Control Synthesis of Singularly Perturbed Fuzzy Systems

Guang-Hong Yang, Senior Member, IEEE, and Jiuxiang Dong

Abstract—This paper considers the problem of designing stabilizing and $H_{\infty}$ controllers for nonlinear singularly perturbed systems described by Takagi–Sugeno fuzzy models with the consideration of the bound of singular perturbation parameter $\epsilon$. For the synthesis problem of simultaneously designing the bound of $\epsilon$ and stabilizing or $H_{\infty}$ controllers, linear matrix inequalities (LMI)-based methods are presented. For evaluating the upper bound of $\epsilon$ subject to stability or a prescribed $H_{\infty}$ performance bound constraint for the resulting closed-loop system, sufficient conditions are developed, respectively. For the stabilizing and $H_{\infty}$ control synthesis without the consideration of improving the bound of $\epsilon$, new design methods are also given in terms of solutions to a set of LMIs. Examples are given to illustrate the efficiency of the proposed methods.

Index Terms—$H_{\infty}$ performance, linear matrix inequalities (LMIs), nonlinear control systems, singularly perturbed systems, stabilizing control, state feedback control, Takagi–Sugeno (T-S) fuzzy models.

I. INTRODUCTION

IN CONTROL engineering applications, it is well known that the multiple time-scale systems or known as singularly perturbed systems often raise serious numerical problems. For the purpose of avoiding the difficulties linked with the stiffness of the equations involved in the design, the singular perturbation design method has been developed [1], where singular perturbation with a small parameter, say $\epsilon$, is exploited to determine the degree of separation between “slow” and “fast” modes of the system, and the so-called reduction technique is proposed to handle these systems. For the stabilization and $H_{\infty}$ control of linear singularly perturbed systems, many important advances have been achieved, see [1]–[6] and the references therein. In particular, the fundamental results are given in [1], [4], and [5].

In recent years, there have been some attempts to address the $H_{\infty}$ control problem for nonlinear singularly perturbed systems. In [7] and [8], the $H_{\infty}$ control for a class of singularly perturbed systems with nonlinearity in the slow variables is examined. A local state feedback $H_{\infty}$ control problem for affine nonlinear singularly perturbed systems is studied in [9]. However, the $H_{\infty}$ control design for general nonlinear singularly perturbed systems still remains as an open research subject.

An important approach to the synthesis problems for nonlinear systems is to model the considered system as Takagi and Sugeno (T-S) fuzzy systems [10], which are locally linear time-invariant systems connected by IF-THEN rules. In [11] and [12], it has shown that the T-S fuzzy systems can approximate any continuous functions at any preciseness, which shows that the T-S fuzzy models can approximate a wide class of nonlinear systems. As a result, the conventional linear system theory can be applied to analysis and synthesis of the class of nonlinear control systems. In recent years, the T-S fuzzy control systems have been studied extensively, and many significant advances have been achieved (see [13]–[15] and the references therein).

For nonlinear singularly perturbed systems, some control synthesis problems have been studied [16]–[19]. In [16] and [18], design methods for the stabilization and $H_{\infty}$ control of nonlinear singularly perturbed systems via state feedback are given in terms of solutions of linear matrix inequalities (LMIs) [20], respectively. A robust state feedback control design is presented in [17]. An LMI-based method of designing $H_{\infty}$ output feedback controllers for uncertain fuzzy singularly perturbed systems is presented in [19].

For the effective applications of the design methods for singularly perturbed control systems, the accurate knowledge of the stability bound $\epsilon^*$ of a singularly perturbed system (i.e., the system is stable for $\epsilon \in [0, \epsilon^*]$) is very important. The characterization and computation of the stability bound have attracted considerable efforts for the past over two decades [21]–[28]. In general, there are two classes of methods to characterize and compute the stability bounds, one is based on frequency domain transfer functions and another is based on state space models. Both of the two methods can provide the exact bounds as shown in [22], [25], and [26]. However, the issue of how to improve the bound $\epsilon^*$ in controller designs has not been addressed in the literature, which undoubtedly is very important for the applications of singularly perturbed system theory.

This paper is concerned with the problem of designing stabilizing and $H_{\infty}$ controllers for nonlinear singularly perturbed systems described by T-S fuzzy models with the consideration of the bound of singular perturbation parameter $\epsilon$. Two LMI-based methods are presented for simultaneously designing
the bound of $\epsilon$ and stabilizing or $H_\infty$ controllers for a fuzzy singularly perturbed system, respectively. For the issue of computing the bound of singularly perturbed parameter $\epsilon$, sufficient conditions are derived for evaluating the upper bound $\epsilon^*$ of $\epsilon$ subject to stability or a prescribed $H_\infty$ performance bound constraint for the resulting closed-loop system for $\epsilon \in (0, \epsilon^*]$, respectively, where the upper bound $\epsilon^*$ can be obtained by solving a generalized eigenvalue problem (GEVP) [20]. For the problem of designing stabilizing and $H_\infty$ controllers without the consideration of improving the bound of $\epsilon$, design methods are also given in terms of solutions to LMIs. The paper is organized as follows. In Section II, the system description, the considered problems, and preliminary lemmas are presented. Section III considers the problem of designing stabilizing controllers and the bound of $\epsilon$, and the case for $H_\infty$ controller design is studied in Section IV. Section V gives examples to illustrate the effectiveness of the new proposed methods. Finally, Section VI concludes this paper.

**Notation:** For a matrix $M$, $\| M \|$ is defined as the largest singular value of $M$. For a square matrix $E$, $H(E)$ is defined as

$$H(E) = E + E^T.$$ 

The symbol $*$ within a matrix represents the symmetric entries

$$[H_{ij}]_{r \times r}^* := \begin{bmatrix} H_{11} & H_{12} & \cdots & H_{1r} \\ H_{21} & H_{22} & \cdots & H_{2r} \\ \vdots & \vdots & \ddots & \vdots \\ H_{r1} & H_{r2} & \cdots & H_{rr} \end{bmatrix}.$$ 

**II. SYSTEM DESCRIPTION AND PROBLEM STATEMENT**

**A. System Description**

The class of nonlinear singularly perturbed systems under consideration is described by the following fuzzy system model:

**Plant Rule i:**

IF $v_1(t)$ is $M_{i1}$ and $v_2(t)$ is $M_{i2}$, ..., $v_p(t)$ is $M_{ip}$

THEN

$$\begin{align*}
\dot{x}_1(t) &= A_{11}^i x_1(t) + A_{12}^i x_2(t) + B_{11}^i u(t) + B_{12}^i u(t) \\
\dot{x}_2(t) &= A_{21}^i x_1(t) + A_{22}^i x_2(t) + B_{21}^i u(t) + B_{22}^i u(t) \\
z(t) &= C_{z1}^i x_1(t) + C_{z2}^i x_2(t) + D_{12}^i u(t)
\end{align*}$$

where $i = 1, 2, \ldots, r$, $M_{ik}$ are fuzzy sets, $v_i(t)$ are the premise variables, $x_1(t) \in \mathbb{R}^{m1}$ and $x_2(t) \in \mathbb{R}^{m2}$ are the state vectors, $u(t) \in \mathbb{R}^m$ is the control input, $w(t) \in \mathbb{R}^k$ is the disturbance, $z(t) \in \mathbb{R}^r$ is the controlled output, the matrices $A_{11}^i, A_{12}^i, A_{21}^i, A_{22}^i, B_{11}^i, B_{12}^i, B_{21}^i, B_{22}^i, C_{z1}^i, C_{z2}^i$ and $D_{12}^i$ are of appropriate dimensions, $r$ is the number of IF-THEN rules, and $\epsilon > 0$ is a small constant.

Denote

$$w_i(v(t)) = \prod_{j=1}^{n} M_{ij}(v_j(t)),$$

$M_{ij}(v_j(t))$ is the grade of membership of $v_j(t)$ in $M_{ij}$, where it is assumed that

$$\sum_{i=1}^{r} w_i(v(t)) > 0, \quad w_i(v(t)) \geq 0, \quad i = 1, 2, \ldots, r.$$ Let

$$\alpha_i(v(t)) = \frac{w_i(v(t))}{\sum_{i=1}^{r} w_i(v(t))}$$

then

$$0 \leq \alpha_i(v(t)) \leq 1, \quad \sum_{i=1}^{r} \alpha_i(v(t)) = 1. \quad (2)$$

$\alpha_i(v(t))$ is said to be normalized membership functions. The T-S fuzzy model of (1) is inferred as follows:

$$\begin{align*}
\dot{x}_1(t) &= \sum_{i=1}^{r} \alpha_i(v(t))(A_{11}^i x_1(t) + A_{12}^i x_2(t) + B_{11}^i u(t) + B_{12}^i u(t)) \\
\dot{x}_2(t) &= \sum_{i=1}^{r} \alpha_i(v(t))(A_{21}^i x_1(t) + A_{22}^i x_2(t) + B_{21}^i u(t) + B_{22}^i u(t)) \\
z(t) &= \sum_{i=1}^{r} \alpha_i(v(t))(C_{z1}^i x_1(t) + C_{z2}^i x_2(t) + D_{12}^i u(t)).
\end{align*}$$

(3)

The system (3) can be rewritten as follows:

$$\begin{align*}
E_{e}\dot{x}(t) &= \sum_{i=1}^{r} \alpha_i(v(t))(A_i \dot{x}(t) + B_{2i} u(t) + B_{1i} w(t)) \\
z(t) &= \sum_{i=1}^{r} \alpha_i(v(t))(C_{1i} \dot{x}(t) + D_{12i} u(t))
\end{align*}$$

(4)

where

$$x(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix},$$

$$A_i = \begin{bmatrix} A_{11}^i & A_{12}^i \\ A_{21}^i & A_{22}^i \end{bmatrix},$$

$$B_{1i} = \begin{bmatrix} B_{11}^i \\ B_{12}^i \end{bmatrix},$$

$$B_{2i} = \begin{bmatrix} B_{21}^i \\ B_{22}^i \end{bmatrix},$$

$$C_{1i} = \begin{bmatrix} C_{z1}^i \\ C_{z2}^i \end{bmatrix},$$

$$D_{12i} = D_{12}^i,$$

$$E_{e} = \begin{bmatrix} I & 0 \\ 0 & \epsilon I \end{bmatrix}.$$ 

In this paper, the concept of parallel distributed compensation (PDC) is used to design fuzzy controller, i.e., the designed fuzzy controller shares the same fuzzy sets with the fuzzy model in
the premise parts (more details can be found in [13]). For the fuzzy model (1), the following state feedback controller [13] is adopted:

**Controller Rule i:**

IF \( v_1(t) \) is \( M_{11} \) and \( v_2(t) \) is \( M_{12}, \ldots, v_p(t) \) is \( M_{1p} \)
THEN \( u(t) = K_i x(t) \).  

(5)

Because the controller rules are same as plant rules, we obtain the state feedback controller as follows:

\[ u(t) = \sum_{i=1}^{r} \alpha_i(v(t))K_i x(t). \]

(6)

Combining (6) and (4), then the resulted closed-loop system is given as follows:

\[
E_{ex}(t) = \sum_{i=1}^{r} \sum_{j=1}^{r} \alpha_i(v(t))\alpha_j(v(t)) \times \left( (A_i + B_{2j}K_j)x(t) + B_{1i}w(t) \right) \\
z(t) = \sum_{i=1}^{r} \sum_{j=1}^{r} \alpha_i(v(t))\alpha_j(v(t)) \times \left( (C_{1i} + D_{12j}K_j)x(t) \right).
\]

(7)

**B. Problem Statement**

In this paper, the following problems will be addressed.

- **Controller Design With Consideration of Bound of \( \epsilon \):**
  1. Find \( K_i (i = 1, \ldots, r) \), and an \( \epsilon^* > 0 \) as big as possible such that the closed-loop system (7) with \( u(t) = 0 \) is asymptotically stable for any \( \epsilon \in (0, \epsilon^*) \) and all \( \alpha_i(v(t)) \) satisfying (2).
  2. Let \( \gamma > 0 \) be a given constant. Find \( K_i (i = 1, \ldots, r) \), and an \( \epsilon^* > 0 \) as big as possible such that the closed-loop system (7) is asymptotically stable and with an \( H_\infty \)-norm less than or equal to \( \gamma \) for any \( \epsilon \in (0, \epsilon^*) \) and all \( \alpha_i(v(t)) \) satisfying (2).

- **Evaluation of Bound of \( \epsilon \):**
  1. Let \( K_i (i = 1, \ldots, r) \) be given. Find an \( \epsilon^* > 0 \) as big as possible such that the closed-loop system (7) with \( u(t) = 0 \) is asymptotically stable for any \( \epsilon \in (0, \epsilon^*) \) and all \( \alpha_i(v(t)) \) satisfying (2).
  2. Let \( \gamma > 0 \) and \( K_i (i = 1, \ldots, r) \) be given. Find an \( \epsilon^* > 0 \) as big as possible such that the closed-loop system (7) is asymptotically stable and with an \( H_\infty \)-norm less than or equal to \( \gamma \) for any \( \epsilon \in (0, \epsilon^*) \) and all \( \alpha_i(v(t)) \) satisfying (2).

**Remark 1:** Problems (i) and (ii) are concerned with simultaneously designing \( K_i (i = 1, \ldots, r) \) and finding the upper bound of \( \epsilon \) with guaranteeing the stability and \( H_\infty \) performance of the closed-loop system (7), respectively. Problems (iii) and (iv) are related to the problem of finding the upper bound of \( \epsilon \) subject to that the closed-loop system (7) is asymptotically stable and with the constraint of an \( H_\infty \) performance bound when \( K_i (i = 1, \ldots, r) \) are given. Moreover, the controller design problems without the consideration of bound of \( \epsilon \) will also be studied in Sections III and IV, respectively.

**C. Preliminaries**

The following preliminaries will be used in the sequel.

For the fuzzy control system (7), let \( \gamma > 0 \) be a constant. If (7) is asymptotically stable, and for any \( w(t) \in L^2[0, \infty) \) (the space of square integrable functions) and \( x(0) = 0 \), the following inequality holds:

\[
\int_0^\infty z^T(t)z(t) dt \leq \gamma^2 \int_0^\infty w^T(t)w(t) dt
\]

then the system (7) is said to be with an \( H_\infty \)-norm less than or equal to \( \gamma \) [29], [30].

Denote

\[
A_{ee} = \sum_{i=1}^{r} \sum_{j=1}^{r} \alpha_i \alpha_j E^{-1}_e (A_i + B_{2j}K_j)
\]

\[
B_{ee} = \sum_{i=1}^{r} \alpha_i E^{-1}_e B_{1i}
\]

\[
C_{ee} = \sum_{i=1}^{r} \sum_{j=1}^{r} \alpha_i \alpha_j (C_{1i} + D_{12j}K_j)
\]

then the closed-loop system (7) can be rewritten as follows:

\[
\dot{x}(t) = A_{ee} x(t) + B_{ee} w(t)
\]

\[
z(t) = C_{ee} x(t).
\]

(8)

The following lemma from [29] gives a sufficient condition for the system (8) to be with an \( H_\infty \)-norm less than or equal to \( \gamma \).

**Lemma 1:** [29] Consider the system (8). If there exists a positive definite matrix \( P \) such that

\[
A^T_{ee} P + PA_{ee} + PB_{ee} B_{ee}^T + \frac{1}{2\gamma^2} C_{ee}^T C_{ee} > 0
\]

holds. Then the system (8) is asymptotically stable and with an \( H_\infty \)-norm less than or equal to \( \gamma \).

**Lemma 2:** If there exist symmetric matrices \( J_{ij}^l, j \neq i, 1 \leq i \leq r, 1 \leq j \leq r \), and matrices \( J_{ij}^l, 1 \leq i < j \leq r, 1 \leq l \leq r \) such that the following LMIs hold:

\[
J_{ij}^l + J_{ji}^l + (J_{ij}^l)^T > 0, \quad 1 \leq i < j \leq r
\]

\[
J_{jj}^l + J_{ij}^l + (J_{ij}^l)^T > 0, \quad 1 \leq i < j \leq r
\]

\[
J_{ij}^l + J_{ji}^l + (J_{ji}^l)^T > 0, \quad 1 \leq i < j < l \leq r
\]

where

\[
J^l = \begin{bmatrix} J_{ij}^l \end{bmatrix}_{r \times r}
\]

\[
J_{ii}^l = 0
\]

\[
J_{ij}^l = (J_{ji}^l)^T
\]

\[
1 \leq i < j \leq r, \quad 1 \leq l \leq r
\]

then

\[
\begin{bmatrix} \alpha_{11} \vdots \alpha_{r1} \end{bmatrix}^T \sum_{l=1}^{r} \alpha_l J^l \begin{bmatrix} \alpha_{11} \vdots \alpha_{r1} \end{bmatrix} \geq 0.
\]

(12)
Proof:

\[
\begin{bmatrix}
\alpha_1 I \\
\vdots \\
\alpha_r I
\end{bmatrix}^T \sum_{l=1}^r \alpha_l J_l 
\begin{bmatrix}
\alpha_1 I \\
\vdots \\
\alpha_r I
\end{bmatrix}
= \sum_{l=1}^r \alpha_l \begin{bmatrix}
\alpha_1 I \\
\vdots \\
\alpha_r I
\end{bmatrix}^T \begin{bmatrix}
J_{11} & \cdots & J_{1r} \\
\vdots & \ddots & \vdots \\
J_{r1} & \cdots & J_{rr}
\end{bmatrix} \begin{bmatrix}
\alpha_1 I \\
\vdots \\
\alpha_r I
\end{bmatrix}
\]

\[
= \sum_{l=1}^r \sum_{j=l+1}^r \alpha_l^2 \alpha_j \begin{bmatrix}
J_{ii}^l + (J_{ij}^l)^T + J_{ji}^l \\
\vdots \\
J_{rr}^l + (J_{ij}^l)^T + J_{ji}^l
\end{bmatrix}
+ \sum_{l=1}^r \sum_{j=l+1}^r \alpha_l \alpha_j \alpha_k \operatorname{He} \begin{bmatrix}
J_{ii}^l + P_{il} + J_{ji}^l \\
\vdots \\
J_{rr}^l + P_{il} + J_{ji}^l
\end{bmatrix}.
\]

From (9) to (11), it follows (12).

Lemma 3: [16] If there exist matrices \( X \) and \( L_i, 1 \leq i \leq r \), with

\[
X = \begin{bmatrix}
X_{11} & 0 \\
X_{21} & X_{22}
\end{bmatrix}
\]

where \( X_{11} \) and \( X_{22} \) are symmetric matrices, satisfying the following LMIs:

\[
X_{11} > 0, \quad X_{22} > 0 \quad (13)
\]

\[
\operatorname{He}(A_i X + B_{2i} L_j + A_j X + B_{2j} L_i) < 0, \quad 1 \leq i \leq j \leq r \quad (14)
\]

then there exists a scaler \( e^* > 0 \) such that, for \( e \in (0, e^*] \), the state feedback controller (6) with

\[
K_i = L_i X_i^{-1}, \quad 1 \leq i \leq r
\]

renders the singularly perturbed fuzzy system (7) asymptotically stable.

Lemma 4: [18] For given \( \gamma > 0 \), if there exist matrices \( X \) and \( L_i, 1 \leq i \leq r \), with

\[
X = \begin{bmatrix}
X_{11} & 0 \\
X_{21} & X_{22}
\end{bmatrix}
\]

where \( X_{11} \) and \( X_{22} \) are symmetric matrices, satisfying the following LMIs, shown in (15) and (16) at the bottom of this page, then there exists a scaler \( e^* > 0 \) such that, for \( e \in (0, e^*] \), the state feedback controller (6) with

\[
K_i = L_i X_i^{-1}, \quad 1 \leq i \leq r
\]

renders the singularly perturbed fuzzy system (7) with an \( H_\infty \) norm less than \( \gamma \).

III. STABILITY BOUND AND STABILIZATION

In this section, a method of simultaneously designing the upper bound of \( \epsilon \) and stabilizing controller gains is derived where the upper bound of singularly perturbed parameter \( \epsilon \) can be improved, which provides a solution to Problem (i). For solving Problem (iii), a method of computing the upper bound of singularly perturbed parameter \( \epsilon \) subject to the stability of the closed-loop system is presented. Moreover, a technique for designing stabilizing controllers for singularly perturbed fuzzy systems without considering of improving the bound of singularly perturbed parameter \( \epsilon \) is also presented in Section III-C.

A. Design of Stability Bound of \( \epsilon \) and Stabilizing Controllers

The following theorem presents a method of simultaneously designing the upper bound of \( \epsilon \) and stabilizing controller gains.

Theorem 1: If there exist matrices \( X, L_i, J^{ik}_{ij}, J^{lk}_{ij}, A, 1 \leq i < j \leq r, J^{ik}_{ii}, J^{lk}_{ii}, 1 \leq i \neq k \leq r \), and positive scalars \( \beta_i, \beta_j, \beta_k, \beta_l \) with

\[
X = \begin{bmatrix}
X_{11} & 0 \\
X_{21} & X_{22}
\end{bmatrix}
\]

where \( X_{11}, X_{22}, A, J^{ik}_{ij}, J^{lk}_{ij}, 1 \leq i \neq k \leq r \) are symmetric matrices, satisfying the following LMIs:

\[
J^{ik}_{ii} + J^{lk}_{ii} + (J^{ik}_{ij})^T > 0, \quad 1 \leq i < j \leq r \quad (17)
\]

\[
J^{ik}_{ij} + J^{lk}_{ij} + (J^{ik}_{ij})^T > 0, \quad 1 \leq i < j \leq r \quad (18)
\]

\[
\operatorname{He} \left( J^{ik}_{ij} + J^{jk}_{ij} + J^{ik}_{jk} \right) > 0, \quad 1 \leq i < j < k \leq r \quad (19)
\]

\[
J^{jk}_{ij} + J^{kj}_{ij} + (J^{jk}_{ij})^T > 0, \quad 1 \leq i < j < k \leq r \quad (20)
\]

\[
\operatorname{He} \left( J^{jk}_{ij} + J^{jk}_{ij} + J^{lk}_{jk} \right) > 0, \quad 1 \leq i < j < k \leq r \quad (21)
\]

\[
[H_{ij}]_{r \times r} + [J^{ik}_{ij}]_{r \times r} < 0, \quad 1 \leq k \leq r \quad (22)
\]

\[
[H_{ij}]_{r \times r} + [J^{ik}_{ij}]_{r \times r} > 0, \quad 1 \leq k \leq r \quad (23)
\]

\[
\begin{bmatrix}
-A & I & 0 \\
0 & -A & I \\
0 & 0 & -A
\end{bmatrix} < 0, \quad \begin{bmatrix}
\beta_1 I & 0 \\
0 & -X_{11} \\
0 & 0 & -X_{22}
\end{bmatrix} < 0 \quad (25)
\]

\[
\begin{bmatrix}
-A & I & 0 \\
0 & -A & I \\
0 & 0 & -A
\end{bmatrix} > 0, \quad \begin{bmatrix}
\beta_1 I & 0 \\
0 & -X_{11} \\
0 & 0 & -X_{22}
\end{bmatrix} > 0 \quad (26)
\]

where

\[
H_{ij} = \begin{bmatrix}
A_i X + B_{2i} L_j + A_j X + B_{2j} L_i \\
B_{1i} + B_{1j} \\
C_i X + D_{1i} L_j + C_j X + D_{1j} L_i \end{bmatrix}, \quad 1 \leq i < j \leq r
\]

and

\[
H_{ji} = H_{ij}^T, \quad 1 \leq i < j \leq r
\]
Theorem 1 presents sufficient conditions under which an upper bound (1/\( \lambda \)) of singularly perturbed parameter \( \epsilon \) and stabilizing controller gains can be obtained. From (27) and \( \hat{\beta}_i > 0, 0.1 \leq i \leq 5, \lambda \) can be minimized by solving the following optimization problem:

\[
\begin{align*}
\text{minimize} & \quad \bar{u}_1 \hat{\beta}_1 + \bar{u}_2 \hat{\beta}_2 + \bar{u}_3 \hat{\beta}_3 + \bar{u}_4 \hat{\beta}_4 + \bar{u}_5 \hat{\beta}_5 \\
\text{subject to} & \quad (17) - (26)
\end{align*}
\]

Combining it with \( \hat{\beta}_i > 0, 1 \leq i \leq 5 \), (28) can be minimized by solving the optimization problem

\[
\begin{align*}
\text{minimize} & \quad a_1 \hat{\beta}_1 + a_2 \hat{\beta}_2 + a_3 \hat{\beta}_3 + a_4 \hat{\beta}_4 + a_5 \hat{\beta}_5 \\
\text{subject to} & \quad (17) - (26)
\end{align*}
\]

where \( a_i \), \( 1 \leq i \leq r \) are positive weighting constants to be chosen. Since the constraints (17) – (26) are of LMIs, the optimization problem can be effectively solved via LMI Control Toolbox [31]. Regarding the issue of how to choose the weighting scalars \( a_i (i = 1, 2, 3, 4, 5) \), generally, one can choose bigger \( a_i \) for rendering \( \hat{\beta}_i \) smaller (\( i = 1, 2, 3, 4, 5 \)). It should be pointed out that the upper bound (1/\( \lambda \)) obtained by solving the previous optimization problem may be conservative. After obtaining stabilizing controller gains, a less conservative bound of \( \epsilon \) can be obtained by Theorem 2 in Section III-B.

B. Computation of Stability Bound of \( \epsilon \)

In this subsection, assume that the controller has been designed. The following theorem gives a technique to estimate the upper bound of singularly perturbed parameter \( \epsilon \) subject to the stability of the closed-loop system.

**Theorem 2:** If there exist matrices \( P, Y, J^k_i, 1 \leq i < j \leq r, J^k_i, 1 \leq i \neq k \leq r \) and a constant \( \lambda > 0 \) with

\[
Y < \lambda P_{11}
\]

\[
\begin{bmatrix}
Y \\
P_{21} \\
P_{22}
\end{bmatrix} > 0
\]

\[
[Q_{aij}]_{r \times r} + [J^k_j]_{r \times r} < 0, \quad 1 \leq k \leq r
\]

\[
[Q_{aij}]_{r \times r} > -\lambda [Q_{aij}]_{r \times r} + [J^k_j]_{r \times r}, \quad 1 \leq k \leq r
\]

where \( J^k_{ii} = 0, J^k_{ij} = (J^k_j)^T, 1 \leq i < j \leq r, 1 \leq k \leq r \). Then, for \( \epsilon \in (0, 1/\lambda) \), the singularly perturbed closed-loop fuzzy system (7) is asymptotically stable, where

\[
Q_{aij} = \frac{1}{2} \text{He} \left( (A_i + B_{2i}K_j)^T P + (A_j + B_{2j}K_i)^T P \right), \quad 1 \leq i, j \leq r
\]

\[
Q_{aij} = \frac{1}{2} \text{He} \left( (A_i + B_{2i}K_j)^T P_a + (A_j + B_{2j}K_i)^T P_a \right), \quad 1 \leq i, j \leq r.
\]

**Proof:** From condition (30) and (31), we have

\[
\begin{bmatrix}
\lambda P_{11} \\
P_{21} \\
P_{22}
\end{bmatrix} > 0,
\]

By \( \epsilon \in (0, 1/\lambda) \), and (36), it follows that

\[
\begin{bmatrix}
\frac{1}{\epsilon} P_{11} \\
P_{21} \\
P_{22}
\end{bmatrix} > 0, \text{ for } \epsilon \in \left( 0, \frac{1}{\lambda} \right).
\]
Pre- and post multiplying (37) by

\[
\begin{bmatrix}
\varepsilon^{(1/2)}I \\
0 \\
\varepsilon^{(1/2)}I
\end{bmatrix}
\]

and its transpose, then the following inequality holds:

\[ P_\varepsilon = \begin{bmatrix} P_{11} & \varepsilon P_{21} \\
\varepsilon P_{21} & P_{22} \end{bmatrix} > 0, \quad \text{for } \varepsilon \in \left(0, \frac{1}{\lambda}\right]. \]

Consider the following Lyapunov function:

\[ V = x^T(t)P_\varepsilon x(t) = x^T(t) \begin{bmatrix} P_{11} & \varepsilon P_{21} \\
\varepsilon P_{21} & P_{22} \end{bmatrix} x(t) \]

then

\[
\dot{V} = x^T(t) \sum_{i=1}^{r} \sum_{j=1}^{r} \alpha_i \alpha_j P_\varepsilon E_\varepsilon^{-1} (A_i + B_{2i}K_j)x(t) + x^T(t) \sum_{i=1}^{r} \sum_{j=1}^{r} \alpha_j (A_i + B_{2i}K_j)^T E_\varepsilon^{-1} P_\varepsilon x(t)
+ x^T(t) \sum_{i=1}^{r} \sum_{j=1}^{r} \alpha_i \alpha_j (P_\varepsilon + \varepsilon P_\varepsilon^T) (A_i + B_{2i}K_j)x(t)
+ x^T(t) \sum_{i=1}^{r} \sum_{j=1}^{r} \alpha_i \alpha_j (A_i + B_{2i}K_j)^T (P_\varepsilon + \varepsilon P_\varepsilon^T) x(t)
= x^T(t) \left( \sum_{i=1}^{r} \sum_{j=1}^{r} \alpha_i \alpha_j P_\varepsilon^T (A_i + B_{2i}K_j) + (A_i + B_{2i}K_j)^T P_\varepsilon \right) x(t)
+ \varepsilon x^T(t) \sum_{i=1}^{r} \sum_{j=1}^{r} \alpha_i \alpha_j \left( P_\varepsilon^T (A_i + B_{2i}K_j) + (A_i + B_{2i}K_j)^T P_\varepsilon \right) x(t)
\]

\[ = x^T(t) \left( \sum_{i=1}^{r} \sum_{j=1}^{r} \alpha_i \alpha_j \frac{1}{2} \text{He}(P_\varepsilon^T (A_i + B_{2i}K_j) + (A_i + B_{2i}K_j)^T P_\varepsilon) \right) x(t)
+ \varepsilon x^T(t) \sum_{i=1}^{r} \sum_{j=1}^{r} \alpha_i \alpha_j \frac{1}{2} \text{He} \left( P_\varepsilon^T (A_i + B_{2i}K_j) + (A_i + B_{2i}K_j)^T P_\varepsilon \right) x(t)
\]

\[ = x^T(t) \left( \sum_{i=1}^{r} \sum_{j=1}^{r} \alpha_i \alpha_j Q_{ij} \right) x(t)
+ \varepsilon x^T(t) \sum_{i=1}^{r} \sum_{j=1}^{r} \alpha_i \alpha_j Q_{aij} x(t)
\]

\[ = x^T(t) \begin{bmatrix} \alpha_1 I \\
\vdots \\
\alpha_\gamma I \end{bmatrix}^T \left( \varepsilon [Q_{aij}]_{r \times r} + [Q_{ij}]_{r \times r} \right) \begin{bmatrix} \alpha_1 I \\
\vdots \\
\alpha_\gamma I \end{bmatrix} x(t) \]

(38)

where \( Q_{ij}, Q_{aij} \) are given by (34) and (35).

On the other hand, from (32) and (33), we have

\[ \varepsilon [Q_{aij}]_{r \times r} + [Q_{ij}]_{r \times r} + [J_{ij}^k]_{r \times r} < 0, \quad 1 \leq k \leq r, \quad \text{for } \varepsilon \in \left(0, \frac{1}{\lambda}\right]. \]

Applying (9)–(11), (39) and Lemma 2, it follows

\[ \begin{bmatrix} \alpha_1 I \\
\vdots \\
\alpha_\gamma I \end{bmatrix}^T \left( \varepsilon [Q_{aij}]_{r \times r} + [Q_{ij}]_{r \times r} + [J_{ij}^k]_{r \times r} \right) \begin{bmatrix} \alpha_1 I \\
\vdots \\
\alpha_\gamma I \end{bmatrix} < 0 \]

combined with (38), we have

\[ \dot{V} < 0, \quad \text{for } \varepsilon \in \left(0, \frac{1}{\lambda}\right], \quad x \neq 0. \]

Thus, for \( \varepsilon \in (0, (1/\lambda)] \), the closed-loop singularly perturbed fuzzy system (7) is asymptotically stable. ■

Remark 3: By Theorem 2, an upper bound of \( \varepsilon \) can be obtained by solving inequalities (9)–(11), (30)–(33) for \( \lambda \). The optimization problem

\[ \text{Minimize } \lambda \text{ subject to (9), (10), (11), (30)–(33)} \]

is a generalized eigenvalue problem (GEVP) [20], which can be effectively solved using LMI Control Toolbox [31]. The problem of computing the bound of \( \varepsilon \) was considered in [17], where a method of finding an interval \([\varepsilon_1, \varepsilon_2]\) so that the system is stable for \( \varepsilon \in [\varepsilon_1, \varepsilon_2] \) was derived. However, to search for small \( \varepsilon \) is related to \( \varepsilon \)-dependent computation, which cannot avoid the difficulties linked with the stiffness of the equations involved in the design.

C. Stabilizing Controller Design Without Considering Stability Bound

In this section, a technique for designing stabilizing controllers for singularly perturbed fuzzy systems without consideration of improving the bound of singularly perturbed parameter \( \varepsilon \), is given as follows.

Theorem 3: If there exist matrices \( X, L_i, J^k_{ij}, 1 \leq i \leq j \leq r, \quad J^k_{ii}, 1 \leq i \neq k \leq r, \) with

\[ X = \begin{bmatrix} X_{11} & 0 \\
X_{21} & X_{22} \end{bmatrix} \]

where \( X_{11}, X_{22}, J^k_{ii}, 1 \leq i \neq k \leq r \) are symmetric matrices, satisfying the following LMIs:

\[ X_{11} > 0, \quad X_{22} > 0 \]
\[ J^k_{ii} + J^k_{ij} + (J^k_{ij})^T > 0, \quad 1 \leq i < j \leq r \]
\[ J^k_{jj} + J^k_{ij} + (J^k_{ij})^T > 0, \quad 1 \leq i < j \leq r \]
\[ \text{He} \left( J^k_{ij} + J^k_{ij} + J^k_{jij} \right) > 0, \quad 1 \leq i < j < l \leq r \]
\[ [T_{ij}]_{r \times r} + [J^k_{ij}]_{r \times r} < 0, \quad 1 \leq k \leq r \]
where
\[ T_{ij} = T_{ji} = \frac{1}{2} H e(A_i X + B_2 L_j + A_j X + B_2 L_i), \]
\[ 1 \leq i \leq j \leq r \]
\[ J_{ij}^k = 0 \]
\[ J_{ij}^k = (J_{ji}^k)^T, \quad 1 \leq i < j \leq r, \quad 1 \leq k \leq r \]
then there exists a scalar \( \epsilon^* > 0 \) such that, for \( \epsilon \in (0, \epsilon^*], \) the state feedback controller (6) with
\[ K_i = L_i X^{-1}, \quad 1 \leq i \leq r. \]
renders the singularly perturbed fuzzy system (7) asymptotically stable.

\textbf{Proof:} By
\[ X = \begin{bmatrix} X_{11} & 0 \\ X_{21} & X_{22} \end{bmatrix} \]
and \( X_{11} > 0, X_{22} > 0, \) we choose \( P = X^{-1}, \) then by matrix invertible formula, it follows that
\[ P = \begin{bmatrix} P_{11} & 0 \\ P_{21} & P_{22} \end{bmatrix} = \begin{bmatrix} X_{11}^{-1} & 0 \\ -X_{21}^{-1} X_{21} X_{11}^{-1} & X_{22}^{-1} \end{bmatrix}. \]
Since \( P_{11} > 0 \) and \( P_{22} > 0, \) there exists a scalar \( \epsilon^*_1 > 0, \) such that \( P_{11} - \epsilon P_{21}^T P_{22}^{-1} P_{21} > 0 \) for \( \epsilon \in (0, \epsilon^*], \) which implies
\[ P_{\epsilon} = \begin{bmatrix} P_{11} & \epsilon P_{21} \\ \epsilon P_{21} & P_{22} \end{bmatrix} > 0, \]
Choose Lyapunov function
\[ V = x^T(t) P_{\epsilon} x(t) \]
then
\[ \dot{V} = x^T(t) \left( \sum_{i=1}^{r} \sum_{j=1}^{r} \alpha_i \alpha_j \text{He} \left( (P_{\epsilon} E_{i}^{-1}(A_i + B_2 K_j)) \right) \right) x(t). \]
On the other hand, substitute
\[ L_j = K_j X \]
for \( L_j \) in (44) and pre- and post multiplying (44) by \( \text{diag}[P_{ij}, \ldots, P_{ij}] \) and \( \text{diag}[P_{ij}, \ldots, P_{ij}], \) then we can obtain
\[ [T_{ij}]_{r \times r} + [J_{ij}^k]_{r \times r} < 0, \quad 1 \leq k \leq r \]
(47)
where
\[ T_{ij} = T_{ji} = \frac{1}{2} H e (P^T (A_i + B_2 K_j + A_j + B_2 K_i)) \]
\[ J_{ij}^k = P^T J_{ij}^k P, \quad 1 \leq i \leq j \leq r. \]
Now, pre- and post multiplying (41)–(43) by \( P^T \) and \( P, \) then it follows that
\[ J_{ij}^k + J_{ij}^k + (J_{ij}^k)^T > 0, \quad 1 \leq i < j \leq r \]
(48)
\[ J_{ij}^k + J_{ij}^k + (J_{ij}^k)^T > 0, \quad 1 \leq i < j \leq r \]
(49)
\[ \text{He} \left( J_{ij}^k + J_{ij}^k + J_{ij}^k \right) > 0, \quad 1 \leq i < j \leq l \leq r. \]
(50)
From (47), we have that there exist a scalar \( \epsilon_2^* > 0 \) such that
\[ [T_{ij}]_{r \times r} + \epsilon [J_{aij}]_{r \times r} + [J_{ij}^k]_{r \times r} \leq 0, \quad \epsilon \in (0, \epsilon_2^*], \]
(51)
\[ 1 \leq k \leq r \]
where
\[ J_{aij} = (1/2) H e \left( P_{a}^T (A_i + B_2 K_j + A_j + B_2 K_i) \right), \]
\[ 1 \leq i \leq j \leq r. \]
By (48)–(50), (51) and Lemma 2, it follows that
\[ \sum_{i=1}^{r} \sum_{j=1}^{r} \alpha_i \alpha_j (\bar{T}_{ij} + \epsilon \bar{T}_{aij}) < 0 \]
i.e.,
\[ \sum_{i=1}^{r} \sum_{j=1}^{r} \alpha_i \alpha_j \frac{1}{2} \text{He} \left( (P + \epsilon P_{a})^T (A_i + B_2 K_j) \right) \]
\[ = \sum_{i=1}^{r} \sum_{j=1}^{r} \alpha_i \alpha_j \text{He} \left( (P + \epsilon P_{a})^T (A_i + B_2 K_j) \right) \]
\[ = \sum_{i=1}^{r} \sum_{j=1}^{r} \alpha_i \alpha_j \text{He} \left( P_{a} E_{i}^{-1}(A_i + B_2 K_j) \right) \]
\[ < 0 \]
combined with (45), then we have
\[ \dot{V} < 0, \quad \text{for } x(t) \neq 0 \]
which implies that the closed-loop singularly perturbed fuzzy system (7) is asymptotically stable, and from (46), we obtain
\[ K_i = L_i X^{-1}, \quad 1 \leq i \leq r. \]
Thus, the proof is complete.

\textbf{Remark 4:} Theorem 3 presents a method for designing state feedback stabilizing controllers for singularly perturbed fuzzy systems. However, the upper bound of singularly perturbed parameter \( \epsilon \) is not addressed in the design.

The following theorem shows that the new method given in Theorem 3 is less conservative than that given in Lemma 3 [16].

\textbf{Theorem 4:} If the condition in Lemma 3 holds, then the condition in Theorem 3 holds.

\textbf{Proof:} Assume that the condition in Lemma 3 holds, then choose \( J_{ij}^k = -(1/2) \text{He} \left( (A_i X + B_2 L_j + A_j X + B_2 L_i) \right), \]
\[ 1 \leq i < j \leq r, \quad J_{ij}^k = 0, \quad 1 \leq i \leq r. \]
Then it follows that conditions (40)–(43) hold from (13) and (14). Moreover, from (14) and the chosen \( J_{ij}^k, \) the following inequalities hold:
\[ [T_{ij}]_{r \times r} + [J_{ij}^k]_{r \times r} = \begin{bmatrix} T_{11} & 0 & \cdots & 0 \\ 0 & T_{22} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & T_{rr} \end{bmatrix} < 0, \]
\[ 1 \leq k \leq r \]
which implies that (44) holds. Therefore, the condition of Theorem 3 holds. Thus, the proof is complete.

IV. BOUND OF $\epsilon$ AND $H_\infty$ CONTROL

In this section, the results in Section III are extended to the case that the closed-loop system is required to be with an $H_\infty$ performance bound. Solutions to Problem (ii) and (iv) formulated in Section II-B are, respectively, given in Section IV-A and IV-B. Moreover, Section IV-C also presented a new method for designing $H_\infty$ controllers without consideration of improving the bound of singularly perturbed parameter $\epsilon$.

A. Design for Bound of $\epsilon$ and $H_\infty$ Performance

In this subsection, a method is given for designing $H_\infty$ controllers with consideration of improving the bound of singularly perturbed parameter $\epsilon$.

**Theorem 5:** For given $\gamma > 0$, if there exist matrices $X, L_i$, $\Lambda$, $J_{ij}, J_{ik}, J_{jk}, 1 \leq i < j \leq r$, $\Lambda_{ii}, 1 \leq i \neq k \leq r$, and positive scalar variables $\beta_1, \beta_2, \beta_3, \beta_4, \beta_5$ with

$$X = \begin{bmatrix} X_{11} & 0 \\ X_{21} & X_{22} \end{bmatrix}$$

where $X_{11}, X_{22}, \Lambda, J_{ij}, J_{ik}, J_{jk}, 1 \leq i \neq k \leq r$ are symmetric matrices, satisfying the following LMIs:

$$J_{ij}^T + J_{ij} + (J_{ij})^T > 0, \quad 1 \leq i < j \leq r \tag{52}$$

$$J_{ii}^T + J_{ij} + (J_{ij})^T > 0, \quad 1 \leq i < j \leq r \tag{53}$$

$$\text{He} \left( J_{ik}^T + J_{ij} + J_{ij}^T \right) > 0, \quad 1 \leq i < j < k \leq r \tag{54}$$

$$J_{ij}^T + J_{ij} + (J_{ij})^T > 0, \quad 1 \leq i < j \leq r \tag{55}$$

$$J_{ij}^T + J_{ij} + (J_{ij})^T > 0, \quad 1 \leq i < j \leq r \tag{56}$$

Denote

$$\lambda = \max \left\{ \| P_2 X \| \left( \beta_5 (\beta_4 + \beta_6) + \sqrt{\beta_5^2 (\beta_4 + \beta_6)^2 + \beta_3 \beta_5} \right), \beta_1 \beta_2 \beta_3 \right\} \tag{62}$$

where $

$$\beta_5 = \max \left\{ \| B_{ij} B_{ij}^T \| : 1 \leq i \leq r \right\} \tag{63}$$

then for $\epsilon \in (0, (1/\lambda)]$, the singularly perturbed closed-loop fuzzy system (8) via state feedback controller

$$u(t) = \sum_{i=1}^{r} \alpha_i K_i x(t)$$

where

$$K_i = L_i X^{-1}$$

is asymptotically stable and with an $H_\infty$ norm less than $\gamma$.

**Proof:** See Appendix A.

**Remark 5:** If the conditions of Theorem 5 hold, then $(1/\lambda)$ is an upper bound of singularly perturbed parameter $\epsilon$, and

$$\lambda = \max \left\{ \| P_2 X \| \left( \beta_5 (\beta_4 + \beta_6) + \sqrt{\beta_5^2 (\beta_4 + \beta_6)^2 + \beta_3 \beta_5} \right), \beta_1 \beta_2 \beta_3 \right\} \tag{62}$$

The problem of minimizing $\lambda$ can be reduced to solving the following optimization problem with LMI constraints:

Minimize $\alpha_1 \beta_1 + \alpha_2 \beta_2 + \alpha_3 \beta_3 + \alpha_4 \beta_4 + \alpha_5 \beta_5$

subject to (52)–(61)

where $\alpha_1, \alpha_2, \alpha_3, \alpha_4$, and $\alpha_5$ are positive weighting constants to be chosen.

B. Computation of Upper Bound of $\epsilon$

In this subsection, we assume that the state feedback gains are given. Then the following theorem gives a method to estimate the upper bound of singularly perturbed parameter $\epsilon$ subject to the closed-loop system with an $H_\infty$ performance bound.

**Theorem 6:** For a given $\gamma > 0$, if there exists $P, Y, J_{ij}, 1 \leq i < j \leq r, 1 \leq i \neq k \leq r$, and a constant $\lambda > 0$ with

$$P = \begin{bmatrix} P_{11} & 0 \\ P_{21} & P_{22} \end{bmatrix}$$

where $P_{11}, P_{22}, J_{ij}, 1 \leq i \neq k \leq r$ are symmetric matrices satisfying (9)–(11) and the following LMIs:

$$Y < \lambda P_{11} \tag{64}$$

$$\begin{bmatrix} Y & P_{11}^T \\ P_{21} & P_{22} \end{bmatrix} > 0 \tag{65}$$

$$\begin{bmatrix} \Phi_{ij} & J_{ij} \end{bmatrix} > 0 \tag{66}$$

$$\begin{bmatrix} \Phi_{ij} & J_{ij} \end{bmatrix} > -\lambda \left( \begin{bmatrix} \Phi_{ij} & J_{ij} \end{bmatrix} \right) \tag{67}$$

(59)
where
\[ \Phi_{ij} = \frac{1}{2} \begin{bmatrix} \Phi_{ij}^{11} & \Phi_{ij}^{12} & \Phi_{ij}^{13} \\ \Phi_{ij}^{21} & 0 & \Phi_{ij}^{23} \\ \Phi_{ij}^{31} & -\Phi_{ij}^{32} & 0 \end{bmatrix}, \]
\[ \Phi_{eij} = \frac{1}{2} \begin{bmatrix} \Phi_{eij}^{11} & 0 & 0 \\ 0 & \Phi_{eij}^{22} & 0 \\ 0 & 0 & \Phi_{eij}^{33} \end{bmatrix}, \quad 1 \leq i, \ j \leq r \]
\[ J_{ii}^e = 0, \quad J_{ij}^k = (J_{ij}^k)^T, \quad 1 \leq i < j \leq r, \quad 1 \leq k \leq r \]

then, for \( \epsilon \in (0, (1/\lambda)] \), the singularly perturbed closed-loop fuzzy system (8) is asymptotically stable and with an \( H_\infty \) norm less than \( \gamma \).

**Proof:** From (64) to (67), it follows that, for any
\( \epsilon \in (0, (1/\lambda)] \)
\[ P_\epsilon = \begin{bmatrix} P_{11} & eP_{12}^T \\ eP_{21} & eP_{22} \end{bmatrix} > 0 \] (68)

and
\[ e[\Phi_{eij}]_{r \times r} \] \[ + [\Phi_{ij}]_{r \times r} + J^k < 0. \]

By (9)–(11), and Lemma 2, which further implies that
\[ \begin{bmatrix} \alpha_1 I \\ \vdots \\ \alpha_r I \end{bmatrix}^T \]
\[ (e[\Phi_{eij}]_{r \times r} + [\Phi_{ij}]_{r \times r}) \begin{bmatrix} \alpha_1 I \\ \vdots \\ \alpha_r I \end{bmatrix} < 0. \] (69)

On the other hand
\[ A_{e1}^T P_\epsilon + P_\epsilon A_{e1} + P_\epsilon B_{e1} C_{e1}^T P_\epsilon + \frac{1}{\gamma^2} C_{e1}^T C_{e1} \]
\[ = \begin{bmatrix} \alpha_1 I \\ \vdots \\ \alpha_r I \end{bmatrix}^T \]
\[ (e[\Phi_{eij}]_{r \times r} + [\Phi_{ij}]_{r \times r}) \begin{bmatrix} \alpha_1 I \\ \vdots \\ \alpha_r I \end{bmatrix}. \]

Thus, from Lemma 1 and (69), the conclusion follows.

**Remark 6:** When the state feedback gains are given, Theorem 6 gives a method to estimate the upper bound of singular perturbation parameter \( \epsilon \) subject to that the closed-loop system is required to be with a prescribed \( H_\infty \) performance bound.

An upper bound of \( \epsilon \) can be obtained by solving inequalities (9)–(11) and (64)–(67) for \( \lambda \). The optimization problem

Minimize \( \lambda \) subject to (9), (10), (11), (64)–(67)
is also a generalized eigenvalue problem (GEVP) [20], which can be effectively solved using LMI Control Toolbox [31].

**C. \( H_\infty \) Controller Design**

In this subsection, a new method is given for designing \( H_\infty \) controllers, but without consideration of improving the bound of singularly perturbed parameter \( \epsilon \).

**Theorem 7:** For given \( \gamma > 0 \), if there exist matrices \( X, L_k \),
\[ J_{ij}^k, 1 \leq i < j \leq r, \]
then there exists a scaler \( \epsilon^* \) such that, for \( \epsilon \in (0, \epsilon^*) \), the singularly perturbed fuzzy system (7) via the state feedback controller (6) with
\[ K_i = L_i X^{-1}, \quad 1 \leq i \leq r. \]
is asymptotically stable and with an \( H_\infty \) norm less than \( \gamma \).

**Proof:** It is similar to Theorem 3, and omitted here. ■

The following theorem shows that the new method given in Theorem 7 is less conservative than that given in Lemma 4 [18].

**Theorem 8:** If the condition of Lemma 4 holds, then the condition of Theorem 7 holds.

**Proof:** It is similar to Theorem 4 and omitted here. ■

**V. EXAMPLE**

We consider an inverted pendulum controlled by a motor via a gear train. It can be described by the following state equations [32]:
\[ x_1(t) = x_2(t) + 0.1w(t) \]
\[ x_2(t) = \frac{g}{m} \sin x_1(t) + \frac{NK_0}{md^2} x_3(t) \]
\[ L_0 x_3(t) = - K_h N x_2(t) - R_0 x_3(t) + u(t) \]
\[ y(t) = x_1(t) + 0.1w(t) \]
\[ z(t) = 0.1 x_1(t) + 0.1 u(t) \]
where $x_1(t) = \dot{\theta}_p(t)$, $x_2(t) = d((\dot{\theta}_p(t))/dt$, $x_3(t) = I_a(t)$, $u(t)$ is the control input, $w(t)$ is the disturbance input, $K_m$ is the motor torque constant, $K_b$ is the back emf constant, and $N$ is the gear ratio. The parameters for the plant are given as $g = 9.8 \text{ m/s}^2$, $l = 1 \text{ m}$, $m = 1 \text{ kg}$, $N = 10$, $K_m = 0.1 \text{ Nm/A}$, $K_b = 0.1 \text{ V/s/rad}$, $R_a = 1 \Omega$ and $L_a = \epsilon \text{ mH}$. Note that the inductance $L_a$ represents the small "parasitic" parameter in the system. Then, we get

$$\dot{x}_1(t) = x_2(t) + 0.1u(t)$$

$$\dot{x}_2(t) = 9.8 \sin x_1(t) + \frac{NK_m}{mL_2} x_3(t)$$

$$\epsilon \dot{x}_3(t) = -x_2(t) - x_3(t) + u(t)$$

$$y(t) = x_1(t) + 0.1u(t)$$

$$z(t) = 0.1x_1(t) + 0.1u(t).$$

Choose the membership functions of the fuzzy sets as

$$M_1(x_1(t)) = 1 - \frac{|x_1(t)|}{\pi}$$

$$M_2(x_1(t)) = \frac{|x_1(t)|}{\pi}.$$

This fuzzy model exactly represents the dynamics of the nonlinear mechanical system under $-\pi \leq x_1(t) \leq \pi$. A T-S fuzzy model can be obtained as follows:

**Plant Rule 1:**

IF $x_1(t)$ is $M_{11}$

THEN $E(\epsilon)y(t) = A_{11}x(t) + B_{21}u(t) + B_{11}w(t)$

$z(t) = C_{11}x(t) + D_{121}u(t);$

**Plant Rule 2:**

IF $x_1(t)$ is $M_{21}$

THEN $E(\epsilon)y(t) = A_{21}x(t) + B_{22}u(t) + B_{12}w(t)$

$z(t) = C_{12}x(t) + D_{122}u(t);$

where

$$E_\epsilon = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \epsilon \end{bmatrix}$$

$$A_{1} = \begin{bmatrix} 9.8 & 0 & 0 \\ 1 & -1 & -1 \\ 0 & 0 & 0 \end{bmatrix}$$

$$A_{2} = \begin{bmatrix} 9.8 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$B_{11} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$B_{12} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$B_{21} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$B_{22} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$C_{11} = \begin{bmatrix} 0.1 \\ 0 \\ 0 \end{bmatrix}$$

$$C_{12} = \begin{bmatrix} 0.1 \\ 0 \\ 0 \end{bmatrix}$$

$$C_{21} = \begin{bmatrix} 0.1 \\ 0.1 \\ 0 \end{bmatrix}$$

$$C_{22} = \begin{bmatrix} 0.1 \\ 0.1 \\ 0 \end{bmatrix}$$

$$D_{121} = 0.1$$

$$D_{122} = 0.1$$

$$\epsilon = 0.1.$$

Lemma 4, Theorems 5 and 7 are applicable for designing fuzzy $H_\infty$ controllers for the system.

By using Lemma 4 and Theorem 7, the obtained optimal $H_\infty$ performance indexes are $\gamma^* = 0.1383$ and $\gamma^* = 0.1308$, respectively, and the corresponding controller gains are given as follows:

$$K_1 = 10^3 \times [-5.3624 -1.6501 -0.0000]$$

$$K_2 = 10^3 \times [-5.3624 -1.6501 -0.0000]$$

(70)

$$K_1 = 10^3 \times [-5.3761 -1.5736 -0.0000]$$

$$K_2 = 10^3 \times [-6.9072 -2.0178 -0.0000].$$

(71)

Now we apply Theorem 5 to design a fuzzy $H_\infty$ controller for the system. Choose weighting scalars as

$$a_1 = 5; \ a_2 = 60; \ a_3 = 3800; \ a_4 = 320; \ a_5 = 13$$

and $\gamma = 0.15$, then the controller gains are given as follows:

$$K_1 = [ -381.0540 -114.7857 -2.5784 ]$$

$$K_2 = [ -413.8125 -123.9202 -2.5778 ].$$

(72)

By using Theorem 6, the upper bounds of $\epsilon$ subject to guaranteeing $H_\infty$ performance bound $\gamma$ can be estimated, and shown in Table 1.

From Table I, it is easy to see that the new proposed design given by Theorem 5 gives a considerable improvement of upper bounds of $\epsilon$. The upper bounds of $\epsilon$ given by Lemma 4 and Theorem 7 are very small.

Assume the initial states are zero, and the disturbance input signal $u(t)$ is shown in Fig. 1. The simulation results of the output $z(t)$ with the controller gains (70)–(72) are given in Figs. 2–4, respectively. The simulations for the square root of the regulated output energy to the disturbance input noise energy are depicted in Figs. 5–7. From the simulation results, it can be seen that the fuzzy controller (72) guarantees good $H_\infty$ performance of the resulting closed-loop system, while the controllers (70) and (71) give poor system responses.
Fig. 1. Disturbance input $w(t)$ used during simulation.

Fig. 4. Trajectory of $z(t)$ via controller (72).

Fig. 2. Trajectory of $z(t)$ via controller (70).

Fig. 5. $\sqrt{\int_0^\infty z^T(t)z(t)dt / \int_0^\infty w^T(t)w(t)dt}$ via controller (70).

Fig. 3. Trajectory of $z(t)$ via controller (71).

Fig. 6. $\sqrt{\int_0^\infty z^T(t)z(t)dt / \int_0^\infty w^T(t)w(t)dt}$ via controller (71).
which further implies that
\[
\begin{bmatrix}
\frac{1}{\lambda}X_{11}^{-1}
& X_{11}

X_{21}^{-1}X_{11}
& X_{22}^{-1}
\end{bmatrix} > 0 \text{ for } \epsilon \in \left(0, \frac{1}{\lambda}\right].
\] (76)

Let \( P = X^{-1} \), then \( P \) can be expressed as follows:
\[
P = \begin{bmatrix}
P_{11} & 0 \\
P_{21} & P_{22}
\end{bmatrix} = \begin{bmatrix}
X_{11}^{-1}

-X_{21}^{-1}X_{11}^{-1}

0

X_{22}^{-1}
\end{bmatrix}.
\]

So (76) is equivalent to
\[
\begin{bmatrix}
\frac{1}{\lambda}P_{11} & P_{21}'

P_{21} & P_{22}
\end{bmatrix} > 0,
\] (77)

Pre- and postmultiplying (77) by
\[
\begin{bmatrix}
\epsilon^{(1/2)}I & 0 \\
0 & \epsilon^{(1/2)}I
\end{bmatrix} > 0
\]
and its transpose, then the following inequality holds:
\[
P_{\epsilon} = \begin{bmatrix}
P_{11} & \epsilon P_{21}' \\
\epsilon P_{21} & P_{22}
\end{bmatrix} > 0,
\] (78)

**Part 2:** From (25), we can obtain
\[
\frac{1}{\beta_{\delta}} I < \Lambda.
\] (79)

From (27), we have
\[
2\beta_{\delta} \lVert P_{\delta} X \rVert \leq \frac{\lambda}{\beta_{\delta}}.
\] (80)

By (17)–(19), (23), and Lemma 2, it follows that
\[
\sum_{i=1}^{r} \sum_{j=1}^{r} \alpha_i \alpha_j H_{ij} < 0
\]
i.e.,
\[
\begin{bmatrix}
-\beta_{\delta} I

\sum_{i=1}^{r} \sum_{j=1}^{r} \alpha_i \alpha_j (A_i X + B_{1i} L_i + A_j X + B_{2i} L_i) * -I
\end{bmatrix} < 0.
\] (81)

Applying the Schur complement to (81), we have
\[
\left( \sum_{i=1}^{r} \sum_{j=1}^{r} \alpha_i \alpha_j \left( X^T A_i^T + L_i^T B_{2i}^T \right) \right)
\times \left( \sum_{i=1}^{r} \sum_{j=1}^{r} \alpha_i \alpha_j \left( A_i X + B_{2i} L_j \right) \right) < \beta_{\delta}^2 I
\]
i.e.,
\[
\left\| \sum_{i=1}^{r} \sum_{j=1}^{r} \alpha_i \alpha_j \left( X^T A_i^T + L_i^T B_{2i}^T \right) \right\| < \beta_{\delta}.
\] (82)
Notice that
\[
\left\| X^T \sum_{i=1}^{r} \sum_{j=1}^{r} \alpha_i \alpha_j Q_{a_1j} X \right\|
\]
\[
= \left\| \sum_{i=1}^{r} \sum_{j=1}^{r} \alpha_i \alpha_j \text{He} \left[ (X^T A_i^T + L_j^T B_2^T) P_a X \right] \right\|
\]
\[
\leq 2 \left\| \sum_{i=1}^{r} \sum_{j=1}^{r} \alpha_i \alpha_j (X^T A_i^T + L_j^T B_2^T) \right\| \| P_a X \| \quad (83)
\]
where \(Q_{a_1j}\) are the same as in (34). Applying (82) to (83), it follows that
\[
\left\| \sum_{i=1}^{r} \sum_{j=1}^{r} \alpha_i \alpha_j X^T Q_{a_1j} X \right\| \leq 2 \beta \| P_a X \|. \quad (84)
\]
By (80) and (84), we have
\[
\left\| \sum_{i=1}^{r} \sum_{j=1}^{r} \alpha_i \alpha_j X^T Q_{a_1j} X \right\| \leq \frac{\lambda}{\beta \overline{\beta}}
\]
which implies
\[
\sum_{i=1}^{r} \sum_{j=1}^{r} \alpha_i \alpha_j X^T Q_{a_1j} X \leq \frac{\lambda}{\beta \overline{\beta}} I. \quad (85)
\]
Pre- and postmultiplying (24) by \([\alpha_1 I, \ldots, \alpha_r I]\) and its transpose, it follows that
\[
\Lambda + \sum_{i=1}^{r} \sum_{j=1}^{r} \alpha_i \alpha_j (\text{He}(A_i X + B_2 L_j))
\]
\[
+ \begin{bmatrix} \alpha_1 I \\ \vdots \\ \alpha_r I \end{bmatrix}^T J^k \begin{bmatrix} \alpha_1 I \\ \vdots \\ \alpha_r I \end{bmatrix} < 0. \quad (86)
\]
Multiplying (86) by \(\alpha_k, 1 \leq k \leq r\) and summing them, it follows
\[
\Lambda + \sum_{i=1}^{r} \sum_{j=1}^{r} \alpha_i \alpha_j (\text{He}(A_i X + B_2 L_j))
\]
\[
+ \sum_{k=1}^{r} \alpha_k \begin{bmatrix} \alpha_1 I \\ \vdots \\ \alpha_r I \end{bmatrix}^T J^k \begin{bmatrix} \alpha_1 I \\ \vdots \\ \alpha_r I \end{bmatrix} < 0.
\]
Combining it with (79), we can obtain
\[
\frac{1}{\beta \overline{\beta}} I + \sum_{i=1}^{r} \sum_{j=1}^{r} \alpha_i \alpha_j (\text{He}(A_i X + B_2 L_j))
\]
\[
+ \sum_{k=1}^{r} \alpha_k \begin{bmatrix} \alpha_1 I \\ \vdots \\ \alpha_r I \end{bmatrix}^T J^k \begin{bmatrix} \alpha_1 I \\ \vdots \\ \alpha_r I \end{bmatrix} < 0.
\]
Applying Lemma 2 to the previous inequality, then
\[
\frac{1}{\beta \overline{\beta}} I + \sum_{i=1}^{r} \sum_{j=1}^{r} \alpha_i \alpha_j (\text{He}(A_i X + B_2 L_j)) < 0
\]
which implies that
\[
\sum_{i=1}^{r} \sum_{j=1}^{r} \alpha_i \alpha_j (\text{He}(A_i X + B_2 L_j)) < 0 \quad (87)
\]
Substituting (85) and (88), we have
\[
\sum_{i=1}^{r} \sum_{j=1}^{r} \alpha_i \alpha_j X^T Q_{a_1j} X
\]
\[
+ \sum_{i=1}^{r} \sum_{j=1}^{r} \alpha_i \alpha_j \lambda (\text{He}(A_i X + B_2 L_j)) < 0. \quad (89)
\]
Pre- and postmultiplying (90) by \(P = X^{-1}\), then we have
\[
\sum_{i=1}^{r} \sum_{j=1}^{r} \alpha_i \alpha_j X^T Q_{a_1j} X + \sum_{i=1}^{r} \sum_{j=1}^{r} \alpha_i \alpha_j (\text{He}(A_i X + B_2 L_j)) X < 0. \quad (90)
\]
where \(Q_{a_1j}\) are the same as in (35). From (87), then
\[
\sum_{i=1}^{r} \sum_{j=1}^{r} \alpha_i \alpha_j Q_{a_1j} < 0, \quad (91)
\]
Combining it and (91), that yields
\[
\sum_{i=1}^{r} \sum_{j=1}^{r} \alpha_i \alpha_j Q_{a_1j} + \epsilon \sum_{i=1}^{r} \sum_{j=1}^{r} \alpha_i \alpha_j Q_{a_1j}
\]
\[
= [\alpha_1 I \ldots \alpha_r I] \begin{bmatrix} (Q_{ij})_{r \times r} + \epsilon [Q_{a_1j}]_{r \times r} \end{bmatrix} \begin{bmatrix} \alpha_1 I \\ \vdots \\ \alpha_r I \end{bmatrix}^T < 0 \text{ for } \epsilon \in \left(0, \frac{1}{\lambda}\right). \quad (92)
\]
Part 3: Choose Lyapunov function
\[
V = x^T(t) P_2 x(t).
\]
Then, by (78) and (92), it follows that \(P_2 > 0\) and
\[
\dot{V} = x^T(t) \begin{bmatrix} \alpha_1 I \\ \vdots \\ \alpha_r I \end{bmatrix}^T (Q_{ij} + \epsilon Q_{a_1j}) \begin{bmatrix} \alpha_1 I \\ \vdots \\ \alpha_r I \end{bmatrix} x(t) < 0,
\]
for \(x(t) \neq 0\) for \(\epsilon \in (0, 1/\lambda)\). Thus, the proof is complete.

\[\blacksquare\]
APPENDIX B
PROOF OF THEOREM 5

Proof: From (58), (62), and Part 1 of the proof of Theorem 3, we have

\[ P_{\epsilon} = \begin{bmatrix} P_{21} & \epsilon P_{22} \\ \epsilon P_{21} & \epsilon P_{22} \end{bmatrix} > 0, \quad \text{for} \quad \epsilon \in \left(0, \frac{1}{\lambda} \right) \]

where \( P = \begin{bmatrix} P_{11} & 0 \\ P_{21} & P_{22} \end{bmatrix} = \begin{bmatrix} X_{11}^{-1} & X_{12}^{-1} \\ -X_{21}^{-1} & X_{22}^{-1} \end{bmatrix} = X^{-1}. \) By (62), it follows that

\[ \frac{1}{\beta_5} \lambda^2 - 2(\beta_4 + \beta_6) \| P_a X \| \lambda - \beta_6 \| P_a X \|^2 \geq 0. \tag{93} \]

On the other hand, from (60), we have

\[ \sum_{i=1}^{r} \sum_{j=1}^{r} \alpha_i \alpha_j \begin{bmatrix} -\beta_4^2 I \\ * \end{bmatrix} \begin{bmatrix} A(X + B_2 K X) + A_i X + B_2 L_i \end{bmatrix} + \frac{1}{2} \begin{bmatrix} \epsilon^2 \| P_a X \| \lambda - \beta_6 \| P_a X \|^2 \end{bmatrix} < 0. \tag{94} \]

Substituting \( K_i X = L_i, \) \( 1 \leq i \leq r, \) then (94) becomes

\[ \begin{bmatrix} -\beta_4^2 I \\ * \end{bmatrix} \begin{bmatrix} \frac{A(X + B_2 K X)^T}{-I} \end{bmatrix} \begin{bmatrix} \epsilon^2 \| P_a X \| \lambda - \beta_6 \| P_a X \|^2 \end{bmatrix} < 0. \tag{95} \]

where

\[ A = A(\alpha) = \sum_{i=1}^{r} \alpha_i A_i, \quad B_2 = B_2(\alpha) = \sum_{i=1}^{r} \alpha_i B_2i, \]
\[ K = K(\alpha) = \sum_{i=1}^{r} \alpha_i K_i. \]

For brief expression, we also denote

\[ B_1 = B_1(\alpha) = \sum_{i=1}^{r} \alpha_i B_{1i}, \]
\[ C_1 = C_1(\alpha) = \sum_{i=1}^{r} \alpha_i C_{1i}, \]
\[ D_{12} = D_{12}(\alpha) = \sum_{i=1}^{r} \alpha_i D_{12i}. \]

Applying the Schur complement to (95), we obtain

\[ (AX + B_2 K X)^T (AX + B_2 K X) < \beta_4^2 I \]

which implies that

\[ \| AX + B_2 K X \| < \beta_4. \tag{96} \]

From (63), it follows that

\[ \| B_1 B_1^T \| \leq \beta_6. \tag{97} \]

Combining (96), (97), and (93), we have

\[ 2 \| AX + B_2 K X \| \| P_a X \| + 2 \| B_1 B_1^T \| \| P_a X \|^2 + \frac{1}{\lambda} \epsilon^2 \| P_a X \| \lambda - \beta_6 \| P_a X \|^2 \leq \frac{1}{\lambda} \beta_5 \]

which further implies that

\[ M_4 < \frac{1}{\beta_5} I, \quad \text{for} \quad \epsilon \in \left(0, \frac{1}{\lambda} \right) \tag{98} \]

where

\[ M_4 = X^T P_4^T (AX + B_2 K X) + (AX + B_2 K X)^T P_a X + B_1 B_1^T P_a X + X^T P_4^T B_1 B_1^T P_a X + \epsilon X^T P_4^T B_1 B_1^T P_a X. \]

On the other hand, from (59), we have

\[ \frac{1}{\beta_5} I < \Lambda. \tag{99} \]

Thus, from (98) and (99), it follows:

\[ M_4 < \lambda \Lambda. \tag{100} \]

By (61) and Lemma 2, we can obtain (101) shown at the bottom of the page. Applying the Schur complement to (101), it follows that

\[ \sum_{i=1}^{r} \sum_{j=1}^{r} \alpha_i \alpha_j T_{ij} + B_1 B_1^T + \frac{1}{\gamma^2} (C_1 X + D_{12 K} X)^T \]
\[ \times (C_1 X + D_{12 K} X) = \begin{pmatrix} \sum_{i=1}^{r} \sum_{j=1}^{r} \alpha_i \alpha_j A_i X + B_2 L_i \\ \sum_{i=1}^{r} \sum_{j=1}^{r} \alpha_i \alpha_j C_{1i} \\ \sum_{i=1}^{r} \sum_{j=1}^{r} \alpha_i \alpha_j D_{12i} 
\end{pmatrix} + \Lambda + B_1 B_1^T
\]
\[ + \frac{1}{\gamma^2} (C_1 X + D_{12 K} X)^T (C_1 X + D_{12 K} X)
\]
\[ = M_3 + \Lambda + \frac{1}{\gamma^2} (C_1 X + D_{12 K} X)^T (C_1 X + D_{12 K} X) \leq 0. \tag{102} \]

where

\[ M_3 = (AX + B_2 K X) + (AX + B_2 K X)^T + B_1 B_1^T + \frac{1}{\gamma^2} (C_1 X + D_{12 K} X)^T (C_1 X + D_{12 K} X) \]

\[ \sum_{i=1}^{r} \sum_{j=1}^{r} \alpha_i \alpha_j T_{ij} = \begin{bmatrix} \sum_{i=1}^{r} \sum_{j=1}^{r} \alpha_i \alpha_j T_{ij} \\ \sum_{i=1}^{r} \sum_{j=1}^{r} \alpha_i B_{2i} \\ \sum_{i=1}^{r} \sum_{j=1}^{r} \alpha_i C_{1i} X + \sum_{i=1}^{r} \sum_{j=1}^{r} \alpha_i \alpha_j D_{12i} L_i \\ 0 \\ \end{bmatrix} - \gamma^2 I < 0. \tag{101} \]
and

\[ L_i = K_i X, \quad 1 \leq i \leq r. \]

By (99), (100), and (102), we have

\[ M_3 + \epsilon M_4 < 0, \quad \epsilon \in \left(0, \frac{1}{\lambda}\right). \]  

(103)

Pre- and postmultiplying (103) by \( P^T \) and \( P \), then we obtain

\[
P^T (A + B_2 K) + (A + B_2 K)^T P + P^T B_1 B_1^T P \\
+ \frac{1}{\gamma^2} (C_1 + D_{12} K)^T (C_1 + D_{12} K) \\
+ \epsilon \left[ P^T \left( A + B_2 K \right) + (A + B_2 K)^T P_a \right] \\
+ \epsilon \left[ P^T B_1 B_1^T P_a + P_a^T B_1 B_1^T P + \epsilon P_a^T B_1 B_1^T P_a \right] < 0
\]

i.e.,

\[
P \dot{A}_{ee} + A_{ee}^T P + P_b B_{ee} B_{ee}^T P + \frac{1}{\gamma^2} C_{ee} C_{ee} < 0.
\]

Then, by Lemma 1, the proof is completed.

---

REFERENCES


Guang-Hong Yang (SM’03) received the B.S. and M.S. degrees in mathematics from Northeast University of Technology, Shenyang, China, in 1983 and 1986, respectively, and the Ph.D. degree in control engineering from Northeastern University, Shenyang, China (formerly, Northeastern University of Technology), in 1994. He is currently a Professor with the College of Information Science and Engineering, Northeastern University. From 1986 to 1995, he was a Lecturer/Associate Professor with Northeastern University of Technology, Shenyang, China. From 2001 to 2005, he was a Research Scientist/Senior Research Scientist with the National University of Singapore, Singapore. His current research interests include fault tolerant control, fault detection and isolation, nonfragile control systems design, and robust control.