

A Brownian Model for Recurrent Earthquakes

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Abstract We construct a probability model for rupture times on a recurrent earthquake source. Adding Brownian perturbations to steady tectonic loading produces a stochastic load-state process. Rupture is assumed to occur when this process reaches a critical-failure threshold. An earthquake relaxes the load state to a characteristic ground level and begins a new failure cycle. The load-state process is a Brownian relaxation oscillator. Intervals between events have a Brownian passage-time distribution that may serve as a temporal model for time-dependent, long-term seismic forecasting. This distribution has the following noteworthy properties: (1) the probability of immediate rerupture is zero; (2) the hazard rate increases steadily from zero at $t = 0$ to a finite maximum near the mean recurrence time and then decreases asymptotically to a quasi-stationary level, in which the conditional probability of an event becomes time independent; and (3) the quasi-stationary failure rate is greater than, equal to, or less than the mean failure rate because the coefficient of variation is less than, equal to, or greater than $1/\sqrt{2} \approx 0.707$. In addition, the model provides expressions for the hazard rate and probability of rupture on faults for which only a bound can be placed on the time of the last rupture.

The Brownian relaxation oscillator provides a connection between observable event times and a formal state variable that reflects the macromechanics of stress and strain accumulation. Analysis of this process reveals that the quasi-stationary distance to failure has a gamma distribution, and residual life has a related exponential distribution. It also enables calculation of “interaction” effects due to external perturbations to the state, such as stress-transfer effects from earthquakes outside the target source. The influence of interaction effects on recurrence times is transient and strongly dependent on when in the loading cycle step perturbations occur. Transient effects may be much stronger than would be predicted by the “clock change” method and characteristically decay inversely with elapsed time after the perturbation.

Introduction

The elastic-rebound model suggests that great tectonic earthquakes may recur at regular time intervals. In the first exposition of this model, Reid (1910) recognized the fundamental nature of what may be called a “crustal strain budget.” A great tectonic earthquake will occur where large elastic strain exists in the crust. The earthquake will substantially relieve crustal strain. Strain will then reaccumulate slowly by steady tectonic forcing and eventually again become large, and another great earthquake will ensue. The duration of an “earthquake cycle” is the ratio of event-strain release to tectonic strain rate.

Nearly a century after its inception, the elastic-rebound model is the canonical “macroscopic” theory of tectonic earthquakes. The corollary notion of great earthquakes that punctuate regular strain cycles continues to play a central role in our understanding of earthquake mechanics. Elastic-rebound theory has formed the cornerstone of long-term

earthquake forecasting, as it has been applied to plate boundaries around the world. Extending this metaphor, one might say, however, that elastic-rebound theory is a loose cornerstone in the shaky edifice of probabilistic earthquake prediction. The problem is that simple elastic rebound as a predictive theory provides a deterministic picture of a single earthquake recurring over time; it has little to say about statistical distributions of actual seismicity in space, time, and size.

Focusing on a single event or a small collection of events assumed to recur in time is a way to apply elastic-rebound principles and avoid some issues of statistical complexity. This approach, which has some observational grounding, has been the basis for long-term forecasting (Lindh, 1983; Bakun and McEvilly, 1984; Sykes and Nishenko, 1984; Bakun and Lindh, 1985) and for regional assessments of seismic hazard (Working Group on California

Earthquake Probabilities, 1988, 1990, 1995, 1999). Ellsworth (1995) reviews evidence for and against the notion of the characteristic earthquake. His analysis of moderate-magnitude seismicity in central California reveals several sequences of recurrent events. Similar analyses by Nadeau *et al.* (1995) and Nadeau and Johnson (1998) also find recurrent microearthquake sequences in seismicity recorded at Parkfield, California. McCalpin and Slemmons (1998) have extended this analysis to paleoseismic evidence for recurrent earthquakes.

Rikitake (1974) formally introduced a probabilistic description of occurrence times for a specific earthquake. He obtained geodetic estimates of the mean and standard deviation of “ultimate strain” for a collection of earthquakes and used these estimates as parameters of a Gaussian distribution, from which he calculated time-dependent recurrence probabilities for several specific earthquakes. Rikitake used the Gaussian distribution for illustration only, stating that “When a probabilistic model for crustal rupture is set out, we may have a more realistic distribution.” Hagiwara (1974) noted that the Gaussian distribution assigns positive probability to negative intervals and is therefore not even a plausible candidate for a stochastic recurrence model. He identified the gamma, lognormal, and Weibull distributions as potential alternatives and chose to apply the Weibull on grounds of “practical convenience” and its popularity in “probabilistic quality control.”

Nishenko and Buland (1987) sought empirically to discern the shape of a generic distribution for recurrence intervals by normalizing 15 putative characteristic earthquake sequences to a common time scale and comparing the resulting empirical distribution of interevent times with several theoretical candidates. Nishenko and Buland, sharing Hagiwara’s preference for convenient and popular models, confined attention to a set of the “usual suspects” in the realm of statistical reliability analysis: the exponential, Weibull, gamma, and lognormal distributions. They drew two important conclusions: first, that the lognormal provided the best fit to the distribution of normalized intervals, and second, that a parameter of the lognormal, the coefficient of variation, is roughly invariant across sequences from different regions with different characteristic time scales. Nishenko and Buland’s results ushered the lognormal distribution into use as a conventional choice for risk assessment (Working Group, 1988, 1990, 1995) and recurrence modeling (Davis *et al.*, 1989; Ogata, 1999).

To the degree that elastic rebound affords a generic macromechanical description of recurrent events, a probabilistic extension of this theory may produce a generic “statistical mechanical” model. This article offers one approach toward building such a model. The basic ideas explored here revolve around addition of random perturbations to the deterministic load path predicted by elastic-rebound theory. Kagan and Knopoff (1987) used the same approach in their development of a time-dependent model for earthquake statistics. After presentation of the basic model, we proceed to

identify the resulting recurrence-interval distribution and to compare this distribution with existing alternatives. We examine how the model can be applied to faults for which the time of the last rupture is unknown but bounded. Finally, we look more closely at the underlying mechanical model for insights into predicted behavior on a single recurrent source and on interactions between sources.

Empirical Analysis

Empirical analysis of earthquake-recurrence-interval data has been used to guide the development of statistical models for the earthquake-recurrence processes (Utsu, 1984; Nishenko and Buland, 1987; Ellsworth, 1995; Ogata, 1999). Although limited in scope, presently available recurrence-interval data suggest that recurrence is not a Poisson process in many cases. This is because recurrence intervals that are short relative to the mean interval are rare. This point, however, is debated by Kagan and Jackson (1999), and at present there is room for a variety of views on the question of whether or not earthquake-recurrence intervals have a strong central tendency or are better described by an exponential distribution. Despite these questions, it has been common practice to use empirical analysis to discern the best probability model from among candidate distributions. Figure 1 illustrates the probability distributions for a number of models that have been used for earthquake forecasting.

Nishenko and Buland (1987) chose a set of models without any physical or theoretical basis and found that the lognormal model provided only a slightly better fit than the gamma or Weibull models. In fact, all of the two-parameter models that they evaluated were statistically consistent with the data set that they assembled. Furthermore, their “large” data set was amalgamated from many small sets, supposedly to sharpen focus on the shape of a generic recurrence distribution. However, the manner by which this artificial data set was concocted—data-dependent normalization to common scale—defeats the very purpose it was intended to serve by grossly altering the shape of the sampling distribution.

Nishenko and Buland’s entire approach, which was also used by McCalpin and Slemmons (1998), is seriously flawed and their goal impossible to achieve. The underlying problems with data-dependent normalization are well understood by statisticians, but apparently not by most earth scientists, if the junior authors of this article are a representative sample. A brief discussion of the issues appears in Appendix 1. The primary flaw is in the “normalization” step, in which each individual recurrence sequence is divided by the sample mean interval between events. The difference is particularly acute when the sample size is small, and this is the case for most earthquake sequences. Normalization defeats the very purpose for which it is intended. It is meant to increase empirical focus on the shape of a common renewal distribution. Instead, it blurs.

Ogata (1999) avoids many of these pitfalls of empirical analysis in the construction of Bayesian estimates of the op-

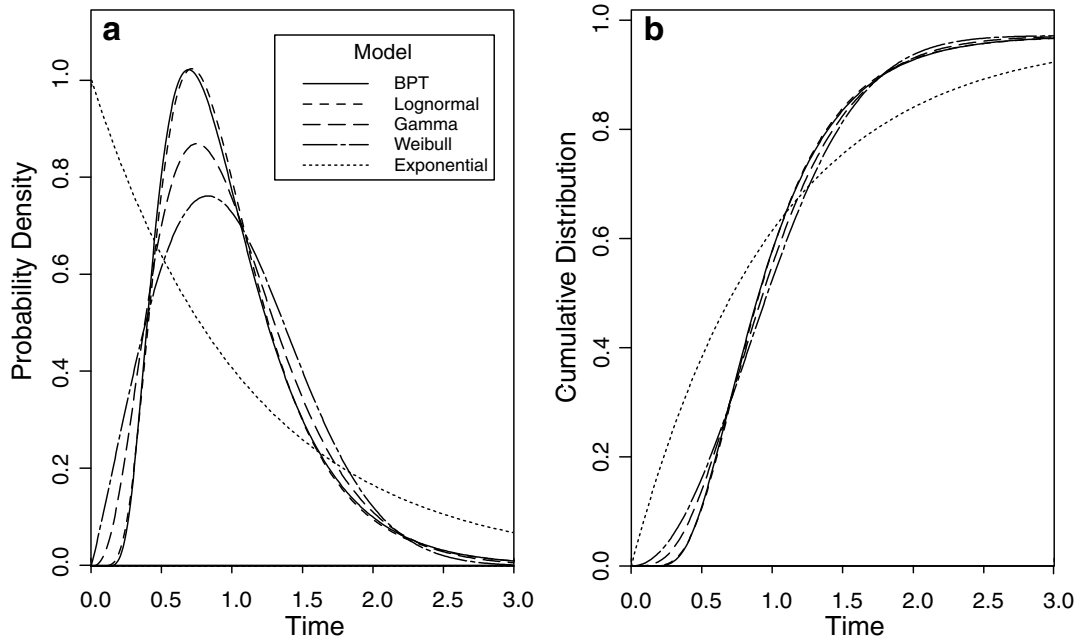


Figure 1. Probability density (a) and cumulative distribution (b) functions of exponential (Poisson), BPT, log-normal, gamma, and Weibull models. All distributions have mean 1 and standard deviation 0.5 (except the exponential distribution).

timum model for a variety of probability distributions. His results strongly suggest that empirical analysis can discriminate against the exponential model, but hope for any elucidation beyond that level is futile with presently available data.

The Brownian Relaxation Oscillator

The analysis of series of events, in particular, failure-time data, has wide application across the sciences (Cox and Lewis, 1966; Kalbfleisch and Prentice, 1980). Throughout the literature, the exponential, lognormal, Weibull, and gamma distributions have been widely used as models for stationary-point processes. As discussed in the preceding section, empirical fitting of these models to data may capture the main features of interest, but they give no insight into the process. Alternatively, models can be constructed on more theoretical grounds that achieve the same measure of performance. In this section, we introduce a simple model of a recurrent-earthquake source.

Consider a fixed, seismic source loaded by steady tectonic forcing and rupturing in repeated occurrences of its characteristic earthquake. The sequence of identical events produced by this model can be described as a point process (Cox and Lewis, 1966). A variable describing the “load state” of a source in cyclic “stick-slip” motion traces, over time, the sawtooth-shaped path of a relaxation oscillator. The load state is at some ground level immediately after an event, increases steadily over time, reaches a failure threshold, “relaxes” instantaneously back to ground level at the next earthquake time, and so on.

The relaxation oscillator portrays an idealized object for which the obvious question, “What is the load state?”, lacks an obvious answer. The load state might be associated with cumulative elastic strain, as originally proposed by Reid (1910), but it could equally well summarize other physical variables, like cumulative moment deficit or cumulative stress. It might also represent the Coulomb stress history at the unknown location where rupture initiates at the conclusion of each cycle, as discussed by Toda *et al.* (1998). At this juncture, the load state is simply a formal representative of rupture potential.

Let $Y_0(t)$ denote the load state at time t , measured on a scale where the postevent ground state is x_0 , the failure state is $x_f = x_0 + \delta$, $\delta > 0$, loading has a constant rate $\lambda > 0$, and $Y_0(0) = x_0$. Relaxation events, that is, earthquakes, occur at times $t_k \triangleq k\delta/\lambda$, $k = 0, 1, 2, \dots$. (The symbol, “ \triangleq ”, read “is defined to equal,” is used when the right-hand side of an equation defines the symbol on the left-hand side.) The function $X_0(t) \triangleq \lambda_0 t$ measures the cumulative applied load from time zero to time t , and $r(t)$, $t \geq 0$ will denote the last relaxation time preceding t . The relaxation oscillator path is

$$\begin{aligned} Y_0(t) &= x_0 + \lambda_0 \cdot [t - r(t)] \\ &= x_0 + X_0(t) - X_0[r(t)] \end{aligned} \quad (1)$$

To effect a statistical mechanical description of elastic rebound, we assume that the ground and failure states, x_0 and x_f , remain fixed while the load path is subject to random perturbations. The cumulative load is the state-valued sto-

chastic process obtained by adding to $X_0(t)$ a random perturbation process, $\varepsilon(t)$:

$$X(t) = X_0(t) + \varepsilon(t) \tag{2}$$

The time of the k -th event, when $X(t)$ first exceeds $k\delta$

$$T_k \triangleq \min\{t : t \geq 0, X(t) \geq k\delta\} \tag{3}$$

is a random variable corresponding to the relaxation time of a stochastic relaxation oscillator. In terms of the random variable for the penultimate relaxation time

$$R(t) \triangleq \max\{T_k : k \geq 0, T_k \leq t\} \tag{4}$$

the stochastic relaxation oscillator is

$$Y(t) = x_0 + X(t) - X[R(t)] \tag{5}$$

Compare the stochastic oscillator in equation (5) with the deterministic version in equation (1).

The sole source of variation in recurrence intervals is the perturbation process, $\varepsilon(t)$, in equation (2). If we specify natural properties of this process, we also specify the natural statistical character of earthquake-recurrence intervals. The deterministic process, X_0 , represents a constant-rate mean path that embodies a macroscopic view of uniform tectonic loading. Formally, the perturbation process ε represents the sum total of all other factors governing eventual rupture of the recurrent source in question. Physically, these factors may include effects of earthquakes external to the target source, aseismic load variations, pore pressure, dilatancy, healing, and general evolution of spatial heterogeneities.

The requirements we impose are that $\varepsilon(t)$ have constant drift and stationary increments. ‘‘Stationary increments’’ simply means that the statistics of $\varepsilon(t)$ do not change with time. The motive for allowing nonzero drift is the possibility of ‘‘spatial bias’’ in the perturbations. Because of its geometry or spatial orientation, a recurrent source may be subject to perturbations, by external seismicity, for instance, that are systematically positive or negative relative to tectonic forcing. Assuming a lack of systematic temporal variation in the perturbation process, symmetry arguments imply constant drift and stationary increments.

The stated requirements do not yield a concrete description of $\varepsilon(t)$. To complete the picture, we note that the deterministic process X_0 satisfies the differential equation $\dot{X}_0 = \lambda_0$. By adding scaled Gaussian white noise, X satisfies the stochastic differential equation

$$\dot{X} = \lambda + \sigma \dot{W} \tag{6}$$

where W is standard Brownian motion and $\sigma \geq 0$ is the perturbation scale parameter. Standard Brownian motion is simply integrated stationary increments, where the distribution of increments is Gaussian, with zero mean and con-

stant variance. Equations (2) and (6) imply that $\varepsilon(t)$ is Brownian motion with diffusion rate σ^2 and drift rate $\beta = \lambda - \lambda_0$. It follows that

$$X(t) = \lambda t + \sigma W(t), t \geq 0 \tag{7}$$

The Brownian perturbation process in equation (7) has constant drift and stationary increments as required. The Gaussian distribution of process increments might be motivated by central-limit arguments, which view perturbations as the sum of many small, independent contributions. Kagan and Knopoff (1987) present seismological arguments for the same Brownian model. Rather than attempting to make this argument precise, we simply stand on Brownian motion as an island of mathematical tractability in a sea of ignorance and potential complexity.

Applying the definition of $X(t)$ in equation (3), event times are seen as ‘‘first-passage’’ or ‘‘hitting’’ times of Brownian motion with drift. The Brownian relaxation oscillators, defined by plugging equations (7), (3), and (4) into equation (5), are a family of stochastic renewal processes identified by four parameters: the mean loading (drift) rate, λ ; the perturbation rate, σ^2 ; the ground state, x_0 ; and the failure state, x_f . We will use the abbreviations BM(x_0, λ, σ) and BRO($x_0, x_f, \lambda, \sigma$), respectively, for Brownian motion and Brownian relaxation oscillator processes. Figure 2 exhibits three simulated BRO(0, 1, 1, σ) paths with $\sigma = 1/4, 1/2, 1$. The qualitative manner in which the parameters govern behavior of these failure processes is fairly clear. Consider, for instance, the typical length of a time interval between failures and

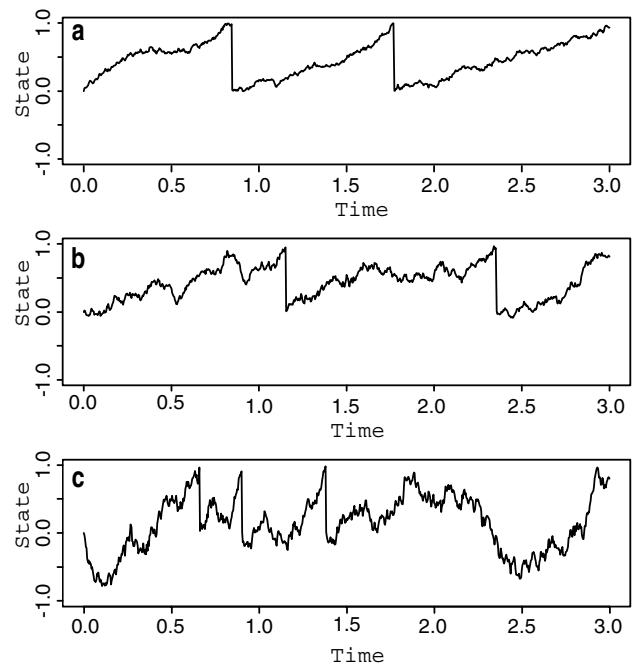


Figure 2. Load-state paths for a Brownian relaxation oscillator with $\lambda = \delta = 1$: (a), $\sigma = 1/4$; (b) $\sigma = 1/2$; (c), $\sigma = 1$.

random variations in lengths of these intervals. The load state must traverse distance δ ($x_f - x_0$) and does so at average rate λ , so recurrence intervals will have average length

$$\mu = \frac{\delta}{\lambda} \tag{8}$$

Relatively small values of σ produce quite regular paths that closely resemble the deterministic relaxation oscillator and generate nearly μ -periodic recurrence. Indeed, the deterministic oscillator with identical recurrence intervals is the $\sigma = 0$ limit of the Brownian relaxation oscillator. Increasing, σ with μ fixed increases aperiodicity. We show subsequently that the variance of recurrence-interval length is

$$\sigma_T^2 = \frac{\sigma^2 \delta}{\lambda^3} \tag{9}$$

and define a dimensionless measure of aperiodicity by

$$\alpha \triangleq \frac{\sigma_T}{\mu} = \frac{\sigma}{\sqrt{\lambda\delta}} \tag{10}$$

Brownian Passage-Time Distributions

In this section, we develop the recurrence properties of the Brownian relaxation oscillator. Let T denote the first passage time to level $x > 0$ by Brownian motion with drift rate $\lambda > 0$ and diffusion rate σ^2 . The probability distribution of T has a well-known closed form with probability density function

$$f(t; x, \lambda, \sigma) = \frac{x}{\sqrt{2\pi\sigma^2 t^3}} \cdot \exp\left\{-\frac{(x - \lambda t)^2}{2\sigma^2 t}\right\}, t \geq 0 \tag{11}$$

This density goes by various names in the literature. Statisticians call it the inverse Gaussian (because its cumulant generating function is the inverse of the Gaussian cumulant generating function), or Wald distribution. Physicists use Brownian passage-time (BPT) distribution, a term we will adopt. Derivation of the density equation (11) dates back at least to Schrödinger in 1915, as documented in a mathematical overview of the inverse Gaussian and related distributions by Seshadri (1993). It includes a historical review and several derivations of the distributional form. Chhikara and Folks (1989) cover basic theory and applications of this distribution at a less technical level.

We may simplify the description of the BPT family by noting the many-to-one correspondence between load-state processes and recurrence-interval distributions. Applying in load-state space the affine transformation, $x \mapsto a + bx$, $b \neq 0$, transforms BRO($x_0, x_f, \lambda, \sigma$) to BRO($a + bx_0, a + bx_f, b\lambda, b\sigma$). Plugging the transformed parameter into equation (11) reveals that the failure-time distribution is un-

changed. Affine invariance of the recurrence-time distribution supports the concept of the relaxation oscillator as a formal model with (infinitely) many possible physical interpretations. It implies that if v is a state variable, then so is any nonsingular affine transformation of v . It also dictates that the family of BPT distributions has a two-dimensional parameter space. We will usually parametrize these distributions by the mean μ and aperiodicity α defined in equations (8) and (10), which render the density in equation (11) as

$$f(t; \mu, \alpha) = \left(\frac{\mu}{2\pi\alpha^2 t^3}\right)^{1/2} \exp\left\{-\frac{(t - \mu)^2}{2\mu\alpha^2 t}\right\} \tag{12}$$

The cumulative distribution function (cdf) of T gives the probability that failure will have occurred by t and may be expressed in terms of the cumulative Gaussian probability function,

$$\Phi(t) \triangleq \frac{1}{\sqrt{2\pi}} \int_{-\infty}^t e^{-x^2/2} dx \tag{13}$$

Let

$$\begin{aligned} u_1(t) &= \alpha^{-1}[t^{1/2}\mu^{-1/2} - t^{-1/2}\mu^{1/2}], \\ u_2(t) &= \alpha^{-1}[t^{1/2}\mu^{-1/2} + t^{-1/2}\mu^{1/2}] \end{aligned} \tag{14}$$

Then the cdf is

$$F(t) \triangleq P\{T \leq t\} = \int_0^t f(\tau) d\tau = \Phi[u_1(t)] + e^{2t\alpha^2} \Phi[-u_2(t)] \tag{15}$$

Calculating the first and second moments of T (see Seshadri, 1993, for instance), we find that $E(T) = \mu$ and $\text{var}(T) = \sigma_T^2 = (\mu\alpha)^2$, so α is the ratio of the standard deviation to the mean, that is, the coefficient of variation of the failure-time distribution. We prefer to call α the aperiodicity, because the coefficient of variation is often used in the seismological literature for the ratio of the sample standard deviation to the sample mean.

The mean interevent time μ or its reciprocal, the mean rate of occurrence, is the natural parameter of first-order interest, as it gauges the typical frequency at which events occur. In our parametrization, μ is a scale parameter. Changing the mean rescales time but does not otherwise alter the shape of the probability distribution. It also parametrizes the exponential recurrence distribution, which generates a “time-independent” Poisson event process. The aperiodicity is chosen as a second parameter for two reasons: it is a natural shape parameter of the BPT family; it is a dimensionless measure of irregularity in the event sequence. Note that a perfectly periodic sequence has $\alpha = 0$.

Time Dependence

The Brownian passage-time distributions quantify occurrence-time probabilities for a steadily loaded system subject to random perturbations. How do these probabilities behave? In particular, what is the effect of elapsed time since the last event on conditional probabilities for the next event? Questions such as these, raised often in previous work on recurrence models (e.g., Cornell and Winterstein, 1988; Davis *et al.*, 1989; Sornette and Knopoff, 1997) are of natural interest, as they pertain to the qualitative character of earthquake forecasts and how these forecasts may evolve over time.

Basic Issues

A density function, $f(t)$, defines overall probabilistic tendency. To examine the effect of elapsed time on occurrence probabilities, we may consider the residual life conditioned on $T > t$. Generally, if V is a positive random time with cdf F , the residual life at elapsed time t has the cdf

$$R_t(v) \triangleq P\{V \leq t + v | V > t\} \tag{16}$$

$$= \frac{F(t + v) - F(t)}{1 - F(t)} \tag{17}$$

The residual life density is

$$r_t(v) \triangleq \frac{dR_t(v)}{dv} \tag{18}$$

$$= \frac{f(t + v)}{1 - F(t)} \tag{19}$$

and the mean residual life is

$$m(t) \triangleq \int_0^\infty vr_t(v)dv \tag{20}$$

In these terms, the question posed by Davis *et al.* (1989), “The longer it has been since the last earthquake the longer the expected time till the next?”, addressed as a “paradox” in earthquake prediction by Sornette and Knopoff (1997) is, “Does $m(t)$ increase with t ?”

The hazard rate, $h(t) \triangleq f(t)/[1 - F(t)]$, describes instantaneous “failure propensity” and may be viewed conceptually as the time change that produces “uniform” failure governed by an exponential distribution; that is, if failure is measured against a clock “ticking” at rate $h(t)$ when real time is t , then time to failure has the $\exp(1)$ distribution.

Issues regarding dependence on elapsed time generally concern the shape of the hazard-rate function and behavior of the residual life as t increases. Quantities of particular

interest are the asymptotic mean residual life and asymptotic hazard rate:

$$m_\infty \triangleq \lim_{t \rightarrow \infty} m(t), \quad h_\infty \triangleq \lim_{t \rightarrow \infty} h(t) \tag{21}$$

The value of $h(0)$ is also important, as it governs the likelihood of immediate rupture after an event. In a probabilistic sense, it measures the extent to which an event “drains” local rupture potential. Elastic-rebound principles imply that if the local crustal strain budget is depleted by an earthquake, then $h(0)$ should be zero.

Time-Dependent Behavior of BPT Distributions

Because the mean acts as a pure scale parameter in the BPT model, the full range of potential distributional shape is covered by varying α . Figure 3 exhibits the densities and hazard (instantaneous failure rate) functions for BPT(1, α), $\alpha = 1/4, 1/2, 1, 2$. It is apparent that small values of α correspond to nearly symmetrical densities with pronounced central tendency near the mean value. Larger values of α , on the other hand, produce densities highly skewed to the right and sharply peaked at a value left of the mean. BPT density functions are always unimodal, with

$$\text{mode} = \mu \left[\left(1 + \frac{9\alpha^4}{4} \right)^{1/2} - \frac{3\alpha^2}{2} \right] < \mu \tag{22}$$

All BPT hazard-rate functions share a general shape. The function $h(t)$ is always zero at $t = 0$; it increases to achieve its maximum value uniquely at some finite time to the right of the density’s mode, and from there it decreases toward an asymptote at level $h_\infty = (2\mu\alpha^2)^{-1}$ (Chhikara and Folks,

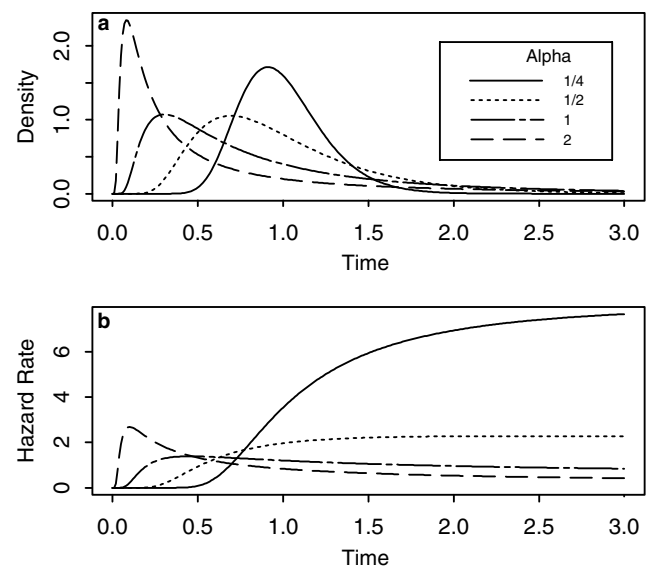


Figure 3. Probability functions for BPT (1, α), $\alpha = 1/4, 1/2, 1, 2$: probability densities (a) and hazard rates (b).

1977). This behavior distinguishes the BPT family from the lognormal model, for which the asymptotic hazard rate is always zero. It also reflects a property of the underlying model, revealing that the Brownian failure process eventually attains a quasi-stationary state in which residual time to failure becomes independent of elapsed time. We examine this property further in a later section.

If we plug the BPT density from equation (12) and the cdf from equation (15) into the definition of the mean residual life in equation (20) and manipulate the resulting expression, we can show that the BPT mean residual life is

$$m(t) = \mu \cdot \frac{\Phi[-u_1(t)] + e^{2/\alpha^2} \Phi[-u_2(t)]}{\Phi[-u_1(t)] - e^{2/\alpha^2} \Phi[-u_2(t)]} - t \quad (23)$$

where u_1 and u_2 are defined in equation (14). Figure 4 illustrates the behavior of the the mean residual life function for BPT(1, α) distributions with $\alpha = 1/4, 1/2, 1$. It decreases over a finite time interval beginning at $t = 0$ and then evolves asymptotically toward

$$m_\infty = h_\infty^{-1} = 2\mu\alpha^2 \quad (24)$$

There is an interesting comparison between the quasi-stationary mean residual life and the unconditional mean. We can see from equation (24) that

$$\mu_\infty \begin{cases} < \\ = \\ > \end{cases} \mu \Leftrightarrow \alpha \begin{cases} < \\ = \\ > \end{cases} 2^{-1/2} \quad (25)$$

The value $\alpha = 2^{-1/2}$ defines a condition in which the mean residual life m_∞ equals the unconditional average recurrence time μ . We will call it quasi-stationary equilibrium and will say more about it in a later section. The alternatives, $\alpha < 2^{-1/2}$ and $\alpha > 2^{-1/2}$, will be called signal dominated and noise dominated, respectively. In these terms, equation (25) says: if the process is signal dominated, then a long (relative to the mean) elapsed time since the last earthquake implies that the expected time until the next earthquake is less than the unconditional average recurrence time; if the process is noise dominated, then a long elapsed time implies a longer-than-average conditional waiting time; and if the process is in equilibrium, then a long elapsed time has no net effect on the conditional expected time until the next earthquake.

Comparison with Alternative Distributions

Four families of renewal-time distributions have commonly appeared as recurrence-time models in the seismological literature: exponential, gamma, Weibull, and lognormal. Use of these distributions has been motivated primarily on the grounds of familiarity, simplicity, and convenience. Here we briefly introduce these alternatives and then compare their properties with the BPT family.

Members of the one-parameter family of exponential

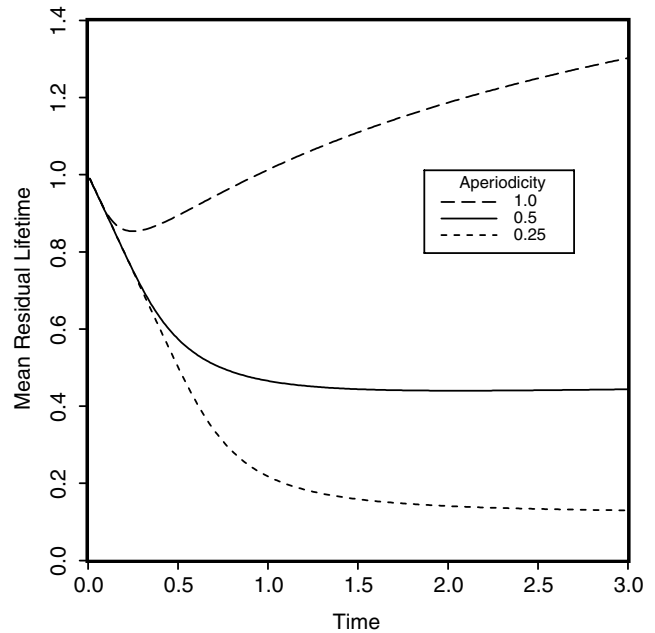


Figure 4. Mean residual life vs. elapsed time for BPT (1, α), $\alpha = 1/4, 1/2, 1$.

distributions are “memoryless” uniquely among continuous distributions. From this property spring many analytical and computational economies: point processes are Poisson, the conditional distribution of N events in a specified interval is uniform when N is given, and so forth. The exponential distribution also maximizes entropy over continuous distributions on the positive half-line with specified mean. In this sense, it represents the “least informative” or “most conservative” choice. Furthermore, the exponential arises from superposition: if failure can occur in any of a large number of independent “modes,” each with small probability, then time to failure has an exponential distribution (see, e.g., Karlin and Taylor, 1975; Barlow and Proschan, 1997). The gamma and Weibull families both contain and generalize the exponential. They also contain the densities of, respectively, the sum and the minimum of n independent exponentials. The Weibull is often referred to as the “weakest link” distribution because of its interpretation in terms of the minimum of exponentials. (A chain breaks when its weakest link fails.) If n links have independent $\exp(\nu)$ failure times, then the chain has a Weibull (ν, n) failure time. A lognormal variable is obtained by taking the exponential of a Gaussian.

Table 1 summarizes relevant properties of the Brownian passage-time family and the four competitors under consideration. The BPT and lognormal families are the only choices with nonmonotonic hazard-rate functions that increase from zero at $t = 0$, then eventually decrease. As we have mentioned, however, the lognormal hazard-rate function decreases to zero, unlike the BPT, which flattens out at a positive level. Weibull hazard-rate functions either start at zero and increase to ∞ or vice versa. Gamma distributions with shape parameter $\theta > 1$ have zero hazard rate at time zero

Table 1
Properties of Candidate Interval Distributions

Name	Probability Density	Shape	$h(0)$	μ/m_∞
Brownian passage time	$\left(\frac{\mu}{2\pi\alpha^2 t^3}\right)^{1/2} \exp\left(-\frac{(t-\mu)^2}{2\alpha^2\mu t}\right)$	$\uparrow\downarrow\rightarrow$	0	$2\alpha^2$
Exponential	$\nu e^{-\nu t}$	\rightarrow	ν^{-1}	1
Gamma	$\frac{\nu^\theta t^{\theta-1} e^{-\nu t}}{\Gamma(\theta)}$	$\downarrow\rightarrow$	∞	θ^{-1}
Weibull	$\theta\nu^\theta t^{\theta-1} \exp(-\nu^\theta t^\theta)$	$\theta > 1$	$\uparrow\rightarrow$	0
		$\theta < 1$	\downarrow	∞
Lognormal	$\left(\frac{1}{\sqrt{2\pi}\sigma t}\right)^{-1} \exp\left(-\frac{(\log t-\mu)^2}{2\sigma^2}\right)$	$\theta < 1$	\downarrow	∞
		$\theta > 1$	\uparrow	0

Columns are as follows: 1, name of parametric family (and indication, as necessary, of parameter range restriction for following columns); 2, parametrized probability density function; 3, shape of failure-rate function: \uparrow = increasing, \downarrow = decreasing, \rightarrow = constant; combination of symbols indicates changing behavior with time, e.g., the BPT failure increases for some time, then decreases, and flattens out to a constant asymptotic level; 4, failure rate at $t = 0$; 5, ratio of mean and quasi-stationary mean recurrence interval.

and increase to a finite asymptotic level that is always smaller than the mean recurrence rate. Hence, these distributions cannot produce behavior equivalent to the signal-dominated regime in the Brownian passage-time family.

Figure 5 shows plots of the hazard-rate functions for the five distributions under discussion (see also Fig. 1 for the corresponding probability density functions [pdfs]). The parameters for each distribution are set so that the mean recurrence time is equal to 1. For all but the exponential, which has only one adjustable parameter, the aperiodicity is equal to 1/2. Although $\alpha = 1/2$ is selected here for illustration, Ellsworth *et al.* (1999) proposed this as a generic aperiodicity based on 37 recurrent earthquake sequences, $-0.7 \leq M \leq 9.2$.

There are several noteworthy comparisons to be made of the probability densities as well (Fig. 1). The lognormal and BPT functions put least weight in the left tail, near $t = 0$. For these two distributions, the density and hazard rate are essentially zero for about the first 25% of the mean recurrence interval. This behavior seems desirable in light of the strain-budget interpretation of elastic rebound and contrasts with the Weibull and gamma functions, which increase relatively steeply from zero at $t = 0$. All density functions are unimodal, with mode left of the mean. The lognormal is the most sharply peaked, followed in order by the BPT, gamma, and Weibull.

For every member of the lognormal family of distributions, the hazard rate goes to zero, and the mean residual life

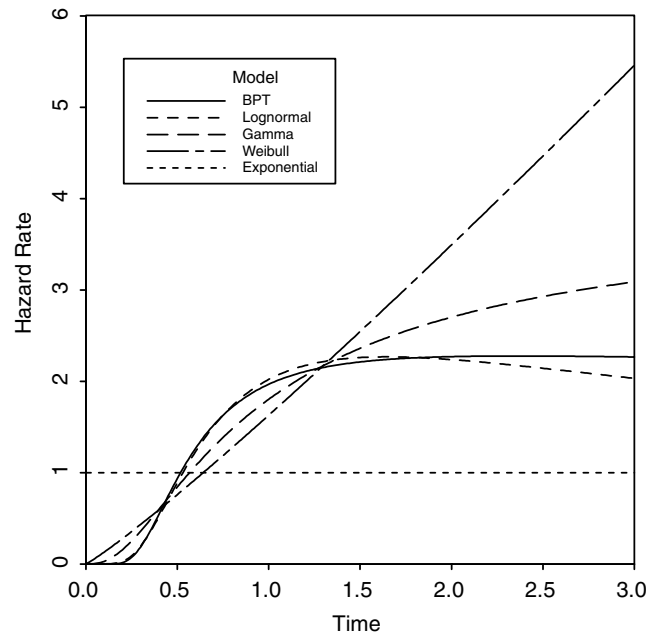


Figure 5. Hazard-rate functions for candidate recurrence distributions. All distributions have mean 1 and standard deviation 0.5 (except the exponential distribution).

increases without bound as $t \rightarrow \infty$. Hence, if the lognormal model were correct, then it would eventually be true that “The longer the time since the last earthquake, the longer the expected time until the next earthquake.” (See Davis *et al.*, 1989, and Sornette and Knopoff, 1997, for a discussion of this issue.) Davis *et al.* (1989) confused this point by using the lognormal model to illustrate that a hazard-rate function estimated from data with a long, current, open interval may decrease as time increases. Any lognormal hazard rate eventually decreases, regardless of parameter values, and estimated hazard-rate functions for other distributions will not necessarily decrease because of a long, open interval.

Barlow and Proschan (1997) point out that its asymptotic properties disqualify the lognormal family as a reliability model when a component subject to failure experiences increasing “wear” with time. This objection would seem to apply more strongly when failure is governed by steadily increasing load on a fault surface. One might contend that when an expected characteristic event is overdue, the stress and moment of that event have likely been dissipated by alternative mechanisms (seismic or aseismic), so a decreased failure rate is appropriate. There may be some validity in this line of reasoning. Faults may exhibit “transient characteristic failure modes,” that is, several repeats of a characteristic event that eventually evolves away (Shaw, 1997). However, models framed purely in terms of recurrent events surely assume that these events will occur. If this assumption is ill founded, then it should be abandoned. Arguing that the occurrence rate for a characteristic event should go to zero because the event itself may not exist defeats understanding by confounding spatial and temporal uncertainty.

Brownian Relaxation Oscillator Distributions

The Brownian passage-time distribution describes the failure probability of a Brownian relaxation oscillator as a function of elapsed time and the statistical properties of the failure time series. In this section, we return to the BRO to examine the probability distribution of load state $Y(t)$. The evolution of $Y(t)$ after reset can be described by solutions to the diffusion equation with positive drift (Seshadri, 1993), conditioned by the necessity to conserve probability.

$$\frac{\partial Y}{\partial t} = -\lambda \frac{\partial Y}{\partial x} + \frac{\sigma^2}{2} \frac{\partial^2 Y}{\partial x^2} \quad (26)$$

The stochastic process in equation (26) is also known as a Weiner process (Grimmett and Stirzaker, 1992).

In equation (7), we defined $X(t)$ as the cumulative applied load from time zero to time t . The stochastic process X is modeled as Brownian motion with drift; it does not reset at event times. X drives the relaxation oscillator Y in equation (5), which is constrained by a “barrier” at level x_f and resets

to x_0 on reaching this barrier. To simplify notation in this section, we change perspective and view the relaxation oscillator in terms of distance to failure. Each cycle begins with Y at level δ , it drifts downward at rate λ toward failure at level zero, and resets to δ on reaching zero. With this orientation, the distribution of $Y(t)$ is supported on $(0, \infty)$.

The probability distribution of X is the well-known Gaussian. The cdf and pdf of $X(t)$ are

$$P_x(x, t) \triangleq P\{X(t) \leq x\} = \Phi\left(\frac{x - \lambda t}{\sigma\sqrt{t}}\right) \quad (27)$$

and

$$p_x(x, t) \triangleq \frac{\partial P_x(x, t)}{\partial x} = (2\pi\sigma^2 t)^{-1/2} \exp\left(-\frac{(x - \lambda t)^2}{2\sigma^2 t}\right) \quad (28)$$

Here we specify the form of the Brownian relaxation oscillator phase distance-to-failure distribution and then explain its name and uses.

Let $R(t)$ denote the maximum event time less than or equal to t , and we define the phase at time $t \leq 0$ as $\tau(t) \triangleq t - R(t)$. Conditioned on event history, the distribution of $[Y(t)|R(t)]$ depends only on the phase, that is, on how much time has elapsed since the last relaxation epoch. In fact, using “ $\stackrel{D}{=}$ ” to denote “equals in distribution,”

$$[Y(t)|R(t)] \stackrel{D}{=} [Y(t)|\tau(t)] \stackrel{D}{=} [\delta - X(\tau)|T > \tau] \quad (29)$$

We write $g(x, \tau)$ and $G(x, \tau)$ for the density function and cdf of the conditioned random variable in equation (29). $\bar{G} \triangleq 1 - G$ is the complimentary cdf, or survivor function, because it gives the probability that failure will occur after time τ . Let

$$A(x, t) = P_x(x, t) - e^{2\lambda\delta/\sigma^2} P_x(x - 2\delta, t) \quad (30)$$

$$a(x, t) \triangleq \frac{\partial A(x, t)}{\partial x} = p_x(x, t) - e^{2\lambda\delta/\sigma^2} p_x(x - 2\delta, t) \quad (31)$$

The complementary distribution function and density function of the conditioned random variable in equation (29) are

$$\bar{G}(x, \tau) \triangleq P\{\delta - X(\tau) \geq x|T > \tau\} = \begin{cases} 1, & x \leq 0, \\ \frac{A(\delta - x, \tau)}{A(\delta, \tau)}, & x > 0, \end{cases} \quad (32)$$

$$g(x, \tau) \triangleq -\frac{\partial G(x, \tau)}{\partial x} \tag{33}$$

$$= \begin{cases} 0, & x \leq 0, \\ \frac{a(\delta - x, \tau)}{A(\delta, \tau)}, & x > 0. \end{cases}$$

The functions in equations (32) and (33) specify probabilities for the distance to failure in the reference frame where the phase is known at time t . This is the perspective of an “earthquake observer” who knows the event history (at least back to the previous earthquake) of rupture times. Figure 6 show plots of the density function defined in equation (33) for $\lambda = \delta = 1$, $\sigma = 1/2, 1$, and $t = 1/2, 1, 2$. Naturally, when $t = 0$, the density is concentrated at $x = \delta$. Thus, equation (33) is the Green’s function for the probability distribution of $Y(t)$.

The complementary cdf and the pdf of the quasi-stationary distance to failure are the $\tau \rightarrow \infty$ limits of equations (32) and (33), respectively. The quasi-stationary distribution is gamma, with rate parameter equal to the signal-to-noise ratio

$$\rho = \frac{\lambda}{\sigma^2} \tag{34}$$

and shape parameter equal to 2. The quasi-stationary distribution and density functions on $x > 0$ are

$$\bar{Q}(x) \triangleq \lim_{\tau \rightarrow \infty} \bar{G}(x, \tau) \tag{35}$$

$$= (1 + \rho x)e^{-\rho x}$$

$$q(x) \triangleq \lim_{\tau \rightarrow \infty} g(x, \tau) \tag{36}$$

$$= \rho^2 x e^{-\rho x}$$

Figure 7 exhibits the quasi-stationary distance-to-failure densities for $\rho = 1, 2$, and 4. It is important to note that the quasi-stationary distribution depends only on the signal-to-noise ratio. The original distance to failure δ has no effect on the distributional form, although it will influence the time required to approach quasi-stationarity. For large values of the signal-to-noise ratio, the quasi-stationary load-capacity distribution concentrates near $x = 0$, indicating that the load-phase process is poised near the failure threshold and an event is imminent. Decreasing ρ spreads out the probability for the residual phase. Let $\xi_\tau \triangleq E[Y(\tau)]$ and $\xi_\infty \triangleq \lim_{\tau \rightarrow \infty} \xi_\tau$. The expected value of the quasi-stationary distribution is the mean of the gamma distribution in equation (36), namely,

$$\xi_\infty = \frac{2}{\rho} = \frac{2\sigma^2}{\lambda} \tag{37}$$

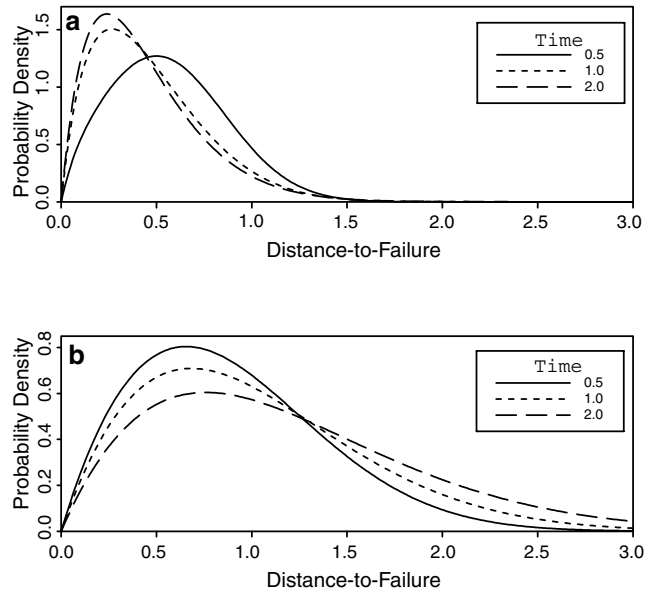


Figure 6. BRO (1, 1, σ) phase distance-to-failure density functions at $t = 1/2, 1, 2$: $\sigma = 1/2$ (a) and $\sigma = 1$ (b).

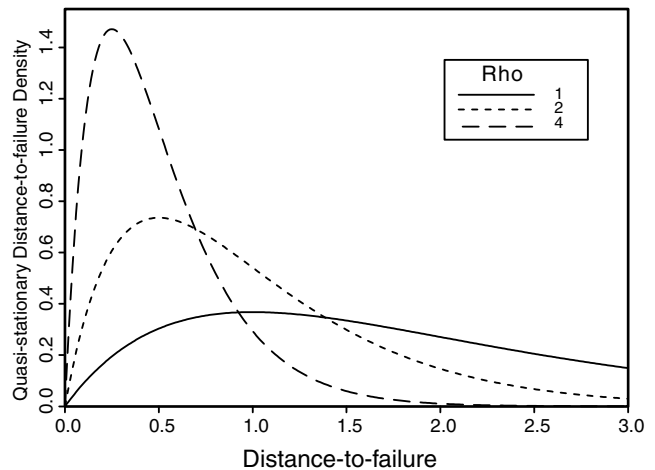


Figure 7. Quasi-stationary distance-to-failure densities. The signal-to-noise ratio is $\rho = \lambda/\sigma^2$. Densities shown are for $\rho = 1, 2, 4$.

How does the expected residual load state after a long elapsed time compare with the initial load state? At reset, $\xi_0 = \delta$, and the ratio of the two is

$$\frac{\xi_\infty}{\xi_0} = \frac{2\sigma^2}{\lambda\delta} = 2\alpha^2 \tag{38}$$

where α^2 , is the aperiodicity parameter of the BPT recurrence distribution. The quasi-stationary equilibrium condition, $\alpha = 2^{-1/2}$ from equation (25), is thus understood in terms of long-term behavior of the load-state process on an unruptured source. A process in quasi-stationary equilibrium tends, when and if it reaches equilibrium, to be fluctuating

about its initial level. When the process is signal dominated ($\alpha < 2^{-1/2}$), it tends to have drifted upward toward the failure threshold, and when it is noise dominated ($\alpha > 2^{-1/2}$), it tends to have drifted away from the failure threshold.

At a given phase in a failure cycle, if the remaining distance-to-the-failure threshold is x , then the conditional residual time-to-failure density is $f(t; x, \lambda, \sigma)$ given in equation (11). The density function of residual time to failure at τ when $Y(\tau)$ is a random variable is obtained by averaging passage-time densities with weighting by the density of $Y(\tau)$, that is,

$$\tilde{f}_\tau(t) = \int_0^\infty f(t; x, \lambda, \sigma) g(x, \tau) dx \quad (39)$$

Taking the limit as $\tau \rightarrow \infty$ in equation (39) yields

$$\tilde{f}_\infty(t) = \int_0^\infty f(t; x, \lambda, \sigma) q(x) dx \quad (40)$$

Plugging the definitions of f from equation (12) and q from equation (36) into equation (40) and simplifying the result yields

$$\tilde{f}_\infty(t) = \frac{\rho\lambda}{2} \exp\left(-\frac{\rho\lambda}{2} t\right) \quad (41)$$

where ρ is the signal-to-noise ratio from equation (34).

The distribution in equation (41) is exponential with rate parameter

$$\frac{\rho\lambda}{2} = \frac{\lambda^2}{2\sigma^2} = (2\alpha^2\mu)^{-1} = h_\infty$$

This result, of course, is implied by the form of the BPT density and is apparent in the asymptotic behavior of the hazard-rate function. Explicit derivation of this exponential density by the argument leading up to equation (41), however, paints a much clearer picture of the origin for this asymptotic behavior as the average of BPT densities weighted by the quasi-stationary gamma density on the residual stress.

Brownian Passage-Time Distributions when the Time Since the Last Earthquake Is Unknown

Thus far, we have used the Brownian passage-time distributions to quantify occurrence-time probabilities when the time of the last earthquake is known. In many applications, however, the time of the last event on a fault segment is merely bounded (known or assumed to have occurred before some specific date). How can we calculate relevant event probabilities for these uninitiated fault segments?

A steady-state solution for the time to failure for a BRO is derived in Appendix 2. In this model, a particular load-state density for distance to failure is chosen as the state of

the BRO at the backward time horizon. For this initial load state, the instantaneous failure rate equals the rate for a Poisson process with the same μ . The load state evolves with time toward the quasi-stationary density in equation (36). Figure 8 compares the failure-time probability density for the uninitiated BRO with the exponential distribution and a BPT model with the same μ and α . Also shown are the corresponding hazard-rate functions. Note that the hazard for the uninitiated BRO evolves from the Poisson hazard rate to the asymptotic value of the BRO equation (24). For the choice of $\alpha = 0.5$, the time elapsed since the backward horizon must be at least $3\mu/2$ to obtain a reasonable approximation of the hazard rate from its asymptotic value.

The steady-state probability preserves, on uninitiated faults, the assumptions and framework applied to faults for which t_0 is known. Hence, this approach brings to cases with incomplete data the full insight and behavioral characterization both derived from and applied to cases with more complete data.

Interactions

Thus far, analysis has revolved around representing the load state of a recurrent source by a stochastic point process in which the only deterministic component is a linear trend. This trend is conceived to represent macroscopic tectonic loading, and all other factors contributing to onset or delay of eventual rupture are lumped into “random” motion. Earthquakes outside the target rupture area may be a primary source of perturbations. Because these earthquakes are generally recorded, their effects might be incorporated explicitly as deterministic factors rather than random fluctuations. This idea relates to the familiar notions of interaction, triggering, and time delay and advance (Harris and Simpson, 1992; Cornell *et al.*, 1993). Cornell *et al.* (1993) present a “phenomenological stochastic model” of mechanically interactive fault segments that captures the essential elements. Their basic model centers on a one-dimensional state variable that is supposed to represent the “stress state” on a recurrent source. For a fault segment “in isolation,” the stress state increases at constant rate, as in our deterministic oscillator. Cornell *et al.* calculate at the target source stress perturbations (which may be positive or negative) due to large earthquakes occurring on other fault segments and add these to the in-isolation loading function. They point out that even deterministic systems of several “mechanically interactive” segments may produce complex temporal rupture patterns that exhibit characteristics of “randomness,” but they do not pursue this point. Instead, they use a Weibull recurrence model for illustration, mentioning the lognormal as an alternative, and model perturbation effects as clock delays or advances—decreases or increases in effective elapsed time.

Interaction in the context of the Brownian relaxation oscillator can be directly incorporated into the theory by adding a step discontinuity to the load state of the phase distance-to-failure process. Addition of a step to the loading

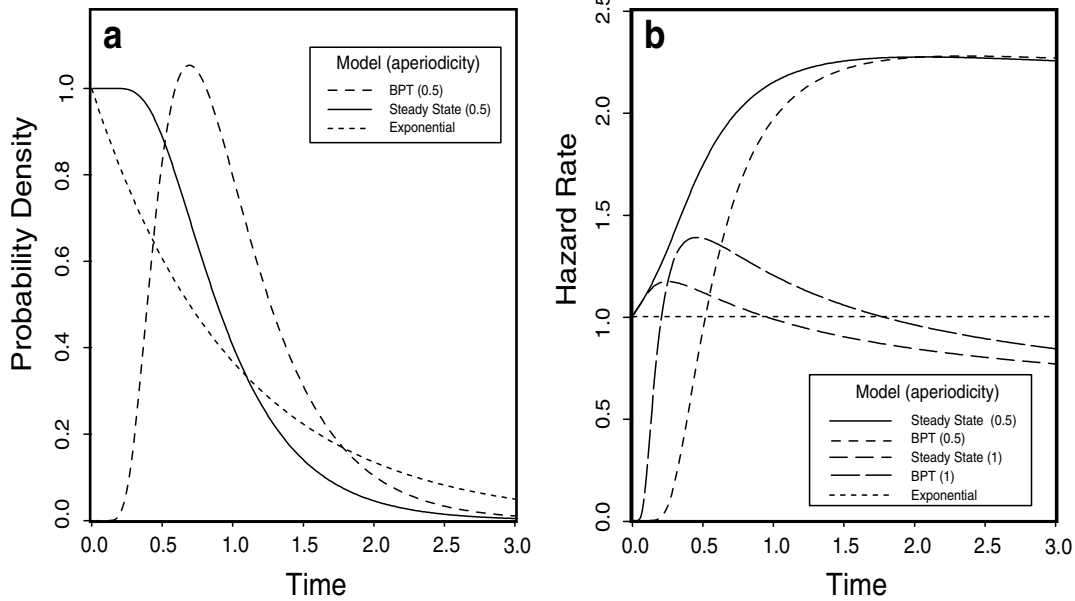


Figure 8. Comparison of density functions (a) and hazard functions (b) for the uninitialized BPT model with the BPT model and exponential model. Mean time is 1 for all models. Aperiodicity of 0.5 and 1.0 is specified.

process will produce one of the three effects illustrated in Fig. 9. A negative step gives an effective delay in recurrence time by reducing the load. A positive step will immediately trigger rupture of a source sufficiently close to failure or else produce a time advance. The precise nature of interaction effects is calculated as in equation (39), by averaging the passage-time density weighted by the phase distance-to-failure distribution.

Suppose a step of size x_s occurs at phase τ_s , and consider the residual life variable $V(\tau_s; x_s)$, which represents remaining time to failure at τ_s , conditioned on adding x_s to $X(\tau_s)$. The probability distribution of $V(\tau_s; x_s)$ may be a pure density obtained by mixing BPT densities, as in equation (39), or it may have positive mass at zero, corresponding to the probability of triggering.

A positive step of size x_s at time τ_s triggers an event instantaneously if and only if $Y(\tau_s) \leq x_s$. The probability of this event is given by the complementary distribution function in equation (32):

$$\begin{aligned}
 P\{\text{trigger}|\tau_s, x_s\} &\triangleq P\{V(\tau_s; x_s) = 0\} \\
 &= P\{Y(\tau_s) \leq x_s\} \\
 &= G(x_s, \tau_s)
 \end{aligned}
 \tag{42}$$

Figure 10 shows contours of BRO(1, 1, 1/2) triggering probabilities as a function of step time and size. Steps sizes range from 0 to 1/4 of failure load, covering the range of effects that might be observed in typical seismological applications. Triggering probabilities for step sizes in this range are small for early phases. For phases beyond the mean, the contours

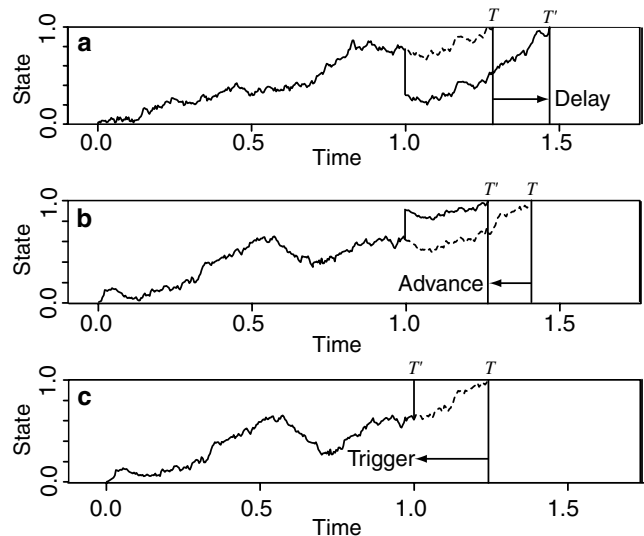


Figure 9. Step effects in Brownian relaxation oscillator. Solid lines show actual state process path. Dotted lines continue path that would have been followed without step. T' is the observed recurrence time, with step in loading process. T is recurrence time that would have been observed without step. Delay caused by negative step (a); advance caused by positive step (b); and trigger (c).

are nearly flat and parallel, again reflecting onset of the time-independent quasi-stationary regime.

If a positive step does not trigger an event, then we know that $Y(\tau_s) > x_s$, and we may write the conditional residual life density as

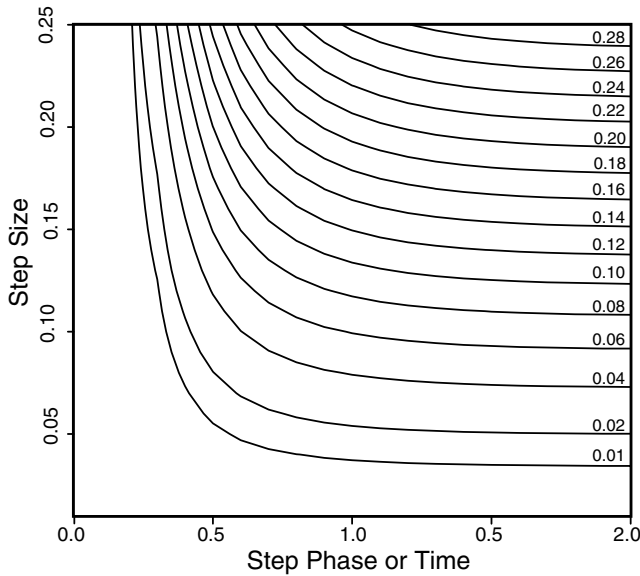


Figure 10. Contours of BRO (1, 1, 1/2) triggering probability vs. step size (expressed as fraction of failure load) and time of step in cycle (phase).

$$\tilde{f}_{\tau_s}(t|x_s, V(\tau_s, x_s) > 0) = \frac{\int_{x_s}^{\infty} f(t; x - x_s, \lambda, \sigma)g(x, \tau_s)dx}{G(x_s, \tau_s)} \quad (43)$$

A negative step in the state process corresponds to a positive step in the distance-to-failure process. If $Y(\tau_s)$ jumps from x to $x + x_s$, the density of $V(\tau_s, -x_s)$ is the mixture

$$\tilde{f}_{\tau_s}(t - x_s) = \int_0^{\infty} f(t; x + x_s, \lambda, \sigma)g(x, \tau_s)dx \quad (44)$$

Figure 11 illustrates the effects of interaction steps on the hazard-rate function of a BRO(1, 1, 1/2) process. In the figure, the thick solid line is the unperturbed hazard rate, and the thin solid lines are hazard rates estimated by the “clock change” method of adjusting the time of the last event by $\pm 10\%$. Dotted and dashed curves correspond to positive and negative steps computed using equations (43) and (44), respectively. The salient facts to be discerned in this depiction are that interaction effects are transient and that their qualitative nature depends on their timing. Early in the recurrence cycle, a step with magnitude equal to 10% of failure load roughly produces a parallel shift in the hazard-rate curve, generating an effective advance or delay in elapsed time, as in the analysis of Cornell *et al.* (1993). At about the mean recurrence time, however, a positive step does not simply advance the clock; it produces a sharp, temporally localized increase in rupture probability. Similarly, a negative step near recurrence time effects a sharp, localized reduction in hazard rate.

This behavior is intuitive. Early in the recurrence cycle, the source has not had sufficient time to reload, and a rela-

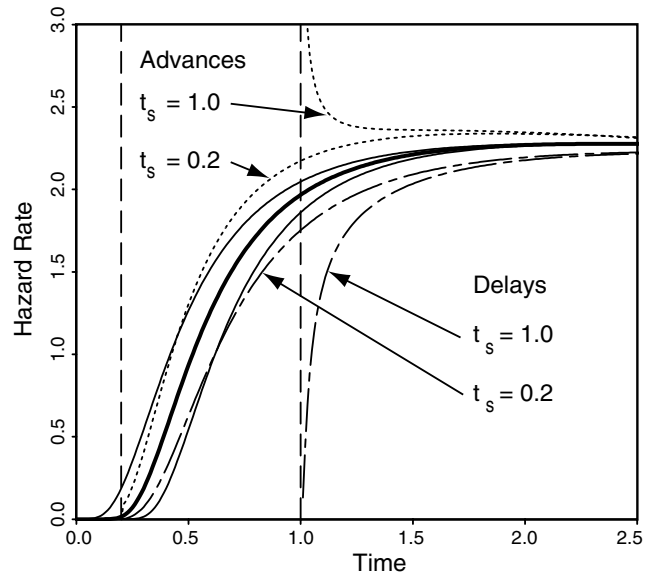


Figure 11. Effects of interaction steps on BRO (1, 1, 1/2) hazard rate. The solid heavy curve is the hazard-rate function for the unperturbed state. Dashed and dotted curves are hazard rates for state perturbed by interaction steps of size $x_s = 0.1$ at phases $t_s = 0.2, 1.0$. Dotted curves correspond to positive steps, dashed curves to negative steps. Solid light lines are hazard rates for “clock advance” (upper) and “clock delay” (lower) of ± 0.1 , which are horizontal translations of the unperturbed hazard rate.

tively small increase or decrease in load will effectively set the clock forward or backward by an amount about equal to mean time for the load to increase by the size of the step. Near the expected rupture time, the load should be sufficiently large that the source is close to rupturing. An abrupt increase in the accumulated load will either push the source over its failure threshold or very near to it, generating an immediate, sharp increase in rupture probability. Likewise, an abrupt decrease will draw the load down, away from the failure threshold, substantially decreasing rupture probability. Over time, conditionally on rupture not occurring, background noise due to the Brownian perturbation process will gradually overtake the deterministic effect of an interaction step, and rupture probabilities will converge back to background levels. Hence, interaction effects in the Brownian passage-time model are transient, and “decay time” depends on the diffusion rate of the background perturbation process relative to the step size.

Physical Interpretations

Brownian relaxation oscillators and Brownian passage-time distributions together afford a dual view of recurrent rupture on a point source. Passage-time distributions describe probabilistic behavior for observable, time-valued variables, that is, earthquake times. Relaxation oscillators describe probabilistic behavior for unobservable, state-

valued variables governing physical rupture processes. We have constructed the load state as a formal variable without precise physical connotation. This view is supported by invariance of the passage-time distribution with respect to affine transformations of state space and also leaves room for various, macroscopically equivalent interpretations. Reid's (1910) original recurrence formulation based on strain accumulation is one view. The relaxation oscillator is equally well a conceptual depiction of Coulomb failure cycles in which effective stress steadily increases until reaching the Coulomb threshold. A perspective viewing the same picture in terms of displacement or moment budget is also valid and fits well into elastic-dislocation models, where the source function for crustal deformation is a distribution of slip on a fault (e.g., Segall and Harris, 1986).

In actuality, the physical quality—strain, stress, or displacement—central to each model interpretation is a spatially variable tensor or vector field. How do we reconcile these fields with the point-process concept? What functional of a nonuniform spatial field is the appropriate one-dimensional summary, the average, the maximum, the value at a specific point (e.g., the eventual nucleation point)? Vexing questions such as these remain hidden when a model is chosen on arbitrary grounds of convenience or familiarity, but they lurk under the surface. The recurrent source is a conceptual idealization, not a tangible object. Perhaps no earthquake ever occurs twice, but sequences of very similar earthquakes do occur (Vidale *et al.*, 1994; Ellsworth, 1995; Nadeau *et al.*, 1995). Modeling time behavior of these sequences by cyclic elastic rebound may generate both theoretical insight and practical assessments of seismic risk as a function of time.

Conceptions of the load-state variable pertain not only to physical interpretations of the deterministic relaxation oscillator but also to how randomness and probabilities enter the picture. The load-state variable in the Brownian relaxation oscillators fluctuates. It can and does move in the sense opposite to tectonic loading and eventual displacement on the fault. The fact that the load state can decrease from its postseismic ground level may seem troubling, but several points must be borne in mind. First, this model idealizes a complex areal or volumetric source to a point. All spatio-temporal fluctuations that may affect the source over the course of a loading cycle must be mapped into the load-state variable. Second, the source generally exists in an area of complex spatial deformation and prolific seismicity. Phenomena such as chemical strengthening or weakening of the fault via solution transport, aftershocks, background seismicity, postseismic strain diffusion, and aseismic relaxation can counteract tectonic forcing and drive down the effective load. Third, there must be a place for randomness in any probabilistic formulation, and in light of the previous point, the loading function would seem to be a good place to put it. Fourth, the scale of possible negative state fluctuations is governed by the diffusion-rate parameter σ^2 , which is typically small relative to the loading rate. Fifth and finally, neg-

ative fluctuations may be eliminated without effect on modeled event-time behavior if we apply a “ratchet” to the Brownian loading process. The ratchet replaces the value of the load state, $Y(t)$, with its cyclic maximum. Figure 12 shows a Brownian relaxation oscillator and the corresponding Brownian ratchet relaxation oscillator. The ratchet process either increases or stays flat, but its passage time to the failure threshold is the same as for its parent process.

Discussion

We have introduced the Brownian relaxation oscillator as a stochastic modification of the deterministic relaxation oscillator model for characteristic earthquakes and identified the Brownian passage time as its recurrence distribution. The Brownian passage-time family differs from the lognormal and other “usual” candidate distributions for long-term earthquake forecasting in that it reflects physical properties underlying earthquake generation, although perhaps only in a gross and simplistic fashion. The model has been applied by the Working Group on California Earthquake Probabilities (1999) and the Earthquake Research Committee (2001) to long-term earthquake forecasts for the San Francisco Bay region and inland faults in Japan, respectively.

The Brownian relaxation oscillator and Brownian passage-time distribution together connect physical notions of largely unobservable loading and failure processes of a point process with observable statistics of event recurrence in time. This connection leads to predictions about the dependence of recurrence times on underlying physical parameters. This connection may be useful in both directions; that is, physical interpretations of the Brownian oscillator may generate testable predictions regarding recurrence behavior, and observed features of recurrence distributions may translate to physical interpretations.

Here is an example to illustrate questions that may be raised. The Brownian oscillator says that the variance in recurrence intervals is $\sigma_T^2 = \sigma^2 \delta / \lambda^3$ equation (9). Suppose that

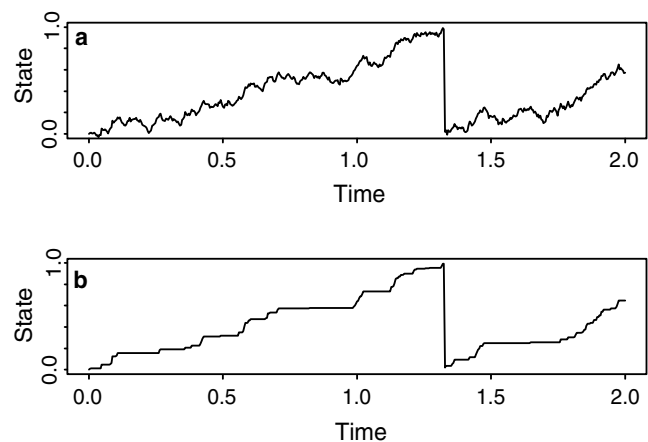


Figure 12. Brownian load-state path (a) and derived Brownian ratchet (b).

the oscillator is a model for stress, λ is the mean stress rate, δ is the stress drop, and σ^2 is the mean stress-diffusion rate. If stress diffusion is due to external seismicity, which occurs with frequency proportional to the loading rate, then $\sigma^2 = \rho\lambda$, where ρ is the mean stress perturbation on the target source from external seismicity. Hence,

$$\sigma^2 = \frac{\rho\lambda\delta}{\lambda^3} = \frac{\rho\delta}{\lambda^2} \quad (45)$$

Now, if ρ and δ are spatially invariant, as would be expected from invariance of the Gutenberg–Richter relation, then the variance in recurrence times will be proportional to λ^{-2} , the standard deviation will be proportional to λ^{-1} , and the aperiodicity α will be constant. Nishenko and Buland (1987), McCalpin and Slemmons (1998), and Ellsworth *et al.* (1999) found empirically that indeed, this parameter has approximately the same value for recurrent sources that differ greatly in location, rupture mechanism, mean recurrence time, and characteristic event magnitude.

Because the recurrence-time distribution of the Brownian oscillator is invariant to affine transformations of load-state space, the model is a formal representative for any load-state process that exhibits mean linear growth and resets to a ground state at the time of an event. It is most natural, perhaps, to conceive that the process represents effective stress reaching the Coulomb threshold at event times and then dropping to a characteristic ground level as rupture relaxes the source region. However, among interesting variables associated with recurrent earthquakes, stress is not the only one that will behave in this manner. Strain and moment are two alternative state variables. Local strain energy and moment deficit both, on average, grow linearly with time, reach a characteristic maximum value at rupture, and renew after rupture.

The Brownian ratchet is a modification of the Brownian loading process that may be more physical and more consistent with the observed behavior of interseismic strain, stress, or displacement. This concept of the load state produces the same recurrence-time distribution as the Brownian model. The idea that other processes and failure criteria may be more physically realistic presents an avenue for variations on the Brownian oscillator theme. The Brownian relaxation oscillator produces events on reaching a static failure threshold. A dynamic failure criterion may be preferred. As an example, suppose that a smooth perturbation is substituted for Brownian motion, and we say that failure occurs when the (stochastic) loading rate exceeds some threshold. This resulting model is a simple stochastic formulation of velocity-weakening friction. The failure-time distribution of this process might be studied analytically or by Monte Carlo simulation.

The Brownian relaxation oscillator may be extended to earthquake models that are not renewal processes. In particular, stochastic versions of the familiar time-predictable and

slip-predictable earthquake models (Shimazaki and Nakata, 1980) derive from randomizing boundary conditions in the Brownian oscillator. Suppose that the size of the event is proportional to δ , the distance between the failure state and ground state. If, rather than failing at fixed level δ , the oscillator fails at a random level selected anew in each failure cycle from some specified probability distribution and returns to a fixed ground state, the model is stochastically slip predictable. On the other hand, if failure always occurs at a fixed level but the ground level is randomized, the model is stochastically time predictable. Rikitake's (1974) and Hagiwara's (1974) stochastic models for "ultimate strain" randomized failure levels but not load state.

It may be the case that a model that randomizes both the failure level and load state will be more physically realistic than a model that randomizes one or the other. Incorporating randomized boundaries into our Brownian failure model would have the effect of decreasing the scale of Brownian perturbations by assigning some variation in recurrence intervals to variability in boundary conditions. Computations are conceptually straightforward. The recurrence-interval distribution will be a mixture of Brownian passage-time distributions with mixing weights determined by the boundary conditions. The drawbacks in pursuing this course are that it requires specification of boundary-value distributions, it complicates estimation and analysis, and it makes empirical discrimination among competing models that much more difficult.

There are numerous potential variations on our model in directions other than randomized boundary values. Among the most immediately interesting are a spatial distribution of coupled Brownian relaxation oscillators and "regression" of the Brownian (or some other stochastic) load state on observed local seismicity and geodetic data. With a spatial distribution of oscillators, one might extend the interaction calculations that we have presented here to cases where a collection of sources interact and derive, by analysis or simulation, the statistics of rupture times on the ensemble system of sources. The idea of regression analysis is to use observed crustal deformation data and local earthquakes to estimate in real time the load state on a potential source of large earthquakes. The model and calculations we have presented here may be a basis for carrying out such a program.

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Appendix 1

The Problem with Data-Dependent Normalization

It is very tempting to normalize a short series of observations by their mean value to facilitate comparison between the series by putting them on a common scale. Making this data-dependent transformation is quite different from simply rescaling. Suppose X_1, X_2, \dots, X_n is a random sample of interval lengths. Let $Y_i = X_i/\bar{X}$, where \bar{X} is the mean of the X_i . We wish to infer the shape of the distribution of X_i by looking at the empirical distribution of the Y_i . However, Y_i has a distribution with a shape much different from that of a deterministically rescaled version of X_i . The difference is particularly acute when the sample size n is small. Exact analysis for the case when X_i are independent and identically distributed $\exp(\lambda)$ shows that the marginal “survival function” of Y_i has the form

$$p_n(y) \triangleq P \{Y_i > y\} = \begin{cases} \left(1 - \frac{y}{n}\right)^{n-1}, & 0 \leq y \leq n \\ 0, & \text{otherwise} \end{cases} \quad (46)$$

Nishenko and Buland’s (1987) implicit assumption is that the distribution in question should be $\exp(1)$, for which the survival function is e^{-y} , $y \geq 0$. As n increases in equation (46), $p_n(y) \rightarrow e^{-y}$ for all y , but for small n , there is marked distinction between $p_n(y)$ and its large-sample limit. In the $n = 2$ case, for instance, which includes a third of Nishenko and Buland’s training sequences, the normalized intervals have a uniform marginal distribution on $[0, 2]$.

When observations are from a nonexponential distribution, the normalized intervals will not have the distribution in equation (46). This type of normalization, however, will always generate a qualitatively poor approximation to the desired distribution.

Even when normalization problems are ignored, we assert that the attempt to discriminate among models by conventional “curve fitting” and goodness-of-fit tests is nearly useless. To appreciate this point, consider the Weibull and lognormal families, along with three additional families of candidate renewal distributions: exponential, gamma, and inverse Gaussian or BPT.

Suppose we observe a renewal process generated by one of these four candidate distributions. We wish to determine the correct one by using a goodness-of-fit test. Figure 1 portrays a set of four pdfs and their corresponding densities. The set is made up of one distribution from each of the four parametric families under consideration. Parameter values for each distribution are chosen so that the four have matching first and second moments. Can we see the differences among these four distributions in samples? The answer is no, at least not with sample sizes that are typical in earthquake analysis.

This is borne out by the results in Table A1: each distribution furnishes the best empirical fit to samples from each distribution. Random samples of size $n = 50$ (much larger than most earthquake data sets) were generated from each candidate distribution, and the sample distribution function was compared with the four candidates by computing the Kolmogorov–Smirnov statistic (maximum vertical discrepancy between cdfs). For each sample, whichever distribution had the smallest Kolmogorov–Smirnov value was deemed the “best fit.” Table 2 illustrates the difficulty of empirical discrimination among these five distributions. These sequences may contain enough information to distinguish the one-parameter exponential model from two-parameter alternatives, but hope for any elucidation beyond that level is futile. We should take some encouragement in this, as whether or not earthquake recurrence has a “memory” is a central question in probabilistic hazards assessment.

Table A1
Goodness-of-Fit Test Results

Sample from	Best Fit				
	BPT	Lognormal	Gamma	Weibull	Exponential
BPT	41.7	24.9	18.1	15.3	0
Lognormal	17.2	77.9	3.9	1.0	0
Gamma	24.8	16.1	25.7	33.3	0.1
Weibull	12.2	10.3	22.1	55.0	0.4
Exponential	0	0.2	0.6	2.3	96.9

Row indicates sampling distribution, and column shows best-fitting theoretical candidate. Entries are percentage frequencies, by sampling distribution, that each candidate distribution provided the best fit. These percentages were computed by generating 1000 samples of size 50 from each candidate distribution and selecting the best fit for each of these 5000 samples by minimizing the Kolmogorov–Smirnov misfit.

Appendix 2

Derivation of the Steady-State Model

Let $N(t)$ denote the cumulative number of events in the time interval (t_0, t) . We are interested in the distribution of random variables $N(t + \Delta) - N(t)$, where typically t is the “present” and Δ is the time span of interest (e.g., 30 years). In particular, the probability of zero events,

$$p_0(t, \Delta) \triangleq P\{N(t + \Delta) - N(t) = 0\} \quad (47)$$

is needed to generate the probability of one or more events by subtraction. The same probability may be stated in terms of the residual life. If $\tau(t) \triangleq t - t_0$ is the current renewal phase (i.e., time since the last event) and $Y_{\tau(t)}$ then the residual life (time remaining until the next event) at t has cdf

$$G_{\tau}(y) \triangleq P\{Y_{\tau} \leq y\} \quad (48)$$

then

$$p_0(t, \Delta) = P \{Y_{\tau(t)} > \Delta\} = 1 - G_{\tau(t)}(\Delta) \quad (49)$$

In most cases, the probability equivalently specified by equations (47) and (49) depends on the renewal phase τ . [An important exception occurs when renewal intervals have an exponential distribution. Then Y_{τ} has the same exponential distribution for all τ , $N(t)$ is a homogeneous Poisson process, and $p_0(t, \Delta)$ is constant as a function of t .]

Within the BPT/BRO framework, a steady-state model for time to failure is obtained as follows. Let $Z(t)$ denote the random state-distance-to-failure variable at time t , and consider the indicator random variables

$$I_z(t) = \begin{cases} 1, & \text{if } Z(t) \leq z \\ 0 & Z(t) > z \end{cases} \quad (50)$$

Let

$$D_{SS}(z) = \lim_{t \rightarrow \infty} \frac{I_z(t)}{t} \tag{51}$$

The steady-state distance-to-failure density is

$$d_{SS}(z) = \begin{cases} (1 - e^{-2\rho z})/\delta, & 0 \leq z \leq \delta \\ e^{-2\rho z}(e^{2\rho\delta} - 1)/\delta, & z > \delta \end{cases} \tag{52}$$

Set $\sigma = \alpha \sqrt{\lambda\delta}$. The BPT density with parameters δ, λ, σ is

$$f_T(t; \delta, \lambda, \alpha) = \frac{\delta}{\sqrt{2\pi\sigma^2 t^3}} \cdot \exp\left\{-\frac{(\delta - \lambda t)^2}{2\sigma^2 t}\right\}, t \geq 0 \tag{53}$$

Suppose at some time b (the backward horizon) we assume that the BRO is in a steady state, that is, we choose a random state such that distance to failure has the density in equation (52). Letting Q denote time to failure from b , the density of Q is

$$f_Q(q; \delta, \lambda, \alpha) = \int_{z=0}^{\infty} f_T(q; z, \lambda, \sigma) \cdot d_{SS}(z; \delta, \lambda, \alpha) dz \tag{54}$$

If there are no renewals in the interval $[b, \tau]$, then the residual life at τ has density

$$g_\tau(y; b, \delta, \lambda, \alpha) = \frac{f_Q(\tau - b + y; \delta, \lambda, \alpha)}{1 - F_Q(\tau - b; \delta, \lambda, \alpha)} \tag{55}$$

The probability density function in equation (55) defines probabilities for residual life, given that the system was started in the steady state at epoch b and no renewals exist in $[b, \tau]$. From this density, the desired probability in equation (49) is

$$p_0(\tau, \Delta) = 1 - G_\tau(\Delta; b, \delta, \lambda, \alpha) \tag{56}$$

To compute this probability, we may integrate equation (55) to get a value for G_τ . This task is simplified by a closed-form expression for the integral in defining f_Q in equation (54). This expression involves the standard Gaussian pdf.

$$\phi(x) \triangleq \frac{1}{\sqrt{2\pi}} e^{-x^2/2}$$

and cdf

$$\Phi(x) \triangleq \int_{-\infty}^x \phi(v) dv$$

Let

$$K(x; a, b) \triangleq a \cdot \left[\Phi\left(\frac{a}{b}\right) - \Phi\left(\frac{a-x}{b}\right) \right] + b \cdot \left[\phi\left(\frac{a}{b}\right) - \phi\left(\frac{a-x}{b}\right) \right] \tag{57}$$

Then

$$f_Q(q; \delta, \lambda, \alpha) = t^{-1} [K(\delta; \lambda t, \alpha\sqrt{\lambda\delta t}) - K(\delta; -\lambda t, \alpha\sqrt{\lambda\delta t}) + (e^{2/\alpha^2} - 1) \cdot (K(\infty; -\lambda t, \alpha\sqrt{\lambda\delta t}) - K(\delta; -\lambda t, \alpha\sqrt{\lambda\delta t}))] \tag{58}$$

Quasi-Steady-State Model

An approximate expression for $p_0(t, \Delta)$ may be obtained when a long time has elapsed relative to the mean recurrence time. Assume that interevent times are BPT (μ, α). If $\tau(t)/\mu$ is large, then the residual life is approximately exponential, with rate $\rho = (2\alpha^2\mu)^{-1}$ and

$$p_0(t, \Delta t) = \frac{e^{-\Delta/(2\alpha^2\mu)}}{e^{-\lambda \cdot \Delta/(2\alpha^2\delta)}} \tag{59}$$

The longer the elapsed time, the better the quasi-steady-state probability approximates the underlying BPT probability. How long is long enough depends on how much approximation error we can tolerate, and this depends on the value of α as well as on μ . If the backward horizon is small in comparison with μ , then the quasi-steady probability is suspect.

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