Analysis of Unstable Operation in a Basic Delta Modulator for PWM Control

Hiroshi SHIMAZU†, Student Member and Toshimichi SAITO†, Member

SUMMARY This paper studies dynamics of a delta modulator for PWM control. In order to analyze the circuit dynamics we derive a one-dimensional return map of switching time. The map is equivalent to a circle map in wide parameter region and its nonperiodic behavior corresponds to undesired asynchronous operation of the circuit. We then present a simple stabilization method of the system operations by means of periodic compulsory switching. The mechanism of the stabilization is considered from viewpoints of bifurcation. Using a simple test circuit, typical operations are confirmed experimentally.

key words: pulse-width modulation, delta modulators, synchronization, circle map, bifurcation

1. Introduction

The pulse-width modulation (ab. PWM) technique is a common method in static inverters whose applications include ac machine drivers and uninterruptable power supply systems [1]. The delta modulator (ab. DM) is known as a basic scheme for an independent voltage-mode PWM circuit [2]–[5]. As compared with other schemes such as sine-triangle and space-vector modulators [6]–[8], the DM has several advantages and disadvantages: it is relatively easy to implement and provides good attenuation of low-order harmonics, however, it has asynchronous and unstable operations [4]. Although steady state of the DMs has been analyzed for improving the disadvantages, the nonlinear dynamics has not been analyzed sufficiently [5]. On the other hand the DMs can be regarded as switched dynamical systems having rich nonlinear phenomena by interaction between continuous and discrete states [9]–[11].

This paper studies the asynchronous operations of the DM from viewpoints of switched dynamical systems. First, we present an equivalent circuit of the DM and derive the dimensionless circuit equation so as to extract essential parameters. Second, we derive a 1-D return map from the circuit equation. In wide parameter range the return map is to be homeomorphism on the unit circle (so called circle map [12]) that is characterized by the rotation number [12]. If the rotation number is irrational, the map exhibits non-periodic orbit: it is the reason of the unstable operation. Third, we apply periodic and compulsory switching to the DM in order to suppress the unstable operations. In this case we clarify that the system can exhibit periodic orbits in wide parameter region used for practical PWM control: the compulsory switching can stabilize the PWM operations. The mechanism of the stabilization is considered from viewpoints of bifurcation. Using a simple test circuit, typical phenomena can be verified in the laboratory. These results provide useful information for understanding dynamics of the DMs and for improving disadvantages of the DM. The preliminary results along these lines can be found in our conference papers [13], [14].

2. The Delta Modulator

Figure 1 shows an equivalent circuit of the delta modulator that can be used as an independent modulator for voltage-mode PWM control. It consists of one capacitor, one nonlinear voltage-controlled current source (ab. VCCS), and one linear current-controlled voltage source (ab. CCVS). The VCCS has time-variant hysteresis characteristics:

\[
I_o(v, t) = \begin{cases} I_s, & \text{for } v \leq V_U(t) \\ -I_s, & \text{for } v \geq V_L(t), \end{cases}
\]

\[
V_U(t) = V_m \sin \omega t + V_b, \quad V_L(t) = V_m \sin \omega t - V_b,
\]

where \(V_U\) and \(V_L\) are sinusoidal threshold signals with dc

![Fig. 1 The equivalent circuit of the delta modulator.](image)
component \( V_b \) and \(-V_b \), respectively \((V_b > 0)\). The output of this hysteresis VCCS is switched from \( I_s \) to \(-I_s \) (respectively, from \(-I_s \) to \( I_s \)) if the capacitor voltage \( v \) reaches the upper threshold \( V_U(t) \) (respectively, the lower threshold \( V_L(t) \)) holding the continuity property of the capacitor voltage \( v \). The circuit dynamics is described by Eq. (2).

\[
C \frac{d}{dt} v = I_o(v,t), \quad V_o = R_o I_o(v,t),
\]

where \( R_o \) is the characteristic resistance of the CCVS. The capacitor voltage \( v \) and sinusoidal signal \( V_m \sin \omega t \) correspond to carrier wave and modulating signal of PWM, respectively. The output voltage \( V_o \) corresponds to controlling signal in the inverters \([1],[5]\). Typical waveforms of \( v \) and \( V_o \) are illustrated in Fig. 1. A simple implementation circuit and basic laboratory measurements can be found in \([3],[4]\). If the parameters are selected suitably the DM can generate synchronous PWM signal, otherwise the DM exhibits asynchronous behavior. It should be pointed out that, although this work discusses only the single-phase case, the discussion can be developed into three-phase cases.

In order to normalize the circuit equation, we define the dimensionless variables and parameters:

\[
\tau = \omega t, \quad x = \frac{v}{V_m}, \quad y = \frac{V_o}{R_o I_s}, \quad a = \frac{I_s}{\omega CV_m}, \quad d = \frac{V_b}{V_m}.
\]

Using these, we obtain the dimensionless form:

\[
\frac{d}{d\tau} x = ah(x, \tau), \quad y = h(x, \tau) = \begin{cases} 1 & \text{for } x \leq X_U(\tau) \\ -1 & \text{for } x \geq X_L(\tau), \end{cases}
\]

where

\[
X_U(\tau) = \sin \tau + d, \quad X_L(\tau) = \sin \tau - d
\]

\[
h(x, \tau) \equiv \frac{1}{I_s} I_o(V_m x, \tau) = \frac{1}{a} X_U(x, \tau) - X_L(x, \tau).
\]

Equation (4) is characterized by two positive parameters \( a \) and \( d \) which can control the slope of the orbit and hysteresis band width, respectively. It should be noted that the original six parameters \((V_m, \omega, V_o, I_s, C, R_o)\) are integrated into these two parameters \( a \) and \( d \). For example, effect of the modulating angular frequency \( \omega \) is included into parameter \( a \). Since the normalized capacitor voltage \( x \) vibrates between \( X_U(t) \) and \( X_L(t) \) as shown in Fig. 1, we can use the following initial condition for calculation of waveforms.

\[
X_L(0) \leq x(0) \leq X_U(0), \quad h(x(0), 0) \in [-1, 1].
\]

3. Switching Time Map and Return Map

In order to analyze the dynamics of Eq. (4), we derive switching time map and return map. As shown in Fig. 1, let \( \tau_n \) be the \( n \)-th switching time at the lower threshold \( X_L \) and let \( \tau'_n \) be the successive switching time at the upper threshold \( X_U \). \( \tau_n \) and \( \tau'_n \) are the \( n \)-th pulse beginning and termination times of the output \( V_o \), respectively. An orbit started from \((x, \tau) = (X_L(\tau_n), \tau_n)\) increases with slope \( a \) and hits \( X_U \) at time \( \tau'_n \). Then the orbit decreases with slope \(-a\), hits the lower threshold \( X_L \) at time \( \tau_n \) and the system repeats in this manner. Since \( \tau_{n+1} \) is determined by \( \tau_n \), we can define the 1-D map \( F \) from positive reals \( R \) to itself.

\[
\tau_{n+1} = F(\tau_n), \quad F(\tau_n + 2\pi) = F(\tau_n) + 2\pi
\]

Let the initial switching time be denoted by \( \tau_0 \). This map is referred to as the switching time map. Figure 2 shows typical shapes of this map. The switching time map \( \tau_{n+1} = F(\tau_n) \) can be calculated by the following implicit equation:

\[
\begin{align*}
X_U(\tau'_n) - X_L(\tau_n+1) &= a(\tau_n+1 - \tau'_n) \\
X_U(\tau'_n) - X_L(\tau_n) &= a(\tau_n - \tau'_n) \\
\sin \tau'_n - \sin \tau_n + 2d &= a(\tau_n+1 - \tau'_n) \\
\sin \tau'_n - \sin \tau_n + 2d &= a(\tau_n - \tau'_n).
\end{align*}
\]

where the minimum roots are adopted as \( \tau_{n+1} \) and \( \tau'_n \). The slope of \( F \) can be calculated by Eq. (8).

\[
DF(\tau_n) \equiv \frac{d\tau_{n+1}}{d\tau_n} = \frac{a + \cos \tau'_n}{a + \cos \tau_n} \cdot \frac{a - \cos \tau_n}{a - \cos \tau'_n}
\]

Noting \(|\cos \tau| \leq 1\), we can say the important characteristics:

- \( F \) is monotone increasing for \( 1 < a \).
- \( F \) can have extremum and discontinuity points for \( 0 < a < 1 \).

Hereafter we consider the following parameter region for simplicity.

\[
1 < a, \quad 0 < d < \frac{\pi}{2}
\]

This condition guarantees two or more switchings within one period \( (\tau_{n+1} - \tau_n < 2\pi) \) and monotone-ness of \( F \). If \( 0 < a < 1 \) and/or \( \frac{\pi}{2} < d \) then the system can exhibit complicated behavior as suggested in Appendix.

Since the system is periodic with period \( 2\pi \), we introduce an angular variable \( \theta_n = \tau_n \mod 2\pi \). Using this, we can define the return map \( f \) from \( I_D \equiv [0, 2\pi] \) to itself.

\[
\theta_{n+1} = f(\theta_n) \equiv F(\theta_n) \mod 2\pi
\]

For the return map, we define periodic orbits. A point \( \theta^o \) is
said to be a $p$-periodic point if $f^p(\theta) = \theta$ and $f^l(\theta) \neq \theta$ for $0 < l < p$, where $f^p$ is the $p$-fold composition of $f$. A 1-periodic point is referred to as a fixed point. A sequence of $p$-periodic points, $\{\theta^p_1, \cdots, \theta^p_n\}$, is said to be a $p$-periodic orbit. A periodic orbit is said to be stable if $|DF^p(\theta)| < 1$, where $DF^p$ is the slope of $f^p$ that is equal to $DF^p$.

Figures 3(a) and (c) show two typical return maps having nonperiodic and periodic orbits, respectively. The nonperiodic return map corresponds to the switching time map in Fig. 2(a). If the return map has stable periodic orbit, the DM exhibits stable synchronous phenomena. Since $F$ is monotone and $F(\tau + 2\pi) = F(\tau) + \pi$ for $a > 1$, the return map is equivalent to a circle map (a homeomorphism on the circle [12]). The circle map is characterized by the rotation number:

$$\rho = \lim_{n \to \infty} \frac{1}{2\pi n} (F^n(\tau_0) - \tau_0), \quad 0 \leq \tau_0 < 2\pi$$

The rotation number corresponds to an averaged switching time interval at $X_L$. Referring to Ref. [12], we can introduce the following results.

- $\rho$ exists and is independent of the initial value $\tau_0$.
- $f$ has a stable periodic orbit if $\rho$ is rational.
- $f$ has a non-periodic orbit if $\rho$ is irrational.

The rational and irrational rotation numbers correspond to the periodic and non-periodic orbits, and then to the synchronous and asynchronous operations of the DM, respectively. Now we can say that the irrational rotation number is the reason of the asynchronous and unstable operation of the DM. Figure 3 shows the characteristics of the rotation number with corresponding diagram of orbits. In the figure we can see that the orbits are non-periodic and the DM exhibits asynchronous behavior for almost all $a > 1$.

4. Stabilizing Non-periodic Operation

In order to stabilize the asynchronous operation of the DM we present a simple method as illustrated in Fig. 4: the output of the hysteresis VCCS is switched to $-I_L$ compulsory at every period end. In this case, the dynamics is described as the following:

$$\frac{d}{d\tau} x = ah(x, \tau)$$

$h(x, \tau)$ is switched from 1 to $-1$ if either $x$ reaches $X_U(\tau)$ or $\tau = 2n\pi$ and $x > X_L(\tau)$. $h(x, \tau)$ is switched from $-1$ to 1 if $x$ reaches $X_L(\tau)$. Let $\tau_{n+1} = F_c(\tau_n)$ denote the switching time map of this system. In order to describe $F_c$ we define a key time $\tau_c \in (0, 2\pi)$: the orbit started from $(\tau_c + 2n\pi, X_L(\tau_c + 2n\pi))$ reaches $(2(n+1)\pi, X_U(2(n+1)\pi))$ before hitting $X_U$ as shown in Fig. 4. Existence of $\tau_c$ is guaranteed by Condition (9). For $2n\pi + \tau_c < \tau_n < 2(n+1)\pi$, all the orbits started from $(\tau_n, X_L(\tau_n))$ are switched compulsory at time $2n\pi$ before hitting $X_U$. For $2n\pi + \tau_c < \tau_n \leq 2(n+1)\pi$ the switching time and its slope are described by

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**Fig. 3** Operation for $d = 0.5$. (a) & (b) Return map and waveform for $a = 1.5$. (c) & (d) Return map and waveform for $a = 1.35$. (e) Characteristics of orbit, (f) Rotation number.

**Fig. 4** Stabilization by periodic compulsory switching.
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Fig. 5  Stabilized operations for \( d = 0.5 \). (a) and (b): Switching time map and return map corresponding to Fig. 2(a) and Fig. 3(a), respectively (\( a = 1.5 \)). (c) and (d): Characteristics of orbit and rotation number calculating formally using Eq. (11). They correspond to Figs. 3(e) and (f), respectively.

\[
X_c(\tau_{n+1}) = X_c(\tau_n) = a(4n\pi - \tau_{n+1} - \tau_n)
\]
\[
(\sin \tau_{n+1} - \sin \tau_n = a(4n\pi - \tau_{n+1} - \tau_n))
\]
\[
DF_c(\tau_n) \equiv \frac{df_{n+1}}{df_n} = -\frac{a - \cos \tau_n}{a + \cos \tau_{n+1}}
\]

(13)

For \( 2n\pi < \tau_n \leq 2n\pi + \tau_c \) the switching time and its slope are described by Eqs. (7) and (8). Introducing an angular variable \( \theta_n = \tau_n \mod 2\pi \), we can define the return map \( f_c \) from \( I_D \) to itself:

\[
\theta_{n+1} = f_c(\theta_n) \equiv F_c(\theta_n) \mod 2\pi
\]

(14)

Figure 5 shows a switching time map and return maps of this stabilized system. It should be noted that \( F_c \) (respectively, \( f_c \)) has an extremum point at \( \tau_n = \tau_c \) (respectively, \( \theta_n = \tau_c \)) and that \( \tau_n = 2\pi \) is a discontinuity point of \( F_c \). As the compulsory switching is applied, the shape of the map on \( (\tau_c, 2\pi] \) is changed and non-periodic behavior in Fig. 3(a) is changed into periodic behavior in Fig. 5(b). Also, all the nonperiodic orbits in Fig. 3(e) is changed into periodic orbits as shown in Fig. 5(c). Such stabilization is confirmed in the parameter region for \( 0 < d < \frac{\pi}{2} \).

Figure 6 shows existence regions of periodic orbit: a stable \( p \)-periodic orbit of the return map exists in the region \( P_p \). The stability and position of each periodic point is calculated numerically using Eqs. (7), (8) and (13). We then characterize borders of the existence regions. Figures 7(a) and (c) show 2- and 3-periodic orbits observed in \( P_2 \) and \( P_3 \), respectively. In these figures, 2- and 3-periodic orbits are confirmed as fixed points of 2- and 3-fold compositions of the return map \( f_c \), respectively. Let \( B_{23} \) denote the border between \( P_2 \) and \( P_3 \). The border is calculated numerically exploiting Eqs. (7) and (13). On \( B_{23} \) we have confirmed that a 2-periodic orbit dies while a 3-periodic orbit is born as shown in Fig. 7(b):

\[
f_c^2(\tau_c) = \tau_c, \quad f_c^2(2\pi) = 2\pi \quad \text{for} \quad (a, d) \in B_{23}
\]

Note again that \( \theta_n = \tau_c \) gives an extremum point of \( f_c \) and \( \tau_n = 2\pi \) is a discontinuity point of \( F_c \). The 2-periodic orbit dies via tangent of extremum points of \( f_c^2 \) with 45 degrees
line $\theta_{n+1} = \theta_n$, and the 3-periodic orbit is born by intersecting of discontinuity point of $F_t$ with the 45 degrees line. Such a scenario continues for all the periodic orbits. Let $B_{p(p+1)}$ denote the border between $P_p$ and $P_{p+1}$. On $B_{p(p+1)}$ we have confirmed for sufficiently large $p$ that a $p$-periodic orbit dies while a $p+1$-periodic orbit is born as the following.

\[ f_{p+1}(\tau) = \tau, \quad f_{p+1}^{2}(2\pi) = 2\pi \quad \text{for} \quad (a, d) \in B_{p(p+1)} \]

5. Experiments

We have fabricated a simple test circuit using the operational transconductance amplifier (ab. OTA, NJM13600) as shown in Fig. 8. The OTA characteristics is assumed to be $I_s = I_s \text{ sgn} \, u\_o$, where $I_s$ is the saturation current and $u\_o$ is the differential input voltage. The positive feedback loop is terminated by the voltage source $U(t)$ through resistor $R_o$. If $U(t) = V_m \sin \omega t$, the output current $I_o$ is characterized by Eq. (1) with $V_b = R_o I_s$: it realizes the hysteresis VCCS. This circuit behaves as the DM. Using an appropriate I/V converter, the current $I_o$ can be transformed to the voltage output $V_o$.

In order to realize the compulsory switching, we subtract the following periodic pulse-train $p(t)$ from $U(t)$.

\[ p(t) = \begin{cases} P_s & t = nT, \\ 0 & t \neq nT, \quad P_s \times 2V_b. \end{cases} \]

If $U(t) = V_m \sin \omega t - p(t)$, the upper threshold at time $nT$ is smaller than the lower threshold at time $nT$:

\[ V_U(nT) = V_b - P_s \leq V_b = V_L(nT). \]  

Since the continuity property of the capacitor voltage $v$ is held, the output current $I_o$ must be $-I_o$ at time $nT$: it realizes the compulsory switching. The subtracter is fabricated using an op-amp. Using this, the non-synchronous operation in Fig. 8(b) is changed into the stable synchronous state in Fig. 8(c).

6. Conclusions

We have studied behavior of DM circuit from viewpoints of switched dynamical systems. Using the 1-D return map, we have clarified that the non-synchronous operation corresponds to the non-periodic orbit by the irrational rotation number. In order to stabilize the operation, we have presented a simple compulsory switching method. The mechanism of the stabilization relates to birth and death of stable periodic orbits of the return map. These results may provide useful information not only for improving performance of DM circuits but also to analyze interesting bifurcation phenomena. Future problems include classification of bifurcation phenomena for wider parameter region and comparison of the compulsory switching method with other stabilization schemes of DM operations [11].

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Appendix

In out of parameter region of (9), the system can exhibit chaotic behavior. We have confirmed chaotic behavior for the DM without/with compulsory switching as shown in Figs. A·1(a) and (b). These examples suggest that the system has rich chaotic behavior and related bifurcation. Detailed analysis of such complicated phenomena is included in future problems.

Fig. A·1 Chaotic behavior. (a) DM for \((a, d) = (0.75, 7.0)\). (b) DM with compulsory switching for \((a, d) = (3.4, 7.0)\).

Hiroshi Shimazu received the B.E. and M.E. degrees in electrical engineering from Hosei University, Tokyo, Japan, in 2003 and 2005, respectively. He joined Murata Manufacturing Co., Ltd., Kyoto, Japan in April 2005. His current research interests are in switched dynamical systems in power electronics.

Toshimichi Saito received the B.E., M.E., and Ph.D. degrees in electrical engineering all from Keio University, Yokohama, Japan, in 1980, 1982 and 1985, respectively. He is currently a Professor at the EECE Department, Hosei University, Tokyo. His current research interests include chaos and bifurcation, artificial neural networks and power electronics. He is a senior member of the IEEE and a member of the INNS.