0^1(\Omega) : \text{div} v = 0 \} \).

Additionally, we introduce the subspaces \( H_0^{1/2} = \{ v \in H^{1/2}(\Gamma) : \int_{\Gamma} v \cdot \vec{n} \ d\Gamma = 0 \} \) and \( H = \{ v \in H^1(\Omega) : \text{div} v = 0 \} \) of \( H^{1/2}(\Gamma) \) and \( H^1(\Omega) \) respectively. Since the functional \( T(u) = \int_{\Gamma} u \cdot \vec{n} \ d\Gamma \) is linear and bounded from \( L^2(\Gamma) \to \mathbb{R} \) and, due to the embedding \( H^{1/2}(\Gamma) \hookrightarrow L^2(\Gamma) \) with continuous injection, also continuous from \( H^{1/2}(\Gamma) \to \mathbb{R} \), we get that the linear subspace \( H_0^{1/2} = \ker(T) \) is closed and consequently a Hilbert space with the scalar product induced by \( H^{1/2}(\Gamma) \). Using the same arguments we can argue that the divergence operator is linear and bounded from \( H^1(\Omega) \to L^2(\Omega) \). Consequently we may also conclude that \( \mathbf{H} \) is a Hilbert space with the \( H^1(\Omega) \) norm.

Considering the regular open bounded domain \( \Omega \subset \mathbb{R}^2 \), the stationary Navier-Stokes equations can be formulated in the following way:

\[
\begin{align}
(2.1) \quad & -\nu \Delta y + (y \cdot \nabla) y + \nabla p = f \\
(2.2) \quad & \text{div } y = 0 \\
(2.3) \quad & y|_{\Gamma} = g,
\end{align}
\]

where \( f \in H^{-1}(\Omega) \), \( g \in H_0^{1/2} \) and \((y \cdot \nabla)y = \left(y_1 \frac{\partial y_1}{\partial x_1} + y_2 \frac{\partial y_1}{\partial x_2}, \ y_1 \frac{\partial y_2}{\partial x_1} + y_2 \frac{\partial y_2}{\partial x_2}\right)\).

Multiplying (2.1) by testfunctions \( v \in \mathcal{V} \), a weak formulation of the Navier-Stokes equations, with the trilinear form \( c : H^1(\Omega) \times H^1(\Omega) \times H^1(\Omega) \to \mathbb{R} \) defined by \( c(u,v,w) = ((u \cdot \nabla)v, w) \) is possible in the following way: find \( y \in \mathbf{H} \) such that

\[
(2.4) \quad a(y,v) + c(y,y,v) = \langle f,v \rangle_{\mathcal{V}',\mathcal{V}}, \text{ for all } v \in \mathbf{V}
\]

\[
(2.5) \quad \gamma_0 y = g,
\]

where \( \gamma_0 \) denotes the trace operator.

Conversely, if \( y \in \mathbf{H} \) satisfies (2.4), then

\[
\langle -\nu \Delta y + (y \cdot \nabla)y - f, v \rangle_{D'(\Omega),D(\Omega)} = 0, \text{ for all } v \in \mathcal{V}
\]

and, consequently (see [14], pg 8), there exists a distribution \( p \in L^2(\Omega) \) such that (2.1) is satisfied in the distributional sense. The equations (2.2) and (2.3) are satisfied in a distributional and trace theorem senses respectively.

Next, we summarize some well-known theoretical results (cf. [15, 16, 14]).
**Theorem 2.1.** Let $\Omega$ be an open bounded set of $\mathbb{R}^2$. The trilinear form $c$ is continuous on $H^1(\Omega) \times H^1(\Omega) \times H^1(\Omega)$ and satisfies:

1. $c(u, v, v) = 0$ for all $u \in H$ with $\gamma_n u = 0$, for all $v \in H^1(\Omega)$.
2. $c(u, v, w) = -c(u, w, v)$ for all $u \in H$ with $\gamma_n u = 0$, for all $v, w \in H^1(\Omega)$.
3. $c(u, v, w) = -c(u, w, v)$ for all $u \in H$, for all $v \in H^1(\Omega)$, for all $w \in V$.
4. $c(u, v, w) = ((\nabla v)^T w, u)$

**Corollary 2.1.** Let $\Omega$ be an open bounded set of $\mathbb{R}^2$. The form $c$ is continuous on $H^1_0(\Omega) \times H^1_0(\Omega) \times H^1_0(\Omega)$.

**Theorem 2.2.** Let $\Omega$ be an open bounded subset of $\mathbb{R}^2$ and let $f \in H^{-1}(\Omega)$. Then, problem (2.4), (2.5), with homogeneous boundary conditions, has at least one variational solution $y \in V$ and there exists a distribution $p \in L^2_0(\Omega)$ such that (2.1), (2.2) and (2.3) are satisfied. Moreover, the solution satisfies the following estimate:

$$\|y\|_V \leq \frac{1}{\nu} \|f\|_{V'}.$$  

**Proof.** For a detailed proof see [15, 16, 14]. □

**Theorem 2.3.** If $\nu > \mathcal{N} \|f\|_{V'}$, where $\mathcal{N} = \sup_{u, v, w \in V} \frac{|c(u, v, w)|}{\|u\|_V \|v\|_V \|w\|_V}$, then the solution for (2.4), (2.5), with homogeneous boundary conditions, is unique.

**Proof.** We refer to [15, 16, 14]. □

**Lemma 2.1.** For every $\varepsilon > 0$, there exists a function $\hat{y} \in H^1(\Omega)$ such that $\text{div } \hat{y} = 0$, $\gamma_0 \hat{y} = g$ and

$$|c(v, \hat{y}, v)| \leq \varepsilon \|v\|^2_V \text{ for all } v \in V.$$

**Proof.** A detailed proof can be found in [14], pg. 175. □

**Theorem 2.4.** Let $f \in H^{-1}(\Omega)$ and $g \in H^{1/2}_0$. Then, there exists at least one solution for the non-homogeneous problem (2.1), (2.2), (2.3).

**Proof.** We give here an outline of the proof. With the help of Lemma 2.1, the existence of a function $\hat{y}$ such that $\text{div } \hat{y} = 0$, $\gamma_0 \hat{y} = g$ is assured. By changing variables $w := y - \hat{y}$, the problem is considered in the space $V$ and has the following form

$$a(w, v) + c(\hat{y}, w, v) + c(w, \hat{y}, v) + c(w, w, v) = (f, v) - a(\hat{y}, v) - c(\hat{y}, \hat{y}, v) \text{ for all } v \in V.$$  

The existence then follows as in the homogeneous case (cf. [16, 14]). □

**Theorem 2.5.** If $\|\hat{y}\|_H$ is sufficiently small, so that

$$|c(v, \hat{y}, v)| \leq \frac{\nu}{2} \|v\|^2_V \text{ for all } v \in V$$

and $\nu$ satisfies $\nu^2 > 4\mathcal{N} \|F\|_{V'}$, with $F = f + \nu \Delta \hat{y} - (\hat{y} \cdot \nabla) \hat{y}$, then there exists a unique solution $(y, p)$ for the problem (2.1), (2.2), (2.3). Additionally the following estimate holds

$$\|y - \hat{y}\|_V \leq \frac{2}{\nu} \|F\|_{V'}.$$  

Extra regularity of the solution can be obtained if the right hand side and the boundary condition are smooth enough.

**Theorem 2.6.** Let \( f \in L^2(\Omega) \) and \( g \in H^{1/2}_0 \cap H^{3/2}(\Gamma) \). Then \( y \in H^2(\Omega) \) and \( p \in H^1(\Omega) \).

**Proof.** The term \((y \cdot \nabla)y\) can also be written as \( \sum_i y_i \partial_i y \) or, equivalently, since \( \text{div} y = 0 \), as \( \sum_i \partial_i (y_i y) \). From Sobolev inequalities we know that \( H^1(\Omega) \hookrightarrow L^{\alpha}(\Omega) \) for all \( 1 \leq \alpha < \infty \) and, hence, \( y_i y_j \in L^{\alpha}(\Omega) \) for all \( 1 \leq \alpha < \infty \). Thus, \( \partial_i (y_i y_j) \in W^{-1,\alpha}(\Omega) \).

Additionally, \( H^{3/2}(\Gamma) \hookrightarrow W^{1+1/q,q}(\Gamma), \) for \( q \geq 2 \) (cf. [17], pg. 218), which implies that \( g \in W^{1-1/\alpha,\alpha}(\Gamma) \) for all \( 1 \leq \alpha < \infty \). Using the regularity results for the non-homogeneous Stokes equations (cf. [18], pg. 18) we get that \( y \in W^{1,\alpha}(\Omega) \) and \( p \in L^{\alpha}(\Omega) \) for all \( 1 \leq \alpha < \infty \). Since \( W^{1,\alpha}(\Omega) \hookrightarrow L^{\infty}(\Omega) \) for \( \alpha > 2 \), \( y_i \partial_i y \) belong to \( L^{\alpha}(\Omega) \), for all \( 1 \leq \alpha < \infty \).

Taking \( \alpha = 2 \) as particular case, we get that \( f - (y \cdot \nabla)y \) belongs to \( L^2(\Omega) \).

Considering that \( g \in H^{1/2}_0 \cap H^{3/2}(\Gamma) \) and applying the regularity results for Stokes equations again, we obtain that \( y \in H^2(\Omega) \) and \( p \in H^1(\Omega) \). \( \square \)

### 3. Optimal control problem

In this section we are concerned with the optimal control of the stationary Navier-Stokes equations, when the control is allowed to act as a pointwise constrained body force on a portion of the domain. We proof the existence of the mentioned optimal solution and derive a first order necessary condition and a second order sufficient condition for the problem. Additionally, the regularity of the optimal control is investigated.

3.1. **Problem statement.** Let \( \hat{\Omega} \) be an open connected subset of \( \Omega \). The mathematical setting of the problem is: find \( (y^*, u^*) \in H \times U_{ad} \) with \( U_{ad} = \{ v \in L^2(\hat{\Omega}) : a \leq v \leq b, \text{a.e.} \} \), which solves

\[
\begin{cases}
\min J(y, u) = \frac{1}{2} \int_{\Omega} |y - z_d|^2 \, dx + \frac{\alpha}{2} \int_{\Omega} |u|^2 \, dx \\
\text{subject to} \\
a(y, v) + c(y, y, v) = (Bu, v) \text{ for all } v \in V \\
\gamma_0 y = g,
\end{cases}
\]

where \( z_d \in L^2(\Omega) \), \( a, b \in L^2(\hat{\Omega}) \), \( g \in H^{1/2}_0 \) and \( B \in L(L^2(\hat{\Omega}), L^2(\Omega)) \), with \( Bu = \begin{cases} u & \text{in } \hat{\Omega} \\ 0 & \text{in } \Omega \setminus \hat{\Omega} \end{cases} \).

To properly incorporate the constraints we define an operator \( G \) as follows:

\[
G : H \times U_{ad} \rightarrow V' \times H^{1/2}_0 \\
(y, u) \mapsto (a(y, \cdot) + c(y, y, \cdot) - (Bu, \cdot) \gamma_0 y - g)
\]
and express the constraints as $G(y, u) = 0$ in $V' \times H_0^{1/2}$. The operator $G$ is Fréchet differentiable and the derivative is given by:

$$G'(y, u)(w, h) = \left( a(w, \cdot) + c(y, w, \cdot) + c(w, y, \cdot) - (Bh, \cdot) \right)$$

(3.1)

3.2. Existence. Let us define $T_{ad} = \{(y, u) \in H \times U_{ad} : G(y, u) = 0\}$

**Theorem 3.1.** There exist an optimal solution for $(P)$.

**Proof.** Taking $u = b$ we obtain, from the existence results for the stationary Navier-Stokes equations, that there exists $y \in H$ which satisfies the mentioned equations. Consequently $(y, u) \in T_{ad}$ and $T_{ad} \neq \emptyset$.

Taking a minimizing sequence $\{(y_n, u_n)\}$ in $T_{ad}$ and considering the properties of $J(y, u)$ we obtain that $\frac{\alpha}{2} \|u_n\|^2 \leq J(y_n, u_n) < \infty$, which implies that $\{u_n\}$ is uniformly bounded.

From the properties of the Navier-Stokes equations and taking the testfunction $w_n = y_n - \hat{y}$ in (2.7), where $\hat{y}$ is such that Lemma 2.1 holds for some $\varepsilon_1 < \nu$, we know that:

$$a(w_n, w_n) + c(w_n, \hat{y}, w_n) = (Bu_n, w_n) - a(\hat{y}, w_n) - c(\hat{y}, \hat{y}, w_n),$$

which implies, using Lemma 2.1, that

$$\nu \|w_n\|_V^2 - \varepsilon_1 \|w_n\|_V^2 \leq \kappa \|Bu_n\| \|w_n\|_V + \nu \|\hat{y}\|_V \|w_n\|_V + N \|\hat{y}\|_V^2 \|w_n\|_V.$$

Considering that $\|v\|_H \leq \rho \|v\|_V$ for all $v \in V$, we get

$$\|w_n\|_H \leq \frac{\theta}{\nu - \varepsilon_1} (\kappa \|u_n\| + \nu \|\hat{y}\|_H + N \|\hat{y}\|_H^2).$$

Taking into account Lemma 2.1, we obtain

$$\|y_n\|_H \leq \frac{\theta}{\nu - \varepsilon_1} (\kappa \|u_n\| + (\nu + \frac{\nu - \varepsilon_1}{\rho}) \|v\|_H + N \|\hat{y}\|_H^2).$$

Since $\{u_n\}$ is bounded in $L^2(\Omega)$, the sequence $\{y_n\}$ is also uniformly bounded in $H$ and, consequently, we may extract a weakly convergent subsequence, also denoted by $\{(y_n, u_n)\}$, such that $u_n \rightharpoonup u^* \in L^2(\Omega)$ and $y_n \rightharpoonup y^*$ in $H$.

The set $U_{ad}$ is convex and closed in $L^2(\Omega)$ and consequently $u^*$ satisfies also the control constraints. In order to see that $(y^*, u^*)$ is solution of the Navier-Stokes equations, the only problem is to pass to the limit in the nonlinear form $c(y_n, y_n, v)$. Due to the weak sequentially continuity of $c(\cdot, \cdot, \cdot)$ (cf. [16], pg. 286), it follows that $c(y_n, y_n, v) \rightharpoonup c(y^*, y^*, v)$. Consequently, due also to the linearity and continuity of the other terms involved, the limit $(y^*, u^*)$ satisfies the constraints. Taking also into consideration that $J(y, u)$ is weakly lower semicontinuous, the result follows. $\square$

3.3. First order necessary condition and optimality system. In this subsection we present a sufficient requirement for the satisfaction of the regular point condition (cf. [19], pg. 50). Thereafter the existence of appropriate Lagrange multipliers is justified and the optimality system for $(P)$ is derived.

**Lemma 3.1.** Let $(y^*, u^*) \in T_{ad}$ be such that $\nu > M(y^*)$, with $M(y) = \sup_{v \in V} \frac{|c(v, v, y)|}{\|v\|_V^2}$. Then $(y^*, u^*)$ is a regular point.
Proof. Given \((d, e) \in \mathbf{V}' \times \mathbf{H}_0^{1/2}\), it is sufficient to show that there exists a pair 
\((w, h) \in \mathbf{H} \times \mathbf{U}_{ad}\) and \(\theta \geq 0\) such that \(G'(y^*, \theta(h - u^*)))(w, h) = (d, e)\).

We begin by fixing \(h = u^*\) and \(\theta = 1\). Then it remains to show the existence of \(w \in \mathbf{H}\) such that:
\[
\begin{align*}
a(w, v) + c(y, w, v) + c(w, y, v) &= \langle d, v \rangle \text{ for all } v \in \mathbf{V} \\
\gamma_0 w &= e.
\end{align*}
\]

By Lemma 2.1, there exists a function \(\bar{w}\) such that \(\gamma_0 \bar{w} = e\), \(\text{div} \bar{w} = 0\). Consequently, by using \(\hat{w} = w - \bar{w}\), the problem is to find \(\hat{w} \in \mathbf{V}\) such that:
\[
\begin{align*}
(3.2) \quad a_1(\hat{w}, v) := a(\hat{w}, v) + c(y, \hat{w}, v) + c(\hat{w}, y, v) &= \langle d, v \rangle \\
&= -a(\hat{w}, v) - c(y, \hat{w}, v) - c(\hat{w}, y, v) =: \langle F, v \rangle \text{ for all } v \in \mathbf{V}.
\end{align*}
\]

Since the bilinearity of \(a_1(\cdot, \cdot)\) follows directly from the properties of \(a(\cdot, \cdot)\) and \(c(\cdot, \cdot, \cdot)\), we just have to verify bicontinuity and coercivity of the form in order to apply the Lax-Milgram theorem.

The bicontinuity follows also from the properties of the other forms in the following way:
\[
\begin{align*}
|a_1(w, v)| &= |a(w, v) + c(y, w, v) + c(w, y, v)| \\
&\leq |a(w, v)| + |c(y, w, v)| + |c(w, y, v)| \\
&\leq \nu \|w\|_\mathbf{V} \|v\|_\mathbf{V} + \mathcal{N}_2 \|y\|_{\mathbf{H}} \|w\|_\mathbf{V} \|v\|_\mathbf{V} + \mathcal{N}_1 \|w\|_{\mathbf{V}} \|y\|_{\mathbf{H}} \|v\|_\mathbf{V} \\
&= (\nu + 2\mathcal{N}_2 \|y\|_{\mathbf{H}}) \|w\|_\mathbf{V} \|v\|_\mathbf{V}.
\end{align*}
\]

For the coercivity,
\[
|a_1(v, v)| = |a(v, v) + c(y, v, v) + c(v, y, v)| = |a(v, v) - c(v, y, v)| \\
\geq |a(v, v)| - |c(v, y, v)| \geq \nu \|v\|_\mathbf{V}^2 - \mathcal{M}(y^*) \|v\|_\mathbf{V}^2 \\
= (\nu - \mathcal{M}(y^*)) \|v\|_\mathbf{V}^2,
\]
and, since by hypothesis \(\nu > \mathcal{M}(y^*)\), the result follows. \(\square\)

Remark 3.1. Comparing Lemma 3.1 with Theorem 2.5, a link between the uniqueness of the Navier-Stokes solution and the satisfaction of the regular point condition can be established. In fact, if \((y, u)\) satisfies the hypothesis of Theorem 2.5, then it also satisfies the regular point condition. To see this, let us note that due to the first hypothesis of Theorem 2.5 and the boundedness of the nonlinear form we obtain
\[
|c(v, y, v)| = |c(v, y - \hat{y}, v) + c(v, \hat{y}, v)| \leq |c(v, y - \hat{y}, v)| + \frac{\nu}{2} \|v\|_\mathbf{V}^2 \\
\leq \mathcal{N} \|y - \hat{y}\|_\mathbf{V} \|v\|_\mathbf{V}^2 + \frac{\nu}{2} \|v\|_\mathbf{V}^2.
\]

Using \(F = f + \nu \Delta \hat{y} - (\hat{y} \cdot \nabla) \hat{y}\) we get from the equations estimates (see Theorem 2.5) that \(\|y - \hat{y}\|_\mathbf{V} \leq \frac{2}{\nu} \|F\|_{\mathbf{V}'}\), which implies
\[
|c(v, y, v)| \leq \frac{2}{\nu} \mathcal{N} \|F\|_{\mathbf{V}'} \|v\|_\mathbf{V}^2 + \frac{\nu}{2} \|v\|_\mathbf{V}^2.
\]
Since by hypothesis \(\nu^2 > 4\mathcal{N} \|F\|_{\mathbf{V}'}\), we obtain that
\[
|c(v, y, v)| < \nu \|v\|_\mathbf{V}^2.
\]
Theorem 3.2. Let \((y^*, u^*) \in H \times U_{ad}\) be an optimal solution for \((P)\) which satisfies \(\nu > M(y^*)\). Then, there exist Lagrange multipliers \((\lambda, \xi) \in V \times H_0^{-1/2}\) such that:

\[
(y^* - z_d, w) + a(u^*, h) + a(\lambda, w) + c(y^*, w, \lambda) + c(w, y^*, \lambda) - (B^* \lambda, h)
+ \langle \xi, \gamma_0 w \rangle_{H_0^{-1/2};H_0^{1/2}} \geq 0, \text{ for all } (w, h) \in H \times C(u^*),
\]

where \(C(u^*) = \{\theta(v - u^*), v \in U_{ad}, \theta \geq 0\}\).

Proof. From the general Lagrange multiplier existence theorem (cf. [19]) follows the existence of \((\lambda, \xi) \in V \times H_0^{-1/2}\) such that

\[
J'(y^*, u^*) + \langle (\lambda, \xi), G'(y^*, u^*) \rangle \geq 0, \text{ for all } (w, h) \in H \times C(u^*).
\]

Considering that

\[
\langle (\lambda, \xi), G'(y^*, u^*) \rangle = a(\lambda, w) + c(y^*, w, \lambda) + c(w, y^*, \lambda) - (B^* \lambda, h) + \langle \xi, \gamma_0 w \rangle_{H_0^{-1/2};H_0^{1/2}},
\]

the result follows.

Next we define the active sets for \((y^*, u^*)\) by

\[
A^a = \{x \in \bar{\Omega} : u^*(x) = a(x) \text{ a.e.}\} \quad \text{and} \quad A^b = \{x \in \bar{\Omega} : u^*(x) = b(x) \text{ a.e.}\}
\]

and the inactive set by

\[
I = \bar{\Omega} \backslash (A^a \cup A^b).
\]

Theorem 3.3. Let \((y^*, u^*)\) be an optimal solution for \((P)\), such that \(\nu > M(y^*)\). Then it satisfies the following optimality system in variational sense:

\[
\begin{cases}
-\nu \Delta y^* + (y^* \cdot \nabla) y^* + \nabla p = B u^* \\
\text{div } y^* = 0 \\
y^*|_{\Gamma} = g \\
-\nu \Delta \lambda - (y^* \cdot \nabla) \lambda + (\nabla y^*)^T \lambda + \nabla \phi = z_d - y^* \\
\text{div } \lambda = 0 \\
\lambda|_{\Gamma} = 0 \\
a u^* = B^* \lambda - \mu \\
a \leq u^* \leq b \text{ a.e.} \\
\mu \geq 0 \text{ on } A^b \\
\mu \leq 0 \text{ on } A^a \\
\mu = 0 \text{ on } I.
\end{cases}
\]

Proof. From the necessary condition, taking \(h = 0\), we obtain that

\[
a(\lambda, w) + c(y^*, w, \lambda) + c(w, y^*, \lambda) + \langle \xi, \gamma_0 w \rangle_{H_0^{-1/2};H_0^{1/2}} = (z_d - y^*, w) \text{ for all } w \in H.
\]

If, additionally, we take the test functions \(w\) such that \(\gamma_0 w = 0\), we get

\[
a(\lambda, w) + c(y^*, w, \lambda) + c(w, y^*, \lambda) = (z_d - y^*, w) \text{ for all } w \in V,
\]
which corresponds to the variational formulation of
\[-\nu \Delta \lambda - (y^* \cdot \nabla) \lambda + (\nabla y^*)^T \lambda + \nabla \phi = z_d - y^*\]
\[\text{div} \lambda = 0\]
\[\lambda|_I = 0.\]

Taking \(w = 0\), we obtain that
\[(\mu, h) := (B^* \lambda - \alpha u^*, h) \leq 0, \text{ for all } h \in C(u^*),\]
which is equivalent (see [20], pg. 4) to the complementarity condition
\[(3.5) \quad \mu \geq 0 \text{ on } A^b, \mu \leq 0 \text{ on } A^a, \mu = 0 \text{ on } I.\]

\[\square\]

3.4. Second order sufficient condition. In this subsection we proof a second order sufficient condition for a stationary pair \((y^*, u^*)\), which is assumed to satisfy the condition \(\nu > \mathcal{M}(y^*)\).

Lemma 3.2. The Lagrange multiplier \(\lambda \in V\) satisfies the following estimate:
\[(3.6) \quad \|\lambda\|_V \leq \kappa \sigma(y) \|y^* - z_d\|,\] where \(\sigma(y) := \frac{1}{\nu - \mathcal{M}(y)}\) and \(\kappa\) is the Poincaré inequality constant.

Proof. From the adjoint equations in (3.4) we obtain that
\[a(\lambda, w) + c(y^*, w, \lambda) + c(w, y^*, \lambda) = (z_d - y^*, w) \text{ for all } w \in V.\]
Taking \(w = \lambda\) as particular case, we get
\[a(\lambda, \lambda) + c(\lambda, y^*, \lambda) = (z_d - y^*, \lambda),\]
which implies that
\[\nu \|\lambda\|_V^2 = (z_d - y^*, \lambda) + c(\lambda, \lambda, y^*) \leq \|y^* - z_d\| \|\lambda\| + \mathcal{M}(y^*) \|\lambda\|_V^2 \]
\[\leq \kappa \|y^* - z_d\| \|\lambda\|_V + \mathcal{M}(y^*) \|\lambda\|_V^2.\]
Consequently, \(\|\lambda\|_V \leq \frac{\kappa}{\nu - \mathcal{M}(y)} \|y^* - z_d\|.\] \[\square\]

Within this framework, a theorem which states the satisfaction of the coercivity property and, thus, of the second order sufficient condition, if \(\|y^* - z_d\|\) is sufficiently small, can be formulated. For that purpose let us introduce the Lagrangian of the problem:
\[\mathcal{L}(y, u, \lambda, \xi) := J(y, u) + \langle (\lambda, \xi), G(y, u) \rangle_{(V \times H_0^{-1/2}, (V' \times H_0^{1/2})^*)} ;\]

Theorem 3.4. If \(\alpha > 4\kappa^3 \sigma(y^*)^3 \mathcal{N} \|y^* - z_d\|\), then there exists some \(\varrho > 0\) such that:
\[(3.7) \quad \langle \mathcal{L}''(y^*, u^*, \lambda, \xi)(w, h), (w, h) \rangle \geq \varrho \|(w, h)\|_{H \times L^2(\tilde{\Omega})}^2 \text{ for all } (w, h) \in H \times C(u^*),\]
with \((w, h) \in \ker(G'(y^*, u^*))\), and \((y^*, u^*)\) is a local optimal solution for \((\mathcal{P})\).
Proof. Using the estimates of the trilinear form we get:
\[
\langle \mathcal{L}''(y^*, u^*, \lambda, \xi)(w, h), (w, h) \rangle = \|w\|^2 + \alpha \|h\|^2 + 2c(w, w, \lambda) \\
\geq \|w\|^2 - 2N \|w\|^2 \|\lambda\|_V + \alpha \|h\|^2.
\]
Considering \((w, h) \in \ker(G'(y^*, u^*))\), we obtain that
\[
\|h\| \geq \frac{1}{\kappa \sigma(y^*)} \|w\|_V
\]
and, consequently,
\[
\langle \mathcal{L}''(y^*, u^*, \lambda, \xi)(w, h), (w, h) \rangle \geq \|w\|^2 - 2N \|w\|^2 \|\lambda\|_V + \alpha \|h\|^2 + \frac{\alpha}{2 \kappa^2 \sigma(y^*)^2} \|w\|^2 \\
\geq \min (1, \frac{\alpha}{2 \kappa^2 \sigma(y^*)^2} - 2N \|\lambda\|_V) \|w\|^2 + \frac{\alpha}{2} \|h\|^2.
\]
Utilizing Lemma 3.2,
\[
\langle \mathcal{L}''(y^*, u^*, \lambda, \xi)(w, h), (w, h) \rangle \geq \varrho \|(w, h)\|^2_{H \times L^2(\tilde{\Omega})},
\]
with \(\varrho = \min (1, \frac{\alpha}{2 \kappa^2 \sigma(y^*)^2} - 2N \|\lambda\|_V - z_d)\).

Since, by hypothesis, \(\alpha > 4\kappa^2 \sigma(y^*)^2 N \|\lambda\|_V - z_d\|\), we get that \(\varrho > 0\) and (3.7) is satisfied. Hence (see [21]), \((y^*, u^*)\) is a local minimum for \((P)\). \(\square\)

3.5. Regularity. In this section we obtain some extra regularity properties of the optimal control, by assuming regularity of the data and the control constraints.

**Lemma 3.3.** The Lagrange multipliers \(\lambda, \phi\) satisfy also the following regularity \(\lambda \in H^2(\Omega)\) and \(\phi \in H^1(\Omega)\).

**Proof.** The Lagrange multiplier \(\lambda\) is solution of the equations
\[
-\nu \Delta \lambda + \nabla \phi = z_d - y^* + (y^* \cdot \nabla) \lambda - (\nabla y^*)^T \lambda \\
\text{div} \lambda = 0 \\
\gamma_0 \lambda = 0.
\]
Since \(\nabla y^* \in L^2(\Omega)\) and \(\lambda \in H^1(\Omega)\), we get, due to the embedding \(H^1(\Omega) \hookrightarrow L^a(\Omega)\), for \(1 \leq \alpha < \infty\), that \((\nabla y^*)^T \lambda \in L^\beta(\Omega)\), with \(\frac{1}{\beta} = \frac{1}{a} + \frac{1}{2}\). Due to the embedding \(W_0^{1,p}(\Omega) \hookrightarrow L^q(\Omega)\) for \(p < 2\) and \(q \in [1, p']\) with \(\frac{1}{p'} = \frac{1}{p} - \frac{1}{2}\), and since \(\frac{1}{\beta} = \frac{1}{a} + \frac{1}{2}\), we may choose the conjugate exponent \(\beta' > 2\) and obtain that \(L^\beta(\Omega) \hookrightarrow W^{-1, \beta'}(\Omega)\). Consequently, \((\nabla y^*)^T \lambda \in W^{-1, \beta'}(\Omega)\), with \(\beta' > 2\).

In a similar way we obtain that \((y^* \cdot \nabla) \lambda \in W^{-1, \beta'}(\Omega)\). Hence, considering the other terms, we get that \(z_d - y^* + (y^* \cdot \nabla) \lambda - (\nabla y^*)^T \lambda \in W^{-1, \beta'}(\Omega)\). Applying the regularity results for the Stokes equations (cf. [18], pg. 18) we get that \(\lambda \in W^{1, \beta'}(\Omega)\).

Since \(W^{1, \beta'}(\Omega) \hookrightarrow L^{\alpha}(\Omega)\), we obtain that \((\nabla y^*)^T \lambda \in L^{\alpha}(\Omega)\). Since \(y^* \in H^1(\Omega) \hookrightarrow L^{2}(\Omega)\), for \(1 \leq \alpha < \infty\) and \(\frac{\alpha}{2} = \frac{\alpha}{a} < \frac{\alpha}{2}\), for \(i, j = 1, 2\) and \(p' > 2\), we get, taking for example \(\alpha = p' = 4\) that \((y^* \cdot \nabla) \lambda \in L^2(\Omega)\). Using the regularity results for the Stokes equations again, we obtain that \(\lambda \in H^2(\Omega)\) and \(\phi \in H^1(\Omega)\). \(\square\)
Under appropriate conditions on the active set, extra regularity for the optimal control can also be obtained. In fact, the existence of a continuous representative of the optimal control on the whole control domain can be proved.

**Lemma 3.4.** Let us introduce the sets $A^b = \{x \in \tilde{\Omega} : (B^*\lambda - ab)(x) \geq 0\}$, $A^a = \{x \in \tilde{\Omega} : (B^*\lambda - a\alpha)(x) \leq 0\}$ and $I = \{x \in \tilde{\Omega} : (B^*\lambda - a\alpha)(x) > 0$ and $(B^*\lambda - ab)(x) < 0\}$. If $A^b \subset \tilde{\Omega}$, $A^a \subset \Omega$, $b \in C(\tilde{\Omega})$ and $a \in C(\tilde{\Omega})$, then $A^b = A^b$, $A^a = A^a$ and $I = I$. Additionally, the active sets are closed and the inactive set is open.

**Proof.** Since by Lemma 3.3 $\lambda \in H^2(\Omega)$ and $H^2(\Omega) \rightarrow C(\tilde{\Omega})$, $\lambda$ has a continuous representative, also denoted by $\lambda$. Taking into account that $B^*$ stands for the restriction operator, $B^*\lambda \in C(\tilde{\Omega})$.

Since on $I$, $B^*\lambda - a\alpha^* = 0$ a.e. and $B^*\lambda$ is continuous, $u^*$ has a continuous representative on $I$. Hence, taking an arbitrary point $x \in I$, we obtain that $(B^*\lambda - ab)(x) < (B^*\lambda - a\alpha^*)(x) = \mu(x) = 0$ and $(B^*\lambda - a\alpha)(x) > (B^*\lambda - a\alpha^*)(x) = \mu(x) = 0$. Consequently $x \in I$ and, hence, $I \subset I$.

If $x \in A^b$, then $(B^*\lambda - ab)(x) = (B^*\lambda - a\alpha^*)(x) = \mu(x) \geq 0$. Thus $x \in A^b$ and, hence, $A^b \subset A^b$. Similarly, if $x \in A^a$, then $(B^*\lambda - a\alpha)(x) = (B^*\lambda - a\alpha^*)(x) \leq 0$ and, consequently, $A^a \subset A^a$.

Since $I \cup \bar{A}^b \cup A^a = I \cup A^b \cup A^a = \tilde{\Omega}$, we get that $I = I$, $A^b = A^b$ and $A^a = A^a$. Because of the continuity of $B^*\lambda - ab$ and $B^*\lambda - a\alpha$ on $\tilde{\Omega}$ and since $\bar{A}^b \subset \tilde{\Omega}$ and $\bar{A}^a \subset \tilde{\Omega}$, we get that $A^b$ and $A^a$ are closed and $I$ is open.

**Theorem 3.5.** Under the hypotheses of Lemma 3.4, $u^*$ has a continuous representative on $\tilde{\Omega}$.

**Proof.** On the open set $I$ we have that $B^*\lambda = a\alpha^*$ and, hence, $u^*|_I \in C(I)$. Next, let $x \in A^b$. If $x$ is an interior point, there exist a neighborhood around $x$, where $u^* = b$ and, since $b$ is continuous, $u^*$ is continuous on the neighborhood. If $x$ is not an interior point, there exist some active and inactive sequences, denoted by $\{x_k^A\}$ and $\{x_k^I\}$, such that $x = \lim_{k \to \infty} x_k^A = \lim_{k \to \infty} x_k^I$.

We shall show that $\lim_{k \to \infty} u^*(x_k^I) = u^*(x) = b(x)$. The second equality $u^*(x) = b(x)$ is obvious since $x \in A^b$. Moreover, we have the following relation:

$$ab(x) \leq B^*\lambda(x) = \lim_{k \to \infty} B^*\lambda(x_k^I) \leq \lim_{k \to \infty} \alpha b(x_k^I) = \alpha b(x).$$

and, hence, $B^*\lambda(x) = \alpha b(x)$. This implies that

$$\alpha \lim_{k \to \infty} u^*(x_k^I) = \lim_{k \to \infty} B^*\lambda(x_k^I) = B^*\lambda(x) = \alpha b(x).$$

Using the same arguments for an arbitrary point $x \in A^a$, the result follows.

4. **Primal-dual active set method**

In this section we apply the primal-dual active set strategy, as stated in [9], for the numerical solution of the constrained distributed optimal control of the Navier-Stokes equations. After the presentation of the algorithm, we investigate sufficient conditions for global and local superlinear convergence of the method.
We begin by reformulating the complementarity condition, expressed by the last three equations in (3.4), in the following way:

\[(4.1) \quad \mu = \max(0, \mu + \alpha(u - b)) + \min(0, \mu + \alpha(u - a)).\]

This last relation suggests a prediction strategy, which, based on the information of iteration \(k - 1\), estimates the active and inactive sets for the following iteration as:

\[A_b^k = \{ x : \mu_{k-1} + \alpha(u_{k-1} - b) > 0 \}, \quad I_k = \{ x : \mu_{k-1} + \alpha(u_{k-1} - b) \leq 0 \leq \mu_{k-1} + \alpha(u_{k-1} - a) \}, \quad A_a^k = \{ x : \mu_{k-1} + \alpha(u_{k-1} - a) < 0 \}.\]

Thus, the complete algorithm, considering \(\bar{\Omega} = \Omega\), can be stated as follows:

**Algorithm 4.1.**

1. Initialization: solve the unconstrained optimal control problem to obtain \(u_0\) and set \(\mu_0 = 0\) and \(k = 1\).

2. Set \(A_b^k = \{ x : \mu_k + \alpha(u_k - b) > 0 \}, \quad A_a^k = \{ x : \mu_k + \alpha(u_k - a) < 0 \}\) and \(I_k = \{ x : \mu_k + \alpha(u_k - b) \leq 0 \leq \mu_k + \alpha(u_k - a) \} \).

3. If \(k \geq 2\), \(A_b^k = A_b^{k-1}\) and \(A_a^k = A_a^{k-1}\) stop.

4. Otherwise find the solution of:

\[(4.2) \begin{cases} 
- \nu \Delta y_k + (y_k \cdot \nabla)y_k + \nabla p_k = \begin{cases} b & \text{in } A_b^k \\
\frac{\lambda_k}{\alpha} & \text{in } I_k \\
a & \text{in } A_a^k 
\end{cases} \\
\text{div } y_k = 0 \\
y_k|\Gamma = g \\
- \nu \Delta \lambda_k - (y_k \cdot \nabla) \lambda_k + (\nabla y_k)^T \lambda_k + \nabla \phi_k = z_d - y_k \\
\text{div } \lambda_k = 0 \\
\lambda_k|\Gamma = 0,
\end{cases}\]

set \(u_k = \begin{cases} b & \text{in } A_b^k \\
\frac{\lambda_k}{\alpha} & \text{in } I_k, \quad \mu_k = \lambda_k - \alpha u_k, \text{ and goto step 2.} \\
a & \text{in } A_a^k
\end{cases}\)

**Theorem 4.1.** If the iterates \(\{y_k\}\) of Algorithm 4.1 satisfy the following conditions:

\[(A1) \quad \overline{M} := \sup_k M(y_k) < \nu,\]

\[(A2) \quad 2N \kappa \sigma(y_k) \| y_k - z_d \| < \frac{\alpha}{4\kappa^2} (\nu - \overline{M})^2, \quad \text{for each } k \geq 0,\]

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then they are the locally unique solutions of

\[
\begin{aligned}
\min J(y, u) &= \frac{1}{2} \int_{\Omega} |y - z_0|^2 \, dx + \frac{\alpha}{2} \int_{\hat{\Omega}} |u|^2 \, dx \\
\text{subject to:} \\
-\nu \Delta y + (y \cdot \nabla)y + \nabla p &= u \\
\text{div } y &= 0 \\
y|_{\Gamma} &= g \\
u &= b \quad \text{on } A_k^b \\
u &= a \quad \text{on } A_k^a.
\end{aligned}
\]

(4.3)

Proof. Under the hypotheses, the system to be solved in step (4) of Algorithm 4.1 is equivalent to the optimality system of the optimal control problem (4.3). The existence of an optimal solution for (4.3) implies the solvability of the system (4.2). Additionally, from Theorem 3.4 and (A2), the second order sufficient condition is satisfied for each \(k \geq 0\) and hence, the solution to (4.3) is locally unique for each \(k\). □

In particular, (A1) is satisfied, as was argued in Remark 3.1, if the iterates satisfy the uniqueness condition of Theorem 2.5. Inequality (A2) holds if the viscosity coefficient \(\nu\) is sufficiently large or the iterates are not so far from the desired state.

In the following theorem we show that the stopping criteria used in step (2) is appropriate. As for the linear case, we prove that if the same active sets are obtained in two consecutive iterations, the algorithm stops at the optimal solution.

**Theorem 4.2.** If there exists \(k \geq 1\) such that \(A_k^a = A_{k+1}^a\) and \(A_k^b = A_{k+1}^b\), then the algorithm stops and the last iterate satisfies the system (4.2), the control constraints \(a \leq u_k \leq b\) and the complementarity condition \(\mu_k = 0\) on \(I_k\), \(\mu_k > 0\) on \(A_k^b\) and \(\mu_k < 0\) on \(A_k^a\).

Proof. By construction, system (4.2) is satisfied. It remains to verify that \(a \leq u_k \leq b\) and \(\mu_k = 0\) on \(I_k\), \(\mu_k > 0\) on \(A_k^b\) and \(\mu_k < 0\) on \(A_k^a\).

On \(I_k\) we know by construction that \(\mu_k = 0\). Since \(A_k^b = A_{k+1}^b\) and \(A_k^a = A_{k+1}^a\),

\[\mu_k + \alpha(u_k - b) = \alpha(u_k - b) \leq 0 \leq \alpha(u_k - a) = \mu_k + \alpha(u_k - a),\]

which implies that \(a \leq u_k \leq b\).

On \(A_k^b\) we have \(u_k = b\) and \(\mu_k + \alpha(u_k + b) > 0\), hence \(\mu_k > 0\). Similarly, on \(A_k^a\) we have \(u_k = a\) and \(\mu_k + \alpha(u_k + a) < 0\), which implies \(\mu_k < 0\). □

Let us now consider two consecutive iterates of Algorithm 4.1. It can then be verified that

\[-\nu \Delta (y_{k+1} - y_k) + (y_{k+1} \cdot \nabla)y_{k+1} - (y_k \cdot \nabla)y_k + \nabla(p_{k+1} - p_k) = u_{k+1} - u_k = \begin{cases} R_{A_{k+1}^b}^b & \text{on } A_{k+1}^b \\ \frac{1}{\alpha}(\lambda_{k+1} - \lambda_k) + R_{I_{k+1}}^b & \text{on } I_{k+1} \\ R_{A_{k+1}^a}^a & \text{on } A_{k+1}^a \end{cases}\]

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\[ \nu \Delta (\lambda_{k+1} - \lambda_k) - (y_{k+1} \cdot \nabla) \lambda_{k+1} + (y_k \cdot \nabla) \lambda_k + (\nabla y_{k+1})^T \lambda_{k+1} - (\nabla y_k)^T \lambda_k + \nabla (\phi_{k+1} - \phi_k) + y_{k+1} - y_k = 0, \]

where

\[
\begin{align*}
R^b_k &= \begin{cases} 
0 & \text{on } A^b_k \cap A^b_{k+1} \\
\frac{b - \lambda_k}{\alpha} = b - u_k < 0 & \text{on } I_k \cap A^b_{k+1} \\
\frac{b - a}{\alpha} < \frac{b}{\alpha} & \text{on } A^a_k \cap A^b_{k+1},
\end{cases} \\
R^k_I &= \begin{cases} 
\frac{b}{\alpha} - \lambda_k = 0 & \text{on } A^b_k \cap I_{k+1} \\
\frac{a}{\alpha} - a \geq 0 & \text{on } A^a_k \cap I_{k+1},
\end{cases} \\
R^k_{A^a} &= \begin{cases} 
\frac{a}{\alpha} - b > \frac{b}{\alpha} & \text{on } A^b_k \cap A^a_{k+1} \\
\frac{a}{\alpha} - \lambda_k = a - u_k > 0 & \text{on } I_k \cap A^a_{k+1} \\
0 & \text{on } A^a_k \cap A^a_{k+1},
\end{cases}
\]

and \( R^k \) is the function whose restrictions to \( A^b_{k+1}, I_{k+1}, A^a_{k+1} \) coincides with \( R^b_k, R^k_I \) and \( R^a_{A^a} \).

4.1. Global convergence. The global convergence of the primal-dual active set method is based on the decreasing properties of the merit functional \( M : L^2(\tilde{\Omega}) \times L^2(\tilde{\Omega}) \to \mathbb{R} \) given by

\[
M(u, \mu) = \alpha^2 \int_{\tilde{\Omega}} \left( |(u-b)^+|^2 + |(a-u)^+|^2 \right) dx + \int_{A^b(u)} |\mu^-|^2 dx + \int_{A^a(u)} |\mu^+|^2 dx,
\]

where \( A^b(u) = \{ x : u \geq b \} \) and \( A^a(u) = \{ x : u \leq a \} \). For the decrease to hold it is necessary to verify an a-priori estimate for the adjoint variable (see [9], pg. 9-14). In the following theorem we verify that under certain conditions the estimate holds for our problem.

**Theorem 4.3.** Let (A1) and (A2) be satisfied. If \( \alpha \) and \( \nu \) are sufficiently large such that \( \kappa^4 < \frac{\alpha}{4} (\nu - M)^2 \), then

\[
\| \lambda_{k+1} - \lambda_k \| < \alpha \rho \| R^k \|, \text{ for each } k = 1, 2, ...
\]

with \( \rho = \frac{\alpha (\nu - M)^2 + 4\kappa^4}{3 \alpha (\nu - M)^2 - 4 \kappa^4} \), and the primal-dual active set method converges.

**Proof.** Considering the Navier-Stokes system in variational form for two consecutive iterates of the algorithm, we obtain that

\[
\nu (\nabla (y_{k+1} - y_k), \nabla w) + c(y_{k+1}, y_{k+1}, w) - c(y_k, y_k, w) = (u_{k+1} - u_k, w),
\]

for all \( w \in V \). Since \( u_{k+1} - u_k = \frac{1}{\alpha} (\lambda_{k+1} - \lambda_k) + R^k \), we get, taking \( w = \delta y = y_{k+1} - y_k \), that

\[
\nu \| \delta y \|^2_V + c(y_{k+1}, y_{k+1}, \delta y) - c(y_k, y_k, \delta y) \leq \frac{1}{\alpha} \| \delta \lambda \| \| \delta y \| + \| R^k \|_{-1} \| \delta y \|_V
\]

\[
\leq \frac{\kappa^2}{\alpha} \| \delta \lambda \|_V \| \delta y \|_V + \| R^k \|_{-1} \| \delta y \|_V,
\]

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where $\kappa$ is the Poincaré inequality constant, $\delta \lambda := \lambda_{k+1} - \lambda_k$ and $\| \cdot \|_{-1}$ is the $H^{-1}(\Omega)$ norm.

Additionally,

$$0 = c(\delta y, \delta y, \delta y) = c(y_{k+1}, \delta y, \delta y) - c(y_k, \delta y, \delta y)$$

$$= c(y_{k+1}, y_{k+1}, \delta y) - c(y_{k+1}, y_k, \delta y) - c(y_k, y_{k+1}, \delta y) + c(y_k, y_k, \delta y),$$

which implies that

$$c(y_{k+1}, y_{k+1}, \delta y) - c(y_k, y_k, \delta y) = c(y_{k+1}, y_k, \delta y) + c(y_k, y_{k+1}, \delta y) - 2c(y_k, y_k, \delta y)$$

$$= c(\delta y, y_k, \delta y) + c(y_k, \delta y, \delta y) \geq -|c(\delta y, \delta y, y_k)|$$

$$\geq -\mathcal{M}(y_k) \| \delta y \|^2_V \geq -\mathcal{M} \| \delta y \|^2_V.$$

Since by hypothesis (A1) holds, we get that

$$\nu \nu - \mathcal{M} \| \delta y \|^2_V \leq \frac{\kappa^2}{\alpha} \| \delta \lambda \|_V + \| R^k \|_{-1}.$$

From the variational formulation of the adjoint equations we get, for two consecutive iterates, that

$$\nu (\nabla \delta \lambda, \nabla w) - c(y_{k+1}, \lambda_{k+1}, w) + c(y_k, \lambda_k, w) + c(w, y_{k+1}, \lambda_{k+1}) - c(w, y_k, \lambda_k) = -\delta y, w,$$

for all $w \in V$. Taking in particular $w = \delta \lambda$,

$$\nu \| \delta \lambda \|^2_V - c(y_{k+1}, \lambda_{k+1}, \delta \lambda) + c(y_k, \lambda_k, \delta \lambda) + c(\delta \lambda, y_{k+1}, \lambda_{k+1}) - c(\delta \lambda, y_k, \lambda_k) = -\delta y, \delta \lambda).$$

Considering that

$$0 = c(\delta y, \delta \lambda, \delta \lambda) = c(y_{k+1}, \delta \lambda, \delta \lambda) - c(y_k, \delta \lambda, \delta \lambda)$$

$$= c(y_{k+1}, \lambda_{k+1}, \delta \lambda) - c(y_{k+1}, \lambda_k, \delta \lambda) - c(y_k, \lambda_{k+1}, \delta \lambda) + c(y_k, \lambda_k, \delta \lambda),$$

we obtain

$$c(y_{k+1}, \lambda_{k+1}, \delta \lambda) - c(y_k, \lambda_k, \delta \lambda) = c(y_{k+1}, \lambda_k, \delta \lambda) + c(y_k, \lambda_{k+1}, \delta \lambda) - 2c(y_k, \lambda_k, \delta \lambda)$$

$$= c(\delta y, \lambda_k, \delta \lambda).$$

Additionally,

$$c(\delta \lambda, \delta y, \delta \lambda) = c(\delta \lambda, y_{k+1}, \delta \lambda) - c(\delta \lambda, y_k, \delta \lambda)$$

$$= c(\delta \lambda, y_{k+1}, \lambda_{k+1}) - c(\delta \lambda, y_{k+1}, \lambda_k) - c(\delta \lambda, y_k, \lambda_{k+1}) + c(\delta \lambda, y_k, \lambda_k),$$

which implies that

$$c(\delta \lambda, y_{k+1}, \lambda_{k+1}) - c(\delta \lambda, y_k, \lambda_k) = c(\delta \lambda, \delta y, \delta \lambda) + c(\delta \lambda, y_{k+1}, \lambda_k) + c(\delta \lambda, y_k, \lambda_{k+1}) - 2c(\delta \lambda, y_k, \lambda_k)$$

$$= c(\delta \lambda, \delta y, \delta \lambda) + c(\delta \lambda, y_k, \lambda_k)$$

$$\geq c(\delta \lambda, y_{k+1}, \delta \lambda) + c(\delta \lambda, \delta y, \delta \lambda).$$

Consequently we get that

$$\nu \| \delta \lambda \|^2_V - c(y_{k+1}, \lambda_{k+1}, \delta \lambda) + c(y_k, \lambda_k, \delta \lambda) + c(\delta \lambda, y_{k+1}, \lambda_{k+1}) - c(\delta \lambda, y_k, \lambda_k)$$

$$= \nu \| \delta \lambda \|^2_V - c(\delta y, \lambda_k, \delta \lambda) + c(\delta \lambda, y_{k+1}, \delta \lambda) + c(\delta \lambda, y_k, \lambda_k)$$

$$\geq \nu \| \delta \lambda \|^2_V - |c(\delta y, \lambda_k, \delta \lambda)| - |c(\delta \lambda, y_{k+1}, \delta \lambda)| - |c(\delta \lambda, \delta y, \delta \lambda)|$$

$$\geq \nu \| \delta \lambda \|^2_V - 2\mathcal{N} \| \delta y \|_V \| \lambda_k \|_V \| \delta \lambda \|_V - \mathcal{M} \| \delta \lambda \|^2_V.
and, hence,
\[(\nu - \overline{M}) \|\delta \lambda\|_V \leq \kappa^2 \|\delta y\|_V + 2N \|\delta y\|_V \|\lambda_k\|_V.
\]
Therefore, using Lemma 3.2 and (A2),
\[\|\delta \lambda\|_V \leq \left(\frac{\kappa^2 + 2N \|\lambda_k\|_V}{\nu - \overline{M}}\right) \|\delta y\|_V \leq \frac{4\kappa^4 + \alpha(\nu - \overline{M})^2}{4\kappa(\nu - \overline{M})} \|\delta y\|_V.
\]
Combining with (4.5) we obtain that
\[(\nu - \overline{M}) \|\delta y\|_V \leq \frac{4\kappa^4 + \alpha(\nu - \overline{M})^2}{4\kappa(\nu - \overline{M})} \|\delta y\|_V + \|R^k\|_{-1}
\]
and, hence,
\[\|\delta y\|_V \leq \beta \|R^k\|_{-1},
\]
with \[\beta = \frac{4\alpha(\nu - \overline{M})^2}{3\alpha(\nu - \overline{M})^2 - 4\kappa^4} \] . Thus,
\[\|\delta \lambda\| \leq \kappa \|\delta \lambda\|_V \leq \frac{4\kappa^4 + \alpha(\nu - \overline{M})^2}{4\kappa(\nu - \overline{M})} \|\delta y\|_V \leq \frac{4\kappa^4 + \alpha(\nu - \overline{M})^2}{4\kappa(\nu - \overline{M})} \|R^k\|_{-1} \leq \alpha \rho \|R^k\|,
\]
where \[\rho = \frac{\alpha(\nu - \overline{M})^2 + 4\kappa^4}{3\alpha(\nu - \overline{M})^2 - 4\kappa^4} \] .

Since by hypothesis \[\kappa^4 < \frac{\alpha}{4}(\nu - \overline{M})^2 \] , we get that \[0 < \rho < 1\] and, consequently, the sufficient condition for global convergence of the nonlinear primal-dual active set method (see [9], pg. 9) is fulfilled.

Under an extra smallness condition over the \(V\)-norm of the adjoint states, an alternative sufficient condition for the decrease of the merit functional is obtained. From Lemma 3.2, the extra condition is fulfilled if the velocity field in each of the primal-dual iterations is not far from the desired state.

**Theorem 4.4.** If \[\|\lambda_{k+1}\|_V, \|\lambda_k\|_V\] are sufficiently small such that
\[(\|\lambda_{k+1}\|_V + \|\lambda_k\|_V)N \|\delta y\|_V^2 < \|\delta y\|^2,
\]
for each \(k = 1, 2, \ldots\), then the primal-dual active set method converges.

**Proof.** Multiplying \(\delta u := u_{k+1} - u_k = \frac{1}{\alpha} \delta \lambda + R^k\) by \(\delta \lambda\) we get
\[\frac{1}{\alpha} \|\delta \lambda\|^2 + (R^k, \delta \lambda) = (\delta u, \delta \lambda).
\]
Using the primal equations in variational form,
\[\frac{1}{\alpha} \|\delta \lambda\|^2 + (R^k, \delta \lambda) = \nu(\nabla \delta y, \nabla \delta \lambda) + c(y_{k+1}, y_{k+1}, \delta \lambda) - c(y_k, y_k, \delta \lambda),\]
which, considering the adjoint equations, yields
\[
\frac{1}{\alpha} \| \delta \lambda \|^2 + (R_k, \delta \lambda) = - \| \delta y \|^2 + c(y_{k+1}, \lambda_{k+1}, \delta y) - c(y_k, \lambda_k, \delta y) - c(\delta y, y_{k+1}, \lambda_{k+1}) \\
+ \alpha (y_k, \lambda_k) + c(y_{k+1}, y_{k+1}, \delta \lambda) - c(y_k, y_k, \lambda_{k+1}) + c(y_k, y_k, \lambda_k).
\]
Utilizing the properties of the trilinear form, we get
\[
\frac{1}{\alpha} \| \delta \lambda \|^2 + (R_k, \delta \lambda) = - \| \delta y \|^2 - c(y_{k+1}, \lambda_{k+1} + \lambda_k, y_k) - c(y_k, \lambda_k, \delta y) \\
+ c(y_k, \delta y, \lambda_{k+1}) + c(y_{k+1}, y_{k+1}, \delta \lambda) + 2c(y_{k+1}, \lambda_{k+1}, y_{k+1}) \\
= - \| \delta y \|^2 + c(\delta y, \lambda_{k+1}, \delta y) + c(\delta y, \lambda_k, \delta y).
\]
Consequently,
\[
\frac{1}{\alpha} \| \delta \lambda \|^2 + \| \delta y \|^2 - c(\delta y, \lambda_{k+1} + \lambda_k, \delta y) = -(R_k, \delta \lambda).
\]
Using the continuity of the trilinear form,
\[
\| R_k \| \| \delta \lambda \| \geq \frac{1}{\alpha} \| \delta \lambda \|^2 + \| \delta y \|^2 - N \| \delta y \|^2 < \| \delta y \|^2,
\]
which completes the proof.

\[\square\]

4.2. Local convergence. The local convergence of the primal dual active set method is based on an interpretation of the strategy as a partial semi-smooth Newton method (cf. [9]). This means that instead of linearizing both nonlinearities, we keep the one from the primal equations and linearize only the one from the complementarity function.

Expessed by equation (4.1), the complementarity condition for our problem can be formulated, using the adjoint equations, as
\[
(4.6) \ F(u) = \alpha(u - b) + \max(0, \alpha b - \lambda) + \min(0, \lambda - \alpha a) = 0.
\]
The primal dual algorithm can then be written as
\[
(4.7) \ \alpha(u_{k+1} - b) - G_{\max}^k (\lambda_{k+1} - \lambda_k) + G_{\min}^k (\lambda_{k+1} - \lambda_k) \\
+ \max(0, \alpha b - \lambda_k) + \min(0, \lambda_k - \alpha a) = 0,
\]
where
\[
G_{\max}^k \phi = \begin{cases} 
0 & \text{on } \mathcal{A}_{b k+1}^a = \{x : \lambda_k - \alpha b > 0\} \\
\phi & \text{on } \mathcal{T}_{b k+1}^a = \{x : \lambda_k - \alpha b \leq 0\},
\end{cases}
\]
\[
G_{\min}^k \phi = \begin{cases} 
\phi & \text{on } \mathcal{A}_{a k+1}^a = \{x : \lambda_k - \alpha a < 0\} \\
0 & \text{on } \mathcal{T}_{a k+1}^b = \{x : \lambda_k - \alpha a \geq 0\},
\end{cases}
\]
Taking into account that, at the optimal solution, (4.6) is satisfied, we obtain
\[
\alpha(u_{k+1} - u^*) - G_{\max}^k(\lambda_{k+1} - \lambda^*) + G_{\min}^k(\lambda_{k+1} - \lambda^*) = \max(0, \alpha b - \lambda^*) - \max(0, \alpha b - \lambda_k) + G_{\max}^k(\lambda^* - \lambda_k) + \min(0, \lambda^* - \alpha a) - \min(0, \lambda_k - \alpha a) - G_{\min}^k(\lambda^* - \lambda_k) := R(u_k).
\]

The optimality condition for the iterate \(k + 1\) can then be expressed as
\[
\alpha(u_{k+1} - u^*) - (\lambda_{k+1} - \lambda^*)_I = R,
\]
where \(R = R(u_k)\) and \(I = \{x : \alpha a(x) \leq \lambda_k(x) \leq \alpha b(x)\}\).

It can be verified (see [9], pg. 16), that the \(\max\) and \(\min\) functions are Newton differentiable from \(L^2(\Omega) \rightarrow L^2(\Omega)\), with \(q > 2\), and, consequently,
\[
\begin{align*}
\|\max(0, \alpha b - \lambda^*) - \max(0, \alpha b - \lambda_k) + G_{\max}^k(\lambda^* - \lambda_k)\|_{L^2} &= o(\|\lambda_k - \lambda^*\|_{L^q}), \\
\|\min(0, \lambda^* - \alpha a) - \min(0, \lambda_k - \alpha a) - G_{\min}^k(\lambda^* - \lambda_k)\|_{L^2} &= o(\|\lambda_k - \lambda^*\|_{L^p}).
\end{align*}
\]

**Theorem 4.5.** Assume that the iterates \(u_k\) converge to the optimal solution \(u^*\). If the optimal adjoint state is sufficiently small such that \(\sqrt{\alpha} > \frac{3}{2} N\sigma(y^*) \|\lambda^*\|_V\), then the primal-dual active set method converges locally superlinearly.

**Proof.** Let us begin by choosing an \(\epsilon\) neighborhood such that
\[
1 - N\frac{\sigma(y^*) \|\lambda_k\|_V}{2\sqrt{\alpha}} \geq \frac{1}{2} - N\frac{\sigma(y^*) \|\lambda^*\|_V}{4\sqrt{\alpha}} > 0
\]
for \(\lambda_k\) with \(\|\lambda_k - \lambda^*\| < \epsilon\).

Multiplying \(u_{k+1} - u^* = \frac{1}{\alpha} (\lambda_{k+1} - \lambda^*)_I + \frac{1}{\alpha} R\) by \(u_{k+1} - u^*\) and proceeding as in the proof of Theorem 4.3, we get
\[
\begin{align*}
\alpha \|u_{k+1} - u^*\|^2 + \|((y_{k+1} - y^*)_I\|^2 &< c(((y_{k+1} - y^*)_I, (\lambda_{k+1} + \lambda^*)_I, (y_{k+1} - y^*)_I) = (R, u_{k+1} - u^*).
\end{align*}
\]

Since \(\|y_{k+1} - y^*\|^2 \leq \sigma(y^*) \|u_{k+1} - u^*\| \|y_{k+1} - y^*\|\) we obtain, using the trilinear form estimates, that
\[
\begin{align*}
\|R\| \|u_{k+1} - u^*\| \geq & \alpha \|u_{k+1} - u^*\|^2 + \|((y_{k+1} - y^*)_I\|^2 - N\|\lambda_{k+1} + \lambda^*\|_V \|(y_{k+1} - y^*)_I\|^2 \\
& \geq \alpha \|u_{k+1} - u^*\|^2 + \|((y_{k+1} - y^*)_I\|^2 \\
& - \frac{N}{\sqrt{\alpha}} \sigma(y^*) \|\lambda_{k+1} + \lambda^*\|_V \|\sqrt{\alpha}(u_{k+1} - u^*)\| \|(y_{k+1} - y^*)_I\| \\
& \geq \alpha \|u_{k+1} - u^*\|^2 + \|((y_{k+1} - y^*)_I\|^2 \\
& - \frac{N}{2\sqrt{\alpha}} \sigma(y^*) \|\lambda_{k+1} + \lambda^*\|_V (\alpha \|u_{k+1} - u^*\|^2 + \|((y_{k+1} - y^*)_I\|^2 \\
& \geq \left(1 - \frac{N}{2\sqrt{\alpha}} \sigma(y^*) \|\lambda^*\|_V + \|\lambda_{k+1}\|_V\right) \|\sqrt{\alpha}(u_{k+1} - u^*), y_{k+1} - y^*\|^2.
\end{align*}
\]
Using (4.11) we obtain
\[ \| R \| \| u_{k+1} - u^* \| \geq \left( \frac{1}{2} - \frac{3}{4} \frac{N}{\sqrt{\alpha}} \sigma(y^*) \| \lambda^* \|_V \right) \| (\sqrt{\alpha}(u_{k+1} - u^*), y_{k+1} - y^*) \|^2. \]

Since by hypothesis \( \sqrt{\alpha} > \frac{3}{2} \frac{N}{\sigma(y^*)} \| \lambda^* \|_V \), we deduce the existence of a constant \( C \) independent of \( k \) and \( I \) such that
\[ (4.13) \quad \| u_{k+1} - u^* \| \leq C \| R \|. \]

Proceeding as in the proof of Theorem 4.3 we also obtain that
\[ \| \lambda^* - \lambda_k \|_V \leq \sigma(y^*) \left( \kappa^2 + 2N \| \lambda_k \|_V \right) \| y_k - y^* \|_V. \]

Additionally we get that
\[ \| y_k - y^* \|_V \leq \kappa \sigma(y^*) \| u_k - u^* \|. \]

Thus, taking into account (4.11),
\[ \| \lambda^* - \lambda_k \|_V \leq \kappa \sigma(y^*) \left( \kappa^2 \sigma(y^*) + 4\sqrt{\alpha} \right) \| u_k - u^* \|. \]

Since \( V \hookrightarrow L^p(\Omega) \), for each \( p \in [1, \infty] \),
\[ (4.14) \quad \| \lambda^* - \lambda_k \|_{L^p} \leq \delta \kappa \sigma(y^*) \left( \kappa^2 \sigma(y^*) + 4\sqrt{\alpha} \right) \| u_k - u^* \|. \]

Considering (4.9), (4.10) and (4.14) we obtain that
\[ \| R \| = o(\| u_k - u^* \|) \]
and, consequently, from (4.13)
\[ \| u_{k+1} - u^* \| = o(\| u_k - u^* \|). \]

\[ \square \]

5. Numerical results

In this section we present some numerical tests, which illustrate the behaviour of the nonlinear primal-dual active set method applied to the constrained distributed control of the Navier-Stokes equations.

As domain we use the channel \((0, 1) \times (0, 0.5)\) and set a step by removing the rectangular sector \((0.5, 1) \times (0, 0.25)\) from the domain. The fluid flows from left to right and has a parabolic inflow boundary condition with maximum value equal to one. For the outflow boundary condition we use the so called ”do nothing” condition (cf. [22]). For the rest of the walls an homogeneous Dirichlet condition is imposed. This problem is referred in the literature as ”forward facing step flow”.

The domain was discretized using a homogeneous staggered grid, with discretization step \( h \), combined with a finite differences scheme. In order to avoid numerical instabilities and to obtain appropriate results for high Reynolds numbers, a first order upwinding scheme is used.

The simulation of the fluid is depicted in Figure 1 for Reynolds number \( Re = 1000 \). As \( Re \) increases, the recirculation bubbles become bigger and stronger. A desirable effect of a control would be to reduce this recirculation effect in order to avoid possible flow separation.

The target of the control is to drive the fluid to an almost linear behavior, specified by the Navier-Stokes flow, with Reynolds number 1, in the channel. The auxiliary
unconstrained optimal control problems obtained in each primal-dual step are solved using a SQP method. For the solution of the linear system in each SQP step we use MATLAB’s sparse solver.

5.1. Example 1. In this first example we use the following parameter values: \( \text{Re} = 1000, \alpha = 0.1, a = \left(\begin{array}{c} -9 \\ -12 \end{array}\right) \) and \( b = \left(\begin{array}{c} 3.5 \\ 2 \end{array}\right) \). The numerical experiment was carried out on a 240 points grid and the optimal control field can be visualized in Figure 2. It can be observed that the control acts principally on the acceleration region before the step, the recirculation sector after the step and at the exit of the channel.

The controlled state can be observed in Figure 3, where the desired effect, of avoiding recirculations, is reached. The cost functional takes the optimal value \( J(y^*, u^*) = 0.06278240 \).

In Figures 4 and 5 the horizontal and vertical components of the control, together with their multipliers, are depicted separately. This representation allows to verify by inspection the satisfaction of the complementarity condition.

In Table 1 the evolution of the method and the cost functional are depicted. The first two columns indicate the primal-dual iteration number and the number of inner SQP iterations needed, respectively. The third column tabulates the value of the cost functional, which decreases with each primal-dual iteration. Finally, the
last two columns show an estimate of the convergence rate and the residual of the complementarity function.

5.2. Example 2. In this example we describe the behavior of the primal-dual active set method, when the control constraints are not constant functions. The parameter data for this experiment are $Re = 500$, $\alpha = 0.1$, $a = \left(\begin{array}{c} -9 \\ -10 - \sin(3\pi x_1 x_2) \end{array}\right)$ and
Table 1. Example 1, $h=1/240$.

<table>
<thead>
<tr>
<th>Iteration</th>
<th>it. SQP</th>
<th>$J(y,u)$</th>
<th>$|u_n-u_{n-1}|_{1}$</th>
<th>$|u_{n-1}-u_{n-2}|$</th>
<th>$|\mathcal{F}(u_n)|_\infty$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>3</td>
<td>0.06280909</td>
<td>-</td>
<td>-</td>
<td>0.0706</td>
</tr>
<tr>
<td>2</td>
<td>3</td>
<td>0.06278349</td>
<td>-</td>
<td>-</td>
<td>0.0131</td>
</tr>
<tr>
<td>3</td>
<td>2</td>
<td>0.06278242</td>
<td>0.01713</td>
<td>0.0022</td>
<td></td>
</tr>
<tr>
<td>4</td>
<td>2</td>
<td>0.06278240</td>
<td>0.06343</td>
<td>5.55 \cdot 10^{-17}</td>
<td></td>
</tr>
</tbody>
</table>

$b = \left(\frac{x_1 x_2 + 2}{2}\right)$. For the numerical test a mesh step of $h = 1/240$ is used. The resulting control field is depicted in Figure 6 and the final controlled state in Figure 7. As in Example 1, it can be observed that the control acts mainly before the step and at the end of the channel. The controlled state obtained does not exhibit recirculation regions.

The evolution of the method is shown in Table 2. It can be observed that the cost functional decreases in each primal-dual step. The local superlinear convergence and the satisfaction of the complementarity system can also be verified from the data.
Table 2. Example 2, h=1/240.

<table>
<thead>
<tr>
<th>Mesh step</th>
<th># PD iterations</th>
</tr>
</thead>
<tbody>
<tr>
<td>1/80</td>
<td>4</td>
</tr>
<tr>
<td>1/120</td>
<td>5</td>
</tr>
<tr>
<td>1/160</td>
<td>5</td>
</tr>
<tr>
<td>1/200</td>
<td>5</td>
</tr>
<tr>
<td>1/240</td>
<td>5</td>
</tr>
</tbody>
</table>

Table 3. Example 2.

In Table 3 the number of iterations needed by the primal-dual active set method for different meshes is tabulated. The data allow us to conjecture a mesh independent behavior of the method, which can be also observed in the linear case (cf. [1, 8]).

REFERENCES


Departamento de Matemática, EPN Quito, Ecuador
E-mail address: juancadelosreyes@gmx.net