Steiner distance stable graphs

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Abstract

Let $G$ be a connected graph and $S$ a nonempty set of vertices of $G$. Then the Steiner distance $d_G(S)$ of $S$ is the smallest number of edges in a connected subgraph of $G$ that contains $S$. Let $k, l, s$ and $m$ be nonnegative integers with $m \geq s \geq 2$ and $k$ and $l$ not both 0. Then a connected graph $G$ is said to be $k$-vertex $l$-edge $(s,m)$-Steiner distance stable, if for every set $S$ of $s$ vertices of $G$ with $d_G(S) = m$, and every set $A$ consisting of at most $k$ vertices of $G - S$ and at most $l$ edges of $G$, $d_{G - A}(S) = d_G(S)$. It is shown that if $G$ is $k$-vertex $l$-edge $(s,m)$-Steiner distance stable, then $G$ is $k$-vertex $l$-edge $(s,m+1)$-Steiner distance stable. Further, a $k$-vertex $l$-edge $(s,m)$-Steiner distance stable graph is shown to be a $k$-vertex $l$-edge $(s-1,m)$-Steiner distance stable graph for $s \geq 3$. It is then shown that the converse of neither of these two results holds.

If $G$ is a connected graph and $S$ an independent set of $s$ vertices of $G$ such that $d_G(S) = m$, then $S$ is called an $I(s,m)$-set. A connected graph is $k$-vertex $l$-edge $I(s,m)$-Steiner distance stable if for every $I(s,m)$-set $S$ and every set $A$ of at most $k$ vertices of $G - S$ and $l$-edges of $G$, $d_{G - A}(S) = m$. It is shown that a $k$-vertex $l$-edge $I(3,m)$-Steiner distance stable graph, $m \geq 4$, is $k$-vertex $l$-edge $I(3,m+1)$-Steiner distance stable.

1. Introduction

In [1] a connected graph is defined to be vertex (edge) distance stable if the distance between nonadjacent vertices is unchanged after the deletion of a vertex (edge) of $G$. A more general definition of an equivalent concept was introduced and studied in [4]. It was shown in [4] that a graph is vertex distance stable if and only if it is edge distance stable. We will thus refer to vertex or edge distance stable graphs as distance stable graphs. From a more general result established in [4], it can be deduced that a graph is distance stable if and only if for every pair $u, v$ of nonadjacent vertices, $|N(u) \cap N(v)| = 0$ or $|N(u) \cap N(v)| \geq 2$. Thus a graph $G$ is distance stable if and only if

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distances between pairs of vertices (in $G$) at distance 2 apart remain unchanged after
the deletion of a vertex or an edge.

Further generalizations of distance stable graphs are studied in [6]. In particular for
nonnegative integers $k$ and $l$ not both zero and $D \subseteq N - \{1\}$ a connected graph $G$ is
defined to be $(k, l, D)$-stable if for every pair $u, v$ of vertices of $G$ that are at distance
distance $d_G(u,v) \in D$ apart and every set $A$ consisting of at most $k$ vertices (of $G - \{u, v\}$) and at
most $l$ edges (of $G$), the distance between $u$ and $v$ in $G - A$ equals $d_G(u,v)$. For a positive
integer $m$, let $N_{s,m} = \{x \in N | x \geq m\}$. In [6] it is established that a graph is $(k, l, \{m\})$
-stable if and only if it is $(k, l, N_{s,m})$-stable. It is further shown that for a positive integer
$x$ a graph is $(k+x, l, \{2\})$-stable if and only if it is $(k,l+x, \{2\})$-stable, but that
$(k,l+x, \{m\})$-stable graphs need not be $(k+x, l, \{m\})$-stable for $m \geq 4$. Graph theory
terminality not presented here can be found in [3].

2. Steiner distance stable graphs

Let $G$ be a connected graph and $S$ a nonempty set of vertices of $G$. Then the Steiner
distance $d_G(S)$ of $S$ is the smallest number of edges in a connected subgraph of $G$ that
contains the vertices of $S$. Such a subgraph must necessarily be a tree and is called a
Steiner tree for $S$. The problem of finding the Steiner distance of a set of vertices in
a (weighted) connected graph $G$ has received considerable attention in the literature
(see, for example, [5, 7]).

The concepts of Steiner distance in graphs and distance stable graphs suggest
another generalization of distance stable graphs. For the remainder of this section
we assume that $k, l, s$ and $m$ are nonnegative integers with $m \geq s \geq 2$ and $k$ and $l$
not both zero. If $S$ is a set of $s$ vertices in a connected graph $G$ such that $d_G(S)=m$,
then $S$ is called an $(s,m)$-set. A connected graph $G$ is said to be $k$-vertex $l$-edge
$(s,m)$-Steiner distance stable if, for every $(s,m)$-set $S$ of $G$ and every set $A$
consisting of at most $k$ vertices of $G - S$ and at most $l$ edges of $G$, $d_G(S)=m$.
Thus $k$-vertex $l$-edge $(2,m)$-Steiner distance stable graphs are the $(k,l, \{m\})$-stable graphs.
Note that if $S$ is a set of $s$ vertices such that $d_G(S)=s-1$ then $d_{G-A}(S)=d_G(S)$
for any set $A$ of at most $k$ vertices of $G - S$ and at most $l$ edges of $G$. For this reason we
require that $m \geq s$.

For any integers $k, l, m$ and $s$ with $m \geq s \geq 2$ and $k$ and $l$ not both 0 there exists a $k$-vertex $l$-edge $(s,m)$-Steiner distance stable graph. To see this let $G$ be
obtained from $m-1$ disjoint copies of $K_{k+1}$, say $H_1, ..., H_{m-1}$, by joining
every vertex of $H_i$ to every vertex of $H_{i+1}$ for $1 \leq i < m-1$ and then adding a vertex
$v_0$ and joining it to every vertex of $H_1$ and a vertex $v_m$ and joining it to every vertex
of $H_{m-1}$. It is not difficult to see that $G$ is $k$-vertex $l$-edge $(s,m)$-Steiner distance
stable.

The next result shows that if distances of $(s,m)$-sets in a connected graph are
preserved after the deletion of certain numbers of vertices and edges, then so are
distances preserved for $(s,d)$-sets where $d \geq m$. 
Theorem 1. If a connected graph $G$ is $k$-vertex $l$-edge $(s,m)$-Steiner distance stable, then it is $k$-vertex $l$-edge $(s,m+1)$-Steiner distance stable.

Proof. Let $S = \{u_1, u_2, \ldots, u_n\}$ be an $(s,m+1)$-set and $A$ a set consisting of at most $k$ vertices of $G - S$ and at most $l$ edges of $G$. Let $T$ be a Steiner tree for $S$ in $G$. Since $m > s - 1$, it follows that $T$ contains a vertex that does not belong to $S$. Let $u_i$ be an end-vertex of $T$ and $v$ a vertex of $T - S$ such that $v$ is the only vertex not in $S$ on the (unique) $u_i - v$ path in $T$. Define $T' = \{u_1, u_2, \ldots, u_{i-1}, u_{i+1}, u_{i+2}, \ldots, u_n, v\}$. Then $T - u_i$ is a tree of size $m$ that contains $S$. So $d_G(S') < m$. If $d_G(S') < m$, then there exists a Steiner tree $H$ of size at most $m - 1$ for $S$. Since $u_i$ is adjacent to some vertex of $S'$ (either $v$ or the vertex of $S$ that follows $u_i$ on the $u_i - v$ path in $T$), $d_G(S) < m$. However, $d_G(S) = m + 1$; so $d_G(S') = m$.

Let $A' = (A - \{v\})$. By hypothesis, $d_{G - A'}(S') = m$. Let $T'$ be a Steiner tree for $S'$ in $G - A'$. Note that $T'$ does not contain $u_i$, otherwise $d_G(S) \leq m$. So, since $T'$ has $m + 1$ vertices and $m + 1 > s$, there exists a vertex $u_j \in S - \{u_i\}$ and $v' \in V(T') - S'$ so that $u_j$ is an end-vertex of $T'$ and $v'$ is the only vertex not in $S'$ on the unique $u_j - v'$ path $P'$ in $T'$.

We consider two cases.

Case 1: Suppose that $u_i$ is adjacent in $G$ to a vertex $w$ of $T'$ different from $u_j$. Now let $S'' = \{u_1, u_2, \ldots, u_{i-1}, u_{i+1}, \ldots, u_n, v\}$. Since $T' - u_i$ together with $u_i$ and the edge $u_iw$ produces a tree of size $m$ that contains $S''$, it follows that $d_G(S'') \leq m$. Note that the vertex following $u_j$ on the $u_j - v'$ path $P'$ belongs to $S''$, call it $x$. If $d_G(S'') < m$, then $d_G(S) \leq m$. Hence $d_G(S'') = m$. Since $S'' \cap A = \emptyset$, the hypothesis implies that $d_{G - A}(S'') = m$. Let $T''$ be a Steiner tree for $S''$ in $G - A$. We have already observed that $u_j$ is adjacent with some vertex $x$ of $T''$. So $T''$ together with $u_j$ and the edge $u_jx$ produces a tree of size $m + 1$ in $G - A$ that contains $S$.

Case 2: Suppose that the only vertex of $T'$ to which $u_i$ is adjacent is $u_j$. Then there is some $u_n \in S$ that is an end-vertex of $T'$ where $u_n \neq u_j$. Otherwise $T'$ is a path with $u_j$ and $v$ as end vertices. However, then $T' - v$ together with $u_i$ and the edge $u_1u_j$ produces a tree of size $m$ that contains $S$.

Case 2.1: Suppose that there exists an end-vertex $u_n$ of $T'$ such that $u_n \neq u_j$ and such that the vertex $w$ adjacent with $u_n$ is not $v$. If $w = v'$ or $w \in S$, let $S'' = \{u_1, u_2, \ldots, u_{n-1}, u_{n+1}, \ldots, u_n, v\}$; otherwise let $S'' = \{u_1, u_2, \ldots, u_{n-1}, u_{n+1}, \ldots, u_n, w\}$. In either case $T' - u_n$ together with $u_i$ and the edge $u_1u_j$ produces a tree of size $m$ that contains $S''$. So $d_G(S'') \leq m$. Note that if $T_{S''}$ is any Steiner tree for $S''$ (in either one of the two cases), then $u_n$ is adjacent with some vertex of $T_{S''}$, since $u_n$ is adjacent with a vertex of $S''$. Therefore $d_G(S') \geq m$, that is, $d_G(S') = m$. Since $A \cap S'' = \emptyset$, it follows from the hypothesis that $d_{G - A}(S') = d_G(S') = m$. Let $T''$ be any Steiner tree for $S''$ in $G - A$. Since $u_n$ is adjacent to a vertex of $S''$ a Steiner tree of size $m + 1$ for $S$ can now be produced in $G - A$.

Case 2.2: Suppose that every end-vertex different from $u_j$ in $T'$ is adjacent with $v$. Let $u_n$ be an end-vertex of $T'$ that is adjacent with $v$. Observe that if $U = \{u_1, u_2, \ldots, u_{n-1}, u_{n+1}, \ldots, u_n, v\}$, then $d_G(U) = m$. Let $A'' = (A - \{v\}) \cup \{u_n\}$ if $v \in A$ and $A'' = A$, otherwise. By hypothesis, $d_{G - A''}(U) = m$. Let $T_U$ be a Steiner tree for $U$ in
Note that $T_U$ does not contain $u_r$. So since $T_U$ has $m+1$ vertices and $m+1 > s$, there exist vertices $u_r \in S - \{u_s\}$ and $v'' \in V(T_U) - U$, so that $u_r$ is an end-vertex of $T_U$ and $v''$ is the only vertex not in $U$ on the unique $u_r-v''$ path $P''$ in $T_U$. Observe that $u_r$ is adjacent to a vertex of $T_U$ other than $u_s$, since $u_r$ is adjacent with $v$. We now proceed as in Case 1 to show that $d_G(A(S)) = m+1$, in this case also.

**Corollary 1.** If a connected graph is $k$-vertex $l$-edge $(s, m)$-Steiner distance stable, then it is $k$-vertex $l$-edge $(s, n)$-Steiner distance stable for all $n \geq m$.

The next theorem implies another result of this type.

**Theorem 2.** If a connected graph $G$ is $k$-vertex $l$-edge $(s, m)$-Steiner distance stable, $m > s \geq 3$, then $G$ is $k$-vertex $l$-edge $(s-1, m)$-Steiner distance stable.

**Proof.** Let $S$ be a set of $s-1$ vertices in $G$ such that $d_G(S) = m$. Suppose that $T_S$ is a Steiner tree for $S$. Let $v \in V(T_S) - S$ and define $S' = S \cup \{v\}$. Then $d_G(S') = m$. Let $A$ be any set of at most $k$ vertices of $G - S$ and at most $l$ edges of $G$. Let $A' = A - \{v\}$. By the hypothesis $d_G(A'(S')) = d_G(S') = m$. Let $T_{S'}$ be a Steiner tree for $S'$ in $G - A'$ and let $v' \in V(T_{S'} - S')$. Such a vertex exists since $T_{S'}$ has $m+1$ (>s) vertices. Now let $S'' = S \cup \{v'\}$ and observe that $S'' \cap A = \emptyset$. Further $d_G(S'') = m$ and $|S''| = s$. So by the hypothesis $d_G(A''(S'')) = d_G(S'') = m$. If $T_{S''}$ is a Steiner tree for $S''$ in $G - A$, then $T_{S''}$ is also a Steiner tree for $S$ in $G - A$, that is, $d_G(A(S)) = d_G(S) = m$.

**Corollary 2.** If a connected graph is $k$-vertex $l$-edge $(s, m)$-Steiner distance stable, $m \geq s \geq 3$, then it is $k$-vertex $l$-edge $(s', m)$-Steiner distance stable for all $s'$ ($2 \leq s' \leq s$).

In Theorem 1 we saw that the condition that a connected graph is $k$-vertex $l$-edge $(s, m)$-Steiner distance stable is sufficient for the graph to be $k$-vertex $l$-edge $(s, m+1)$-Steiner distance stable. The next result shows that this condition in not necessary.

**Theorem 3.** For any integers $k, s$ and $m$ such that $s \geq 2$, $2s - 2 \geq m \geq s$ and $k \geq 1$, there exists a graph $G$ which is $k$-vertex $0$-edge $(s, m+1)$-Steiner distance stable, but not $k$-vertex $0$-edge $(s, m)$-Steiner distance stable.

**Proof.** Let $r = s + (m-s+3)k+k-1$. Define $G$ to be the $k$th power $C_r^k$ of the cycle $C_r$. That is, $G$ is obtained from $C_r$ by joining every vertex $u$ on $C_r$ to every vertex $v$ on $C_r$ with $d_G(u, v) \leq k$. Suppose $C_r = v_0, v_1, \ldots, v_{r-1}, v_0$.

We show first that $G$ is $k$-vertex $0$-edge $(s, m+1)$-Steiner distance stable. Let $S$ be a set of $s$ vertices of $G$ such that $d_G(S) = m+1$. We show that the subgraph $\langle S \rangle_G$ induced by $S$ contains at least $m-s+3$ components. Suppose that $\langle S \rangle_G$ has $m-s+i$ components where $i \geq 4$. Then $G$ has at least $(m-s+i)k+s$ vertices. But $(m-s+i)k+s \geq s+(m-s+3)k+k$, contrary to the assumption that $G$ has $s+(m-s+3)k+k-1$ vertices. However, since $G$ is a power of a cycle, each vertex in
a Steiner tree $T_S$ for $S$ that does not belong to $S$ has degree 2 in $T_S$ and is joined to one vertex from each of two components of $\langle S \rangle_G$. Hence $\langle S \rangle_G$ has at most $m-s+3$ components, otherwise $d_G(S) > m+1$. Thus $\langle S \rangle_G$ has exactly $m-s+3$ components $G_1, G_2, \ldots, G_{m-s+3}$. Since $G$ is a power of a cycle, each $G_i$ has a spanning path. Let $p_i$ be the order of $G_i$ for $1 \leq i \leq m-s+3$. Let $v_{i,1}, v_{i,2}, \ldots, v_{i, p_i}$ denote the order of the vertices of $G_i$ as they appear on $C_r$ as we proceed in clockwise order about $C_r$. We may assume that $G_1, G_2, \ldots, G_{m-s+3}$ are labelled so that $v_{i, p_i}$ precedes $v_{i+1,1}$ on $C_r$ as we proceed in clockwise order about $C_r$. For $1 \leq i \leq m-s+3$, $G_i$ has at most $m-s+3$ components $\langle G_i \rangle_G = G_i, G_1, \ldots, G_{m-s+3}$. Since $G$ is a power of a cycle, each $G_i$ has a spanning path. Let $p_i$ be the order of $G_i$ for $1 \leq i \leq m-s+3$. Let $v_{i,1}, v_{i,2}, \ldots, v_{i, p_i}$ denote the order of the vertices of $G_i$ as they appear on $C_r$ as we proceed in clockwise order about $C_r$. We may assume that $G_1, G_2, \ldots, G_{m-s+3}$ are labelled so that $v_{i, p_i}$ precedes $v_{i+1,1}$ on $C_r$ as we proceed in clockwise order about $C_r$.

We show next that $G$ is not $k$-vertex 0-edge $(s, m)$-Steiner distance stable.

Let $S = \{v_0, v_{(k+1)}, v_{2(k+1)}, \ldots, v_{(m-s+1)(k+1)}, v_{(m-s+1)(k+1)+1}, \ldots, v_{(s+1)(k+1)} \}$. Then $|S| = s$, $d_G(S) = m + 1 > d_G(S)$. So $G$ is not $k$-vertex 0-edge $(s, m)$-Steiner distance stable.

If we let $m = s-1$ in the construction of the proof of Theorem 3, we obtain a graph that is $k$-vertex 0-edge $(s, s)$-Steiner distance stable and not $k$-vertex 0-edge $(s-1, s-1)$-Steiner distance stable.

We now show that the converse of Theorem 2 does not hold.

**Theorem 4.** For $s \geq 3$ there is a graph which is 1-vertex 0-edge $(s-1, s-1)$-Steiner distance stable but not 1-vertex 0-edge $(s, s)$-Steiner distance stable.

**Proof.** Let $G_1 \cong K_{1,s}$ where $V(G_1) = \{x, v_1, v_2, \ldots, v_s\}$ and $x$ has degree $s$ in $G_1$. Let $G_2 \cong K_4$ with $V(G_2) = \{w_1, w_2, w_3, w_4\}$. Define $G$ to be the graph obtained from $G_1 \cup G_2$ by adding the edges in $\{v_i w_j|1 \leq i, j \leq s, i \neq j\}$. Then $|S| = s$, $d_G(S) = m$. If $A = \{v_1, v_2, \ldots, v_s\}$, then $d_G(A) = m + 1 > d_G(S)$. So $G$ is not 1-vertex 0-edge $(s, s)$-Steiner distance stable.

Suppose now that $S'$ is any $(s-1, s-1)$-set. Then $\langle S' \rangle_G$ is not connected. If $|S' \cap V(G_2)| \geq 2$, then $|S' \cap \{v_1, v_2, \ldots, v_s\}| = 0$. Moreover, $x \in S'$. Let $S' = \{x, w_3, \ldots, w_s\}$. After deleting any vertex $u$ of $G$ not in $S'$, either $v_1$ or $v_2$ still belongs to the resulting graph. Thus in this case $d_{G-u}(S') = d_G(S') = s-1$ for all $(s-1, s-1)$-sets $S'$ and for all $u \in V(G) - S'$.

If $|S' \cap V(G_2)| = 1$, say $w_1 \in S'$ and $x \in S'$, then necessarily $S' \cap \{v_1, v_2, \ldots, v_s\} \subseteq \{v_1\}$. After the deletion of any vertex $u$ from $G$ that does not belong to $S'$, either $v_2$ or $v_3$ still belongs to the resulting graph, which implies that $d_{G-u}(S') = d_G(S') = s-1$ in this case also.
If \( |S' \cap V(G_2)| = 1 \), say \( w_1 \in S' \) and \( u \notin S' \), then \( v_1 \in S' \). After deleting any vertex \( u \in V(G_1) - S' \) from \( G_2 \) a graph results that contains either \( w_2 \) or \( w_3 \). So \( d_{G_u}(S') = d_G(S') \) in this case as well.

Finally suppose \( S' \subseteq \{v_1, v_2, \ldots, v_n\} \). Then we may assume \( v_1 \notin S' \). After deleting any vertex \( u \in V(G) - S' \) from \( G_2 \), a graph results that contains either \( w_1 \) or \( x \). So in this case \( d_{G_u}(S') = d_G(S') \).

Since the graph \( G \) of the proof of Theorem 4 is 1-vertex 0-edge \((s-1, s-1)\)-Steiner distance stable, it follows, by Theorem 1, that \( G \) is 1-vertex 0-edge \((s-1, s)\)-Steiner distance stable. Since \( G \) is not 1-vertex 0-edge \((s, s)\)-Steiner distance stable, it follows that the converse of Theorem 2 does not hold in general.

Recall it was shown in [4], for a positive integer \( k \), that a graph is \((k, 0, \{2\})\)-stable if and only if it is \((0, k, \{2\})\)-stable. So a graph is \( k \)-vertex 0-edge \((2, 2)\)-Steiner distance stable if and only if it is \( 0 \)-vertex \( k \)-edge \((2, 2)\)-Steiner distance stable. The next result shows that the necessity of this condition has an extension to \((3, 3)\)-sets.

**Theorem 5.** For a positive integer \( k \), a graph \( G \) is \( k \)-vertex 0-edge \((3, 3)\)-Steiner distance stable if it is \( 0 \)-vertex \( k \)-edge \((3, 3)\)-Steiner distance stable.

**Proof.** Suppose \( G \) is not \( k \)-vertex 0-edge \((3, 3)\)-Steiner distance stable. Let \( S \) be a \((3, 3)\)-set for which there exists a set \( A \) of at most \( k \) vertices in \( G - S \) such that \( d_{G - A}(S) > d_G(S) = 3 \). Let \( A = \{v_1, v_2, \ldots, v_l\} \) be a minimal such set. Observe that \( \langle S \rangle_G \) is not connected, since \( d_G(S) = 3 \). Let \( u \) be a vertex of \( S \) that is not adjacent to either of the remaining two vertices of \( S \).

Let \( A_i = A - \{v_i\} \) for \( 1 \leq i \leq l \). By our choice of \( A \), \( d_{G - A_i}(S) = d_G(S) = 3 \). So necessarily \( uw_i \in E(G) \) for all \( i \). Moreover, every Steiner tree for \( S \) in \( G \) must contain one of the edges in \( B = \{w_1, w_2, \ldots, w_l\} \). Thus \( d_{G - B}(S) > d_G(S) \), implying that \( G \) is not \( 0 \)-vertex \( l \)-edge \((3, 3)\)-Steiner distance stable. \( \Box \)

The converse of Theorem 5 does not hold. Let \( H_1 \cong K_{s-1} - uv \) for some pair \( u, v \) of vertices and \( H_2 \cong K_2 \) where \( V(H_2) = \{x, y\} \). Let \( G \) be obtained from \( H_1 \cup H_2 \) by adding the edges \( ux \) and \( vy \). Then \( G \) is 1-vertex 0-edge \((s, s)\)-Steiner distance stable, but \( G \) is not 0-vertex 1-edge \((s, s)\)-Steiner distance stable. To see this let \( z \in V(H_1) - \{u, v\} \). Then \( d_G(\{x, y, z\}) = 3 \) but \( d_{G - xy}(\{x, y, z\}) = 4 \).

Theorem 5 cannot be extended to \((s, s)\)-sets for \( s \geq 4 \). To see this, let \( G \cong (K_2 \cup K_{s-2}) + K_1 \), i.e., \( G \) is obtained by joining a new vertex to every vertex in disjoint copies of \( K_2 \) and \( K_{s-2} \). Then \( G \) is 0-vertex 1-edge \((s, s)\)-Steiner distance stable but \( G \) is not 1-vertex 0-edge \((s, s)\)-Steiner distance stable.

### 3. Independent Steiner distance stable graphs

In this section we focus our attention on independent sets of vertices of a graph. Our first result shows that in a certain sense the problem of finding Steiner trees for sets of
independent vertices is equivalent to the problem of finding the Steiner trees of sets of vertices that are not necessarily independent. Let \( \Pi_1 \) be the problem of finding a Steiner tree for a nonempty set of vertices of a connected graph and \( \Pi_2 \) the problem of finding a Steiner tree for a nonempty independent set of vertices of a connected graph. Let \( G \) be a connected graph and \( S \) a nonempty set of vertices of \( G \). Suppose \( G_1, G_2, \ldots, G_n \) are the components of \( \langle S \rangle_G \). Let \( R(G; S) \) be the graph with vertex set \((V(G) - S) \cup \{v_1, v_2, \ldots, v_n\}\) (where \( v_i \) corresponds to \( G_i, 1 \leq i \leq n \)) and edge set \( \{uw | u \in E(G - S)\} \cup \{uv_i | u \in V(G - S) \) and \( u \) is adjacent in \( G \) to some vertex of \( G_i \}). Thus \( R(G; S) \) is the contraction of \( G \) that results from the partition \((\bigcup_{i=1}^{n} V(G_i)) \cup \{u \in V(G - S)\}\).

**Theorem 6.** There is an (efficient) algorithm that solves \( \Pi_1 \) is and only if there is an (efficient) algorithm that solves \( \Pi_2 \).

**Proof.** Clearly if there is an algorithm that solves \( \Pi_1 \) then in particular such an algorithm finds a Steiner tree for an independent set of vertices. So there is an algorithm for solving \( \Pi_2 \).

Suppose now that there is an algorithm for solving \( \Pi_2 \). Let \( G \) be a connected graph and \( S \) a nonempty set of vertices. Suppose \( G_1, G_2, \ldots, G_n \) are the components of \( \langle S \rangle_G \). Let \( v_i \) correspond to \( G_i \) in \( R(G; S) \). Observe that \( \{v_1, v_2, \ldots, v_n\} \) is an independent set of vertices in \( R(G; S) \). So by applying the algorithm that solves \( \Pi_2 \) to \( R(G; S) \) we obtain a Steiner tree \( T \) for \( \{v_1, v_2, \ldots, v_n\} \) in \( R(G; S) \). Let \( T' \) be the tree obtained from \( T \) by replacing each \( v_i \) with a spanning tree of \( G_i \) (1 \( \leq i \leq n \)) and every edge of the type \( u \) where \( u \in V(G - S) \) with an edge of \( G \) that joins u to some vertex of \( G_i \). Then \( T' \) is a tree that contains \( S \). So \( |E(T')| \geq d_G(S) \). We show next that \( T' \) is a Steiner tree for \( S \). Let \( T_S \) be a Steiner tree for \( S \) such that the number of components \( m \) of \( \langle S \rangle_{T_S} \) is a minimum. Clearly \( m \geq n \). We show that \( m = n \) by showing that \( \langle V(G_i) \rangle_{T_S} \) is connected for each \( i \) (1 \( \leq i \leq n \)). Suppose that \( \langle V(G_i) \rangle_{T_S} \) contains at least two components. Let \( T_1 \) and \( T_2 \) be two such components such that an edge \( e \) of \( G_i \) (that does not belong to \( T_S \)) joins a vertex \( x \) of \( T_1 \) to a vertex \( y \) of \( T_2 \). Since \( x \) and \( y \) belong to \( S \), \( T_S \) contains an \( x-y \) path \( P \). So \( P \) together with the edge \( e \) produces a cycle \( C \). This cycle must contain a vertex and hence an edge that does not belong to any \( G_j \) (1 \( \leq j \leq n \)). Let \( f \) be such an edge. Then \( T'_S = T_S + xy - f \) is a Steiner tree for \( S \) such that \( \langle S \rangle_{T'_S} \) has fewer components than \( \langle S \rangle_{T_S} \). This contradicts our assumption. Thus \( m = n \) and \( |E(\langle V(G_i) \rangle_{T_S})| = |V(G_i)| - 1 \) for 1 \( \leq i \leq n \). Since \( R(T_S; S) \) is a tree of \( R(G; S) \) that contains \( S \), it follows that \( |E(R(T_S; S))| \geq |E(T)| \). Thus \( |E(T')| = |E(T)| + \sum_{i=1}^{n} (|V(G_i)| - 1) \leq |E(R(T_S; S))| + \sum_{i=1}^{n} (|V(G_i)| - 1) = |E(T_S)| = d_G(S) \). Thus by an earlier remark \( |E(T')| = d_G(S) \).  

Theorem 6 also follows directly from the nearest vertex reduction test described by Beasley [2].

The concepts presented in Section 2 and Theorem 6 suggest the next topic. If \( G \) is a connected graph and \( S \) an independent set of \( s \) vertices of \( G \) such that \( d_G(S) = m \), then
S is called an $Z(s, m)$-set. A connected graph is defined to be $k$-vertex $l$-edge $Z(s, m)$-Steiner distance stable if, for every $Z(s, m)$-set $S$ and every set $A$ of at most $k$ vertices of $G - S$ and at most $l$ edges of $G$, $d_{G - A}(S) = m$. The next result establishes an analogue of Theorem 1 with respect to $I(3, m)$-sets.

**Theorem 7.** If $G$ is a $k$-vertex $l$-edge $I(3, m)$-Steiner distance stable graph with $m \geq 4$, then $G$ is a $k$-vertex $l$-edge $I(3, m + 1)$-Steiner distance stable graph.

**Proof.** Let $U = \{u_1, u_2, u_3\}$ be an $I(3, m + 1)$-set and let $A$ be a set of at most $k$ vertices of $G - S$ and at most $l$ edges of $G$. Let $T$ be a Steiner tree for $U$. Since $m \geq 4$ there exists an end-vertex $u_i$ of $T$ such that $d_G(u_i, u_j) \geq 3$ for all $u_j \in U - \{u_i\}$. Suppose $u_i = u_1$. Let $v$ be the vertex adjacent with $u_1$ in $T$ and let $U' = (U - \{u_1\}) \cup \{v\}$. Then $U'$ is an $I(3, m)$-set. Let $A' = A - v$. Then $d_{G - A}(U') = m$. We now consider two cases.

**Case 1:** Whenever $T'$ is a Steiner tree for $U'$ in $G - A'$, then $N_T(u_2) \cap N_T(u_3) = \emptyset$. Let $T'$ be a Steiner tree for $U'$ in $G - A'$ and $v'$ a vertex adjacent with $u_2$ in $T'$. Then $U'' = \{u_1, u_3, v'\}$ is an $I(3, m)$-set. Since $v' \notin A$, it follows that there is a Steiner tree of size $m$ for $U''$ in $G - A$. Such a tree together with $u_2$ and the edge $u_2v'$ produces a Steiner tree for $U$ of size $m + 1$ in $G - A$.

**Case 2:** There exists a Steiner tree $T'$ for $U'$ in $G - A'$ such that $N_T(u_2) \cap N_T(u_3) \neq \emptyset$. Let $y \in N_T(u_2) \cap N_T(u_3)$. Observe, in this case since $d_G(U) = m + 1 \geq 5$, that $d_G(u_1, u_j) \geq m - 1$ for $j = 2, 3$. Note further that $d_G(u_1, u_2)$ and $d_G(u_1, u_3)$ cannot both be $m + 1$. So suppose $d_G(u_1, u_2) \leq m$.

**Case 2.1:** Suppose $d_G(u_1, u_2) = m$. Let $z$ be the vertex adjacent with $v$ in $T'$. Then $U_1 = \{u_1, z, u_2\}$ is an $I(3, m)$-set. Note that $z \notin A$ since $z$ belongs to $T' - v$. So there exists a Steiner tree $T_1$ of size $m$ for $U_1$ in $G - A$. Since $d_G(u_1, u_2) = m$, the tree $T_1$ is necessarily a path. Further, $d_T(u_1, z) = 2$. Let $v'$ be a vertex that is adjacent with both $u_1$ and $z$ in $T_1$. Then $T_1 - v$ together with $u_1$ and $v'$ and the edges $u_1v'$ and $v'z$ produces a Steiner tree of size $m + 1$ in $G - A$ for $U$.

**Case 2.2:** Suppose $d_G(u_1, u_2) = m - 1$. Let $w$ be a vertex that is adjacent with $u_2$ on a shortest $u_1 - u_2$ path. Let $A' = A - \{w\}$. Let $U_2 = \{u_1, w, u_3\}$. Then $d_G(U_2) \leq m + 1$. If $d_G(U_2) = m$, then there exists a Steiner tree $T_2$ of size $m$ for $U_2$ in $G - A'$. Let $v'$ be adjacent with $u_1$ in $T_2$. Then $v' \notin A$. Note that neither $u_3$ nor $u_2$ is adjacent with $v'$. Also a shortest $v' - u_2$ path together with the path $u_2, y, u_3$ produces a connected graph of size at most $m$ that contains $U_3 = \{u_3, u_2, v'\}$. So $d_G(U_3) \leq m$. However, $d_G(U_3) \neq m$ otherwise $d_G(U) \leq m$. Let $T_3$ be a Steiner tree for $U_3$ of size $m$ in $G - A$. Then $T_3$ together with $u_1$ and the edge $u_1v'$ is a Steiner tree of size $m + 1$ for $U$ in $G - A$.

Suppose now that $d_G(U_3) = m + 1$. Then $uw_3$ and $wy$ are not edges of $G$. Since a shortest $u_1 - w$ path together with the path $w, u_2, y$ produces a connected graph of size at most $m$ that contains $U_4 = \{u_1, w, y\}$, it follows that $d_G(U_4) \leq m$. However, $d_G(U_4) \neq m$, otherwise it follows, since $u_3$ is adjacent with $y$, that $d_G(U_2) \leq m$. So $U_4$ is an $I(3, m)$-set. Let $T_4$ be a Steiner tree of size $m$ in $G - A'$. If $w$ is an end-vertex of $T_4$, then $T_4 - w$ together with $u_2$ and $u_3$ and the edges $u_2y$ and $u_3y$ produces a Steiner tree of size $m + 1$ for $U$ in $G - A$. 

Suppose thus that $T_n$ is a path that contains $w$ as an internal vertex. Then the $u_1\rightarrow w$ path of $T_n$ has length $m - 2$. Let $v'$ be the vertex adjacent with $u_1$ on $T_4$. Then $U_5 = \{v', u_2, u_3\}$ is an $I(3, m)$-set and $v' \notin A$. So there exists a Steiner tree $T_5$ of size $m$ for $U_5$ in $G - A$. Thus $T_5$ together with $u_1$ and the edge $u_1v'$ produces a Steiner tree for $U$ in $G - A$. □

It remains an open problem to determine if a $k$-vertex $l$-edge $I(s, m)$-Steiner distance stable graph $m \geq 4$, is a $k$-vertex $l$-edge $I(s, m + 1)$-Steiner distance stable graph where $s \geq 4$.

In closing we wish to remark that it is our belief that the problem of determining whether a graph $G$ is $k$-vertex $l$-edge $(s, m)$-Steiner distance stable (for $m \geq s \geq 3$ and $k$ and $s$ nonnegative integers not both 0) is NP-hard.

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References