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THE APPROXIMATE SOLUTION OF HIGH-ORDER LINEAR FRACTIONAL DIFFERENTIAL EQUATIONS WITH VARIABLE COEFFICIENTS IN TERMS OF GENERALIZED TAYLOR POLYNOMIALS

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Abstract-In this paper, we have developed a new method called Generalized Taylor collocation method (GTCM), which is based on the Taylor collocation method, to give approximate solution of linear fractional differential equations with variable coefficients. Using the collocation points, this method transforms fractional differential equation to a matrix equation which corresponds to a system of linear algebraic equations with unknown Generalized Taylor coefficients. Generally, the method is based on computing the Generalized Taylor coefficients by means of the collocation points. This method does not require any intensive computation. Moreover, It is easy to write computer codes in any symbolic language. Hence, the proposed method can be used as effective alternative method for obtaining analytic and approximate solutions for fractional differential equations. The effectiveness of the proposed method is illustrated with some examples. The results show that the method is very effective and convenient in predicting the solutions of such problems.

Keywords- Fractional differential equation, Taylor collocation method, Adomian decomposition method, Homotopy perturbation method, Variational iteration method, Fractional differential transformation method

1. INTRODUCTION

Interest in the concept of differentiation and integration to non–integer order has existed since the development of the classical calculus [1]. By implication, mathematical modeling of many physical systems are governed by linear and nonlinear fractional differential equations in various applications in fluid mechanics, viscoelasticity, chemistry, physics, biology and engineering.

Since many fractional differential equations are nonlinear and do not have exact analytical solutions, various numerical and analytic methods have been used to solve these equations. Recently, Adomian decomposition method (ADM) [2-5], variational iteration method [2,6,7], homotopy perturbation method [8-10], homotopy analysis method [11-24-25], fractional differential transformation (FDTM) [12-14] and fractional difference method [2] have been used to obtain analytical approximate solutions of fractional differential equations.

In this paper, we aim to present a new generalization of the Taylor collocation method that will extend the application of the method to linear fractional differential equations with variable coefficients:

$$\sum_{k=0}^{m} P_k(x_i) D_*^{k\alpha} y(x_i) = f(x_i), \quad a \le x \le b,\ n-1 \le m\alpha < n \tag{1.1}$$

with initial conditions

$$D_*^i y(a) = \lambda_i,\ \ i = 0,1,\ldots,n-1;\ a \le c \le b \tag{1.2}$$

which $P_k(x)$ and $f(x)$ are functions defined on $a \le x \le b$, the real coefficient λ_i is a appropriate constant.

The new technique will be named as Generalized Taylor collocation method (GTCM) and is based on Taylor collocation method [15,16,23], generalized Taylor's formula [17] and Caputo fractional derivative [18]. Using the collocation points, the GTCM transforms the given fractional differential equation and initial conditions to matrix equation with (including) unknown Generalized Taylor coefficients. By means of the obtained matrix equations and Maple 13 programme, the Generalized Taylor coefficients can be computed.

2. BASIC DEFINITIONS

In this section, we present some basic definitions and important properties of fractional calculus [2,3,5,6,8-10,13,17,18].

Definition 1. A real function f(x), $x > 0$, is said to be in the space C_μ, $\mu \in R$ if there exists a real number $p > \mu$, such that $f(x) = x^p f_1(x)$, where $f_1(x) \in C(0,\infty)$, and it is said to be in the space C_μ^n iff $f^{(n)} \in C_\mu$, $n \in N$.

Definition 2. The fractional integral operator ${}_aJ_x^\alpha$ (Riemann–Liouville operator) of order $\alpha \ge 0$ of a function $f \in C_\mu$, $\mu \ge -1$, is defined by

$$\begin{aligned} {}_aJ_x^\alpha f(x) &= \frac{1}{\Gamma(\alpha)}\int_a^x (x-u)^{\alpha-1} f(u)du \quad x \ge a \\ {}_aJ_x^0 f(x) &= f(x) \end{aligned} \tag{2.1}$$

Definition 3. A fractional derivative of arbitrary order ${}_aD_x^\alpha$ with $m-1 \le \alpha < m$, $m \in N$, can be defined through fractional integration of order $m-\alpha$ as follows:

$$\begin{aligned} {}_aD_x^\alpha f(x) &= {}_aD_x^m\, {}_aJ_x^{m-\alpha} f(x) \\ &= \left\{ \begin{array}{ll} \dfrac{d^m}{dx^m}\left[\dfrac{1}{\Gamma(m-\alpha)}\displaystyle\int_a^x \frac{f(u)}{(x-u)^{\alpha+1-m}}du\right], & m-1 < \alpha < m \\ \dfrac{d^m}{dx^m} f(x), & \alpha = m \end{array} \right\} \end{aligned} \tag{2.2}$$

$${}_aD_x^0 f(x) = f(x)$$

Equations (2.1) and (2.2) are known as the Riemann–Liouville integral and the Riemann–Liouville derivative for $a = 0$.

Definition 4. The fractional derivative of f(x) in the Caputo sense is defined as follows:

$${}_aD_*^{\alpha} f(x) = {}_aJ_x^{m-\alpha}\ {}_aD_x^m f(x)$$
$$= \begin{cases} \dfrac{1}{\Gamma(m-\alpha)}\displaystyle\int_a^x \frac{f^{(m)}(u)}{(x-u)^{\alpha+1-m}}du, & m-1<\alpha<m,\ m\in N, \\ & x>0 \ and\ f(x)\in C_{-1}^m \\ \dfrac{d^m}{dx^m} f(x), & \alpha = m \end{cases}$$

Some basic properties of the fractional operator are listed below [6,11,12,13,14]:

$$D_*^{\alpha} f(x) = J^{m-\alpha} D^m f(x) \neq D^m J^{m-\alpha} f(x) = D^{\alpha} f(x)$$

$$D_*^{\alpha} f(x) = D^{\alpha}\left(f(x) - \sum_{k=0}^{m-1} \frac{x^k}{k!} f^{(k)}(0^+) \right)$$

$$D_*^{\alpha}(\lambda f(x) + \mu g(x)) = \lambda D_*^{\alpha} f(x) + \mu D_*^{\alpha} g(x) \quad \lambda, \mu \text{ are constants}$$

$D_*^{\alpha} D_*^{\beta} f(x) = D_*^{\alpha+\beta} f(x) = D_*^{\beta} D_*^{\alpha} f(x)$, $\forall \alpha, \beta \in R^+$, if f(x) is sufficiently smooth.

$$D_*^{\alpha}(x^j) = \begin{cases} 0, & if\ \mathrm{j}\in N\cup\{0\}\ and\ j<m \\ \dfrac{\Gamma(j+1)}{\Gamma(j+1-\alpha)} x^{j-\alpha}, & if\ \mathrm{j}\in N\ and\ j\geq m\ or\ j\notin N\ and\ j>m \end{cases}$$

$$D_*^{\alpha} C = 0 \text{ for any constant C}$$

$$D_*^{\alpha} J^{\alpha} f(x) = f(x)$$

$$J^{\alpha} D_*^{\alpha} f(x) = f(x) - \sum_{k=0}^{m-1} \frac{t^k}{k!} f^{(k)}(0^+), \quad m-1<\alpha\leq m, \ m\in N$$

Theorem 1. [20] (Generalized Taylor formula) Suppose that ${}_aD_*^{k\alpha} f(x)\in C(a,b]$ for $k=0,1,...,n+1$, where $0<\alpha\leq 1$. Then

$$f(t) = \sum_{i=0}^{n} \frac{(x-a)^{i\alpha}}{\Gamma(i\alpha+1)} \left[D_*^{i\alpha} f(x) \right]_{x=a} + R_n^{\alpha}(x,a) \tag{2.3}$$

with

$$R_n^{\alpha}(x,a) = \frac{(x-a)^{(n+1)\alpha}}{\Gamma((n+1)\alpha+1)} \left[{}_aD_*^{(n+1)\alpha} f(x) \right]_{x=\xi}, \quad \xi\in[a,x], \quad \forall x\in(a,b]$$

where

$$D_*^{n\alpha} = D_*^{\alpha} D_*^{\alpha} ... D_*^{\alpha} \quad \text{n times.}$$

DESCRIPTION OF THE METHOD

In this section, we use a Taylor collocation method to solve fractional differential equations. This method is very useful and can be used to solve many important fractional differential equations. The basic ideas of the Taylor collocation method [15,16,23] is to develop and apply to the $m\alpha$th-order linear fractional differential equation with variable coefficients. We adopt Caputo's definition, which is a modification of the Riemann–Liouville definition and has the advantage of dealing properly with initial value problems, for the concept of the fractional derivative.

In the first, we consider Eq.(1.1) and Eq.(1.2)
and we assume that the solution of this equation can be expressed as a truncated Generalized Taylor series

$$y(x)=\sum_{k=0}^{N}\frac{(x-c)^{k\alpha}}{\Gamma(k\alpha+1)}D_*^{k\alpha}y(x)\Big|_{x=c}, \quad a\le x,c\le b \quad N\ge m \tag{3.1}$$

Let us consider the $m\alpha$th-order fractional differential equation with variable coefficients (1.1) and find the truncated series expansions of each term in expression (1.1) at $x=c$ and their matrix representations. We first consider the desired solutions $y(x)$ of (1.1) defined by a truncated Generalized Taylor series (3.1). Then can be written matrix form

$$\left[y(x)\right]=XM_0A \tag{3.2}$$

where

$$X=\begin{bmatrix}1 & (x-c)^{\alpha} & (x-c)^{2\alpha} & \cdots & (x-c)^{N\alpha}\end{bmatrix}_{N+1}$$

$$A=\begin{bmatrix}y(c) & D_*^{\alpha}y(c) & D_*^{2\alpha}y(c) & \cdots & D_*^{N\alpha}y(c)\end{bmatrix}_{N+1}^{t}$$

$$M_0=\begin{bmatrix}\frac{1}{\Gamma(1)} & 0 & 0 & \cdots & 0\\ 0 & \frac{1}{\Gamma(\alpha+1)} & 0 & \cdots & 0\\ 0 & 0 & \frac{1}{\Gamma(2\alpha+1)} & \cdots & 0\\ \vdots & \vdots & \vdots & \vdots & \vdots\\ 0 & 0 & 0 & \cdots & \frac{1}{\Gamma(N\alpha+1)}\end{bmatrix}_{(N+1)\times(N+1)}$$

Generalized Taylor collocation method is based on computing the Generalized Taylor coefficients by means of the collocation points are thereby finding the matrix A containing the unknown Generalized Taylor coefficients.
Firstly, we substitute the collocation points defined by

$$x_i=a+i\frac{b-a}{N},\ i=0,1,2,\cdots,N\ ,\ x_0=a, x_N=b \tag{3.3}$$

into (3.1) matrix representation to obtain

$$\left[y(x_i)\right]=X_iM_0A\,;\,i=0,1,...,N \tag{3.4}$$

and

$$\begin{aligned}\left[y(x_0)\right]&=X_0M_0A\\ \left[y(x_1)\right]&=X_1M_0A\\ &\vdots\\ \left[y(x_N)\right]&=X_NM_0A\end{aligned} \tag{3.5}$$

Thus, we obtain a new matrix form

$$Y^{(0)}=CM_0A, \tag{3.6}$$

where

$$Y^{(0)}=\begin{bmatrix} y(x_0) & y(x_1) & y(x_2) & \dots & y(x_N)\end{bmatrix}^T$$

and

$$C=\begin{bmatrix} X_0 & X_1 & \cdots & X_N\end{bmatrix}^T=\begin{bmatrix} 1 & (x_0-c)^{\alpha} & (x_0-c)^{2\alpha} & \cdots & (x_0-c)^{N\alpha} \\ 1 & (x_1-c)^{\alpha} & (x_1-c)^{2\alpha} & \cdots & (x_1-c)^{N\alpha} \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ 1 & (x_N-c)^{\alpha} & (x_N-c)^{2\alpha} & \cdots & (x_N-c)^{N\alpha}\end{bmatrix}_{(N+1)x(N+1)}$$

Similarly, the matrix representation of the functions $D_*^{k\alpha}y(x)$ becomes

$$D_*^{k\alpha}y(x)\Rightarrow Y^{(k)}=CM_kA\ ,\quad k=0,1,\ldots,m\le N \tag{3.7}$$

where

$$M_1=\begin{bmatrix} 0 & \frac{1}{\Gamma(1)} & 0 & \cdots & 0 \\ 0 & 0 & \frac{1}{\Gamma(\alpha+1)} & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \vdots & \frac{1}{\Gamma((N-1)\alpha+1)} \\ 0 & 0 & 0 & \cdots & 0\end{bmatrix}_{(N+1)\times(N+1)}$$

$$M_k=\begin{array}{c} \overbrace{\qquad}^{k+1\ column} \\ \begin{bmatrix} 0 & 0 & \cdots & \frac{1}{\Gamma(1)} & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 & \frac{1}{\Gamma(\alpha+1)} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 & 0 & \cdots & \left.\frac{1}{\Gamma((N-k)\alpha+1)}\right\} \\ 0 & 0 & \cdots & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 & 0 & \cdots & 0\end{bmatrix}_{(N+1)\times(N+1)}\end{array}\begin{array}{l} \\ \\ \\ \dashrightarrow N-k.\ row \\ \\ \\ \\ \end{array} ,k=2,3,\ldots,N \tag{3.8}$$

and for $x=x_i,\ i=0,1,\ldots,N$ (1.1) to obtain

$$\sum_{k=0}^{m}P_k(x_i)D_*^{k\alpha}y(x_i)=f(x_i),\quad a\le x\le b, n-1\le\alpha<n$$

Then we can write the system (1.1) in the matrix form

$$\begin{gathered} P_0(x_0)y(x_0)+P_1(x_0)D_*^{\alpha}y(x_0)+\cdots+P_m(x_0)D_*^{m\alpha}y(x_0)=f(x_0) \\ P_0(x_1)y(x_1)+P_1(x_1)D_*^{\alpha}y(x_1)+\cdots+P_m(x_1)D_*^{m\alpha}y(x_1)=f(x_1) \\ \vdots \\ P_0(x_N)y(x_N)+P_1(x_N)D_*^{\alpha}y(x_N)+\cdots+P_m(x_N)D_*^{m\alpha}y(x_N)=f(x_N) \end{gathered}$$

and

$$P_0Y^{(0)}+P_1Y^{(1)}+\cdots+P_mY^{(m)}=F$$

or briefly

$$\sum_{k=0}^{m} P_k Y^{(k)} = F \tag{3.9}$$

where

$$P_k = \begin{bmatrix} P_k(x_0) & 0 & \cdots & 0 \\ 0 & P_k(x_1) & \cdots & \vdots \\ \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & \cdots & P_k(x_N) \end{bmatrix}_{(N+1)\times(N+1)} ,F = \begin{bmatrix} f(x_0) \\ f(x_1) \\ \vdots \\ f(x_N) \end{bmatrix}_{(N+1)\times 1}$$

Then we can write the matrix (3.7) and (3.9)

$$\left\{\sum_{k=0}^{m} P_k C M_k \right\} A = F \tag{3.10}$$

Now let us form the matrix representation of the conditions. Using the conditions (1.2) and (3.7), we have

$$[y(x)] = XM_0 A \ , \ k = 0,1,\ldots,m \le N$$

into equation $x = a$

$$[y(a)] = \left[1 \quad (a-c)^{\alpha} \quad (a-c)^{2\alpha} \quad \ldots \quad (a-c)^{N\alpha}\right] M_0 A = HM_0 A$$

If h=a-c, H matrix

$$H = \left[1 \quad h^{\alpha} \quad h^{2\alpha} \quad \ldots \quad h^{N\alpha}\right]$$

Then we can write $D_*^{k\alpha} y(x)$ for x=a, in the matrix form

$$\left[D_*^{k\alpha} y(x)\Big|_{x=a}\right] = \left[D_*^{k\alpha} y(a)\right] = HM_k A \tag{3.11}$$

Substituting the matrix representations (3.11) into the (1.2), we obtain

$$HM_i A = \lambda_i \quad i = 0,\ldots,n-1$$

Let us define U_i as

$$U_i = HM_i \equiv \left[u_{i0} \quad u_{i1} \quad \ldots \quad u_{iN}\right] \ ,i = 0,1,\ldots,n-1 \tag{3.12}$$

Thus, the matrix forms of conditions (1.2) become

$$U_i A = \lambda_i \ ,i = 0,1,\ldots,n-1 \tag{3.13}$$

3. METHOD OF SOLUTION

Let us consider the fundamental matrix equation (3.10) corresponding to the $m\alpha$ th-order linear fractional differential equation with variable coefficients (1.1). We can write Eq. (3.10) in the form

$$WA = F \tag{4.1}$$

where

$$W = \left\{\sum_{k=0}^{m} P_k C M_k \right\} \tag{4.2}$$

The augmented matrix of (4.1) becomes

$$\left[W;F\right]=\begin{bmatrix} w_{00} & w_{01} & \cdots & w_{0N} & ; & f(x_0) \\ w_{10} & w_{11} & \cdots & w_{1N} & ; & f(x_1) \\ \vdots & \vdots & \cdots & \vdots & ; & \vdots \\ w_{N0} & w_{N1} & \cdots & w_{NN} & ; & f(x_N) \end{bmatrix}_{(N+1)\times(N+2)} \tag{4.3}$$

We now consider the matrix equations (3.13) corresponding to conditions (1.2). Then the augmented matrix of (3.13) becomes

$$\left[U_i;\lambda_i\right]=\left[u_{i0} \quad u_{i1} \quad \ldots \quad u_{iN};\lambda_i\right] \tag{4.4}$$

Consequently, to find the unknown Generalized Taylor coefficients $D_*^{k\alpha}y(c)$, $k=0,1,\ldots,m$ related with the approximate solution of the problem consisting of (1.1) and conditions (1.2), by replacing the m row matrix (4.4) by the last m rows of augmented matrix (4.3) ,we have new augmented matrix

$$\left[W^*;F^*\right]=\begin{bmatrix} w_{00} & w_{01} & \cdots & w_{0N} & ; & f(x_0) \\ \vdots & \vdots & \cdots & \vdots & ; & \vdots \\ w_{N-m,0} & w_{N-m,1} & \cdots & w_{N-m,N} & ; & f(x_{N-m}) \\ u_{00} & u_{01} & \cdots & u_{0N} & ; & \lambda_0 \\ u_{10} & u_{11} & \cdots & u_{1N} & ; & \lambda_1 \\ \vdots & \vdots & \cdots & \vdots & ; & \vdots \\ u_{m-1,0} & u_{m-1,1} & \cdots & u_{m-1,N} & ; & \lambda_{m-1} \end{bmatrix}$$

or the corresponding matrix equation

$$W^*A=F^* \tag{4.5}$$

so that

$$W^*=\begin{bmatrix} w_{00} & w_{01} & \cdots & w_{0N} \\ \vdots & \vdots & \cdots & \vdots \\ w_{N-m,0} & w_{N-m,1} & \cdots & w_{N-m,N} \\ u_{00} & u_{01} & \cdots & u_{0N} \\ u_{10} & u_{11} & \cdots & u_{1N} \\ \vdots & \vdots & \cdots & \vdots \\ u_{m-1,0} & u_{m-1,1} & \cdots & u_{m-1,N} \end{bmatrix}, F^*=\begin{bmatrix} f(x_0) \\ \vdots \\ f(x_{N-m}) \\ \lambda_0 \\ \lambda_1 \\ \vdots \\ \lambda_{m-1} \end{bmatrix}, A=\begin{bmatrix} y^{(0)}(c) \\ y^{\alpha}(c) \\ \vdots \\ y^{N\alpha}(c) \end{bmatrix}$$

If $\det W^*\neq 0$, we can write equation (4.5) as

$$A=\left(W^*\right)^{-1}F^*$$

And the matrix A is uniquely determined. Thus, the $m\alpha$ th-order linear fractional differential equation (1.1) with variable coefficients under conditions (1.2) has a unique solution which is given by the truncated generalized Taylor solution

$$y(x)=\sum_{k=0}^{N}\frac{(x-c)^{k\alpha}}{\Gamma(k\alpha+1)}D_*^{k\alpha}y(x)\Big|_{x=c}+R_N^{\alpha}(x,c);\ \ a\leq x,c\leq b.\ \ N\geq m$$

If we set $u_{ij} = 0$ and $\lambda_i = 0$ in the matrix $\left[W^*;F^*\right]$, we can obtain the general solution of equation (1.1). If $\det W \neq 0$ in the matrix $\left[W;M_0F\right]$, we can obtain the particular solution of equation (1.1).
Since the obtained generalized Taylor series solution is an approximate solution of fractional differential equation, it must be satisfied approximately; i.e, for $x_i, i = 0,1,2,...$

$$E(x_i) = \left|y(x_i) - \tilde{y}(x_i)\right| = \left|y(x_i) - \sum_{k=0}^{N} \frac{(x_i - c)^{k\alpha}}{\Gamma(k\alpha+1)} D_*^{k\alpha} y(x)\Big|_{x=c}\right| \cong 0$$

or

$$E(x_i) = 10^{-j\alpha} \text{ (j positive integer).}$$

If max $10^{-j\alpha}$ (j, any positive integer) is prescribed, then the truncation limit N is increased until the difference $E(x_i)$ at each of the points x_i becomes smaller than the prescribed $10^{-j\alpha}$. The results obtained for N = 4(4)12 using the Generalized Taylor collocation method discussed in section 2 are shown in table 1. For the convergence criteria and error estimates of solutions, see [19,20].

4. NUMERICAL EXAMPLES

To demonstrate the effectiveness of the method, we consider here some linear fractional differential equations, some of which have been considered by other methods.
Example 1: Consider the Bagley–Torvik equation that governs the motion of a rigid plate immersed in a Newtonian fluid [21]

$$a_0 D_*^2 y(x) + a_1 D_*^{3/2} y(x) + a_2 y(x) = f(x)$$

we consider the case $f(x) = a_2(x+1), a_0 = 1, a_1 = 1$ and $a_2 = 1$ with the following initial conditions

$$y(0) = 1, y'(0) = 1.$$

Approximate the solution $y(x)$ by Generalized Taylor series

$$y(x) = \sum_{k=0}^{8} \frac{x^{k\alpha}}{\Gamma(k\alpha+1)} D_*^{k\alpha} y(x)\Big|_{x=0}$$

where α *=1/2.*
Fundamental matrix relation of this equation is

$$(CM_4 + CM_3 + CM_0)A = F$$

and collocation points

$$x_0 = 0,\ x_1 = \frac{1}{3},\ x_2 = \frac{2}{3},\ x_3 = 1.$$

The new augmented matrix of example 1 becomes

$$\begin{bmatrix} 1 & 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 \end{bmatrix}$$

Finally, we find the fourth order approximation solution as

$$y(x) = 1 + x$$

Numerical results will not be presented since the exact solution is evaluated.

Example 2: Consider the problem [22]

$$D_*^{\beta} y(x) = -y(x),\ 1 \le \beta \le 2$$

with initial conditions

$$y(0) = 0, y'(0) = 0$$

which is known to have the exact solution $E_\beta(-x^\beta)$, ($E_\beta(x) = \sum_{k=0}^{\infty} \frac{x^k}{\Gamma(\beta k + 1)}$ Mittag-Leffler Function)

In the interval $0 \le x \le 1$, approximate the solution $y(x)$ by Generalized Taylor series

$$y(x) = \sum_{k=0}^{8} \frac{x^{k\alpha}}{\Gamma(k\alpha + 1)} D_*^{k\alpha} y(x)\Big|_{x=0}$$

where α *=1/2.*

For $\beta = \frac{3}{2}$, Fundamental matrix relation of this equation is

$$(CM_3 - CM_0)A = 0$$

and collocation points

$$x_0 = 0,\ x_1 = \frac{1}{8},\ x_2 = \frac{1}{4},\ x_3 = \frac{3}{8},\ x_4 = \frac{1}{2},\ x_5 = \frac{5}{8},\ x_6 = \frac{3}{4},\ x_7 = \frac{7}{8},\ x_8 = 1.$$

The new augmented matrix of example 2 becomes

$$[W^*;F^*] = \begin{bmatrix} 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & ; & 0 \\ 1 & 0.39891 & 0.125 & 1.0332 & 0.40672 & 0.12666 & 0.033569 & 0.0078719 & 0.0016723 & ; & 0 \\ 1 & 0.56415 & 0.25 & 1.0940 & 0.5954 & 0.2594 & 0.09663 & 0.031922 & 0.0095656 & ; & 0 \\ 1 & 0.69096 & 0.375 & 1.1728 & 0.76127 & 0.40091 & 0.18154 & 0.073086 & 0.026735 & ; & 0 \\ 1 & 0.79782 & 0.5 & 1.2659 & 0.92282 & 0.55319 & 0.28677 & 0.1326 & 0.055795 & ; & 0 \\ 1 & 0.89201 & 0.625 & 1.3717 & 1.0873 & 0.71792 & 0.41236 & 0.2119 & 0.099279 & ; & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & ; & 1 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & ; & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & ; & 0 \end{bmatrix}$$

Thus, the approximate solution is

$$y(x) = 1 + 1.1284x^{(1/2)} + 0.98753x + 0.81043x^{(3/2)} + 0.48018x^2 - 0.090682x^{(5/2)}$$
$$+ 1.1656x^3 - 0.93338x^{(7/2)} + 0.46121x^4$$

Comparison of numerical result with the exact solution is shown in figure 1.

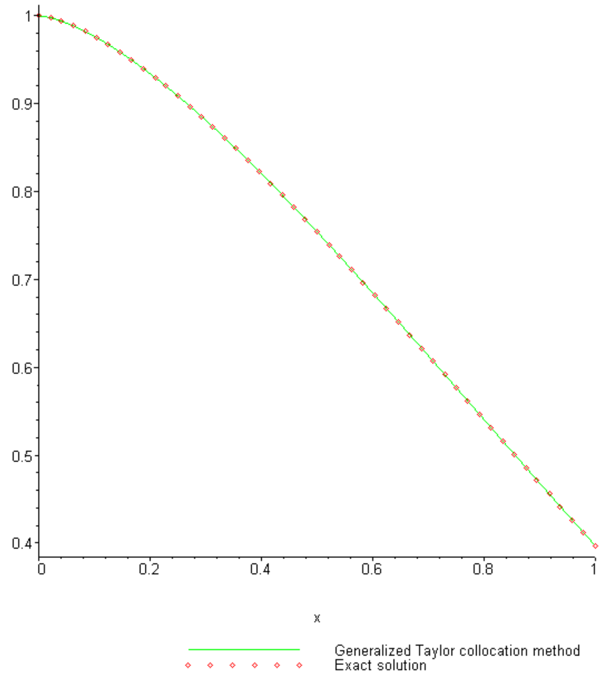

Figure 1. The comparison of the GTCM approximation and the exact solution $E_{\alpha}(-x^{\alpha})$ in the interval $0 \le x \le 1$.

Table 1. Error analysis of Example 2 for the x value

x	Exact Solution	Present Method (c=0)					
		N=4	Error	N=8	Error	N=12	Error
0	1.00000000	1.00000000	0	1.00000000	0	1.00000000	0
0.2	0.93403621	0.93587485	0.00183864	0.93403524	0.97009E-6	0.93403617	0.44805E-7
0.4	0.82005638	0.82232699	0.00227061	0.82005506	0.13247E-5	0.82005632	0.60316E-7
0.6	0.68452989	0.67881003	0.00571985	0.68452840	0.14884E-5	0.68452982	0.67627E-7
0.8	0.54041695	0.51226600	0.02815094	0.54041451	0.24344E-5	0.54041688	0.69751E-7
1.0	0.39662936	0.32670703	0.06992232	0.39660815	0.00002	0.39662929	0.6657E-7

Examples 3:

We consider the composite fractional oscillation equation [2,12]

$$D_*^2 y(x) - A D_*^\beta y(x) - B y(x) = 8, \quad 0 \le x \le 1, \; 0 \le \beta \le 2$$

with the initial conditions

$$y(0) = 0, \; y'(0) = 0$$

Taking $A=-1, B=-1, \alpha=0.5, \beta=1.5, N=16$ and using GTCM. Then, Fundamental matrix relation

$$(CM_4 + CM_3 + CM_0)A = F$$

and $y(x)$ is evaluated as

$$y(x) = 4x^2 - 2.406151x^{5/2} + 1.320610x^3 - 0.615105x^{7/2} - 0.251940x^4 + 0.727149x^{9/2} - 1.011534x^5 + 0.959356x^{11/2} - 0.536680x^6 + 0.081857x^{13/2} + 0.105271x^7 - 0.072321x^{15/2} + 0.015014x^8$$

Numerical results with comparison to Ref. [2, 12] are given in Table 2.

Table 2. The comparison of the GTCM with Adomian decompositon method, fractional differential transform method and the exact solution in the interval $0 \le x \le 1$.

x	$\beta=1.5$			
	y_{ADM}	Y_{FDTM}	y_{GTCM}	y_{Exact}
0.0	0.000000	0.000000	0.000000	0.000000
0.1	0.036478	0.033507	0.033507	0.033507
0.2	0.140640	0.125221	0.125221	0.125221
0.3	0.307485	0.267609	0.267609	0.267609
0.4	0.533284	0.455435	0.455435	0.455435
0.5	0.814757	0.684335	0.684335	0.684335
0.6	1.148840	0.950393	0.950392	0.950392
0.7	1.532571	1.249959	1.249959	1.249959
0.8	1.963033	1.579557	1.579557	1.579557
0.9	2.437331	1.935832	1.935832	1.935832
1.0	2.952567	2.315526	2.315526	2.315526

Comparison of numerical result with the exact solution are showed in figure 2

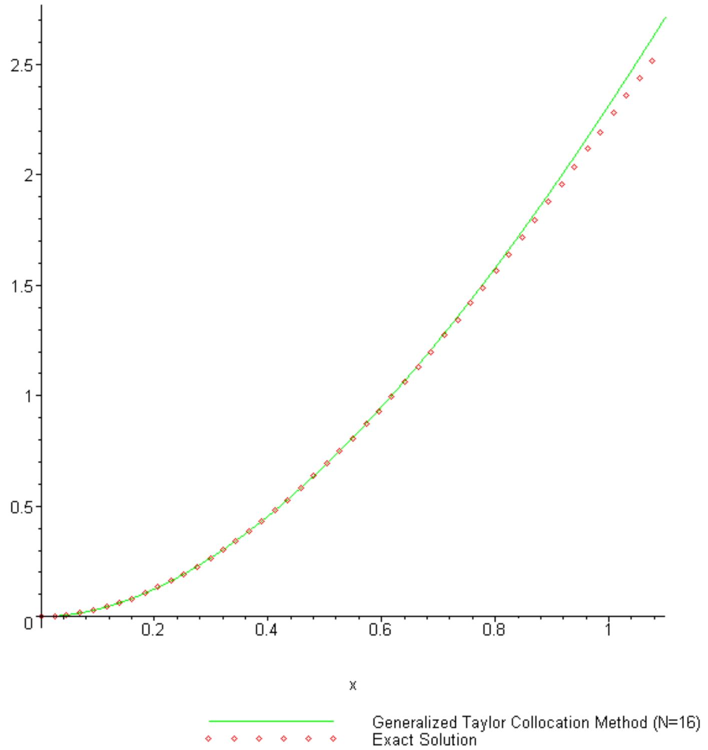

Figure 2. The comparison of the GTCM approximation and the exact solution [2] in the interval $0 \le x \le 1$.

6. CONCLUSION

In general, it is difficult to find the exact solutions of fractional differential equations. So, it is needed to approximate solution methods. In this study, a new generalization of Taylor collocation method is presented for the numerical solutions of linear fractional equations. Mentioned method transforms linear differential equations into a algebraically system which is dependent on collocation points. It is easy to write pc codes which are related to obtained system for necessary computation.

The examples which have exact solutions have been used to show the efficiency of results of method. Graphics and numerical results show that this method is extremely effective and practical for this sort of approximate solutions of differential equations.

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