

Neumann boundary conditions from Born-Infeld dynamics

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Abstract

We would like to show that certain excitations of the F-string/D3-brane system can be shown to obey Neumann boundary conditions by considering the Born-Infeld dynamics of the F-string (viewed as a 3-brane cylindrically wrapped on an S_2). In the paper by Callan and Maldacena it was shown that excitations which are normal to both the string and the 3-brane behave as if they had Dirichlet boundary conditions at the point of attachment. Here we show that excitations which are coming down the string with a polarization along a direction parallel to the brane are almost completely reflected just as in the case of all-normal excitations, but the end of the string moves freely on the 3-brane, thus realizing Polchinski's open string Neumann boundary condition dynamically. In the low energy limit $\omega \rightarrow 0$, i.e. for wavelengths much larger than the string scale only a small fraction $\sim \omega^4$ of the energy escapes in the form of dipole radiation. The physical interpretation is that a string attached to the 3-brane manifests itself as an electric charge, and waves on the string cause the end point of the string to freely oscillate and therefore produce e.m. dipole radiation in the asymptotic outer region.

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1 Setup

Callan and Maldacena [1] showed that the Born-Infeld action, when taken as the fundamental action, can be used to build a configuration with a semi-infinite fundamental string ending on a 3-brane³, whereby the string is actually made out of the brane wrapped on S_2 (see also [2]). The relevant action can be obtained by computing a simple Born-Infeld determinant, dimensionally reduced from 10 dimensions

$$L = -\frac{1}{(2\pi)^3 g_s} \int d^4x \sqrt{1 - \vec{E}^2 + (\vec{\partial}x_9)^2}, \quad (1)$$

where g_s is the string coupling ($\alpha' = 1$).

The above mentioned theory contains 6 scalars x_4, \dots, x_9 , which are essentially Kaluza-Klein remnants from the 10-dimensional $N = 1$ electrodynamics after dimensional reduction to 3 + 1 dimensions. As is well known these extra components of the e.m. field A_4, \dots, A_9 describe the transverse deviations of the brane x_4, \dots, x_9 .

The solution, which satisfies the BPS conditions, is necessarily also a solution of the linear theory, the $N = 4$ *super*-Electrodynamics,

$$\vec{E} = \frac{c}{r^2} \vec{e}_r, \quad \vec{\partial}x_9 = \frac{c}{r^2} \vec{e}_r, \quad x_9 = -\frac{c}{r}, \quad (2)$$

where $c = \pi g_s$ sets the distance scale, and in what follows we drop it. Here the scalar field represents the geometrical spike, and the electric field insures that the string carries uniform NS charge along it. The RR charge of the 3-brane cancels out on the string, or rather the tube behaves as a kind of RR dipole whose magnitude can be ignored when the tube becomes thin. Also, it is seen in [1] that the infinite electrostatic energy of the point charge can be reinterpreted as being due to the infinite length of the attached string. The energy per unit length comes from the electric field and corresponds exactly to the fundamental string tension.

Polchinski, when he introduced D-branes as objects on which strings can end, required that the string have Dirichlet (fixed) boundary conditions for coordinates normal to the brane, and Neumann (free) boundary conditions for coordinate directions parallel to the brane [5, 3, 4]. It was shown in [1] that small fluctuations which are normal to both the string and the brane are mostly reflected back with a *phase shift* $\rightarrow \pi$ which indeed corresponds to Dirichlet boundary condition. See also [8] and [6] for a supergravity treatment of this problem.

In this paper we will show that P-wave excitations which are coming down the string with a polarization along a direction parallel to the brane are almost completely reflected just as in the case of all-normal excitations, but the end of the string moves

³ We choose the favorite case of the 3-brane because, first of its non-singular behaviour in SUGRA, and secondly it is suggestive of our own world which is after all 3-dimensional.

freely on the 3-brane, thus realizing Polchinski's open string Neumann boundary condition dynamically. As we will see a superposition of excitations of the scalar x_9 and of the e.m. field reproduces the required behaviour, e.g. reflection of the geometrical fluctuation with a *phase shift* $\rightarrow 0$ (Neumann boundary condition)⁴.

In addition we observe e.m. dipole radiation which escapes to infinity from the place where the string is attached to the 3-brane. We shall see that in the low energy limit $\omega \rightarrow 0$, i.e. for wavelengths much larger than the string scale a small fraction $\sim \omega^4$ of the energy escapes to infinity in the form of e.m. dipole radiation. The physical interpretation is that a string attached to the 3-brane manifests itself as an electric charge, and waves on the string cause the end point of the string to freely oscillate and therefore produce e.m. dipole radiation in the asymptotic outer region of the 3-brane. Thus not only in the static case, but also in a more general dynamical situation the above interpretation remains valid. This result provides additional support to the idea that the electron may be understood as the end of a fundamental string ending on a D-brane.

2 The Lagrangian and the equations

Let us write out the full Lagrangian which contains both electric and magnetic fields, plus the scalar $x_9 \equiv \phi$

$$L = - \int d^4x \sqrt{Det} ,$$

$$\text{where } Det = 1 + \vec{B}^2 - \vec{E}^2 - (\vec{E} \cdot \vec{B})^2 - (\partial_0\phi)^2(1 + \vec{B}^2) + (\vec{\partial}\phi)^2$$

$$+ (\vec{B} \cdot \vec{\partial}\phi)^2 - (\vec{E} \times \vec{\partial}\phi)^2 + 2\partial_0\phi(\vec{B}[\vec{\partial}\phi \times \vec{E}]) \quad (3)$$

We will proceed by adding a fluctuation to the background values (2) :

$$\vec{E} = \vec{E}_0 + \delta\vec{E}, \quad \vec{B} = \delta\vec{B}, \quad \phi = \phi_0 + \eta .$$

Then keeping only terms in the *Det* which are linear and quadratic in the fluctuation we will get

$$\delta Det = \delta\vec{B}^2 - \delta\vec{E}^2 - (\vec{E}_0\delta\vec{B})^2 - (\partial_0\eta)^2 + (\vec{\partial}\eta)^2$$

$$+ (\delta\vec{B}\vec{\partial}\phi)^2 - (\vec{E}_0 \times \vec{\partial}\eta)^2 - (\delta\vec{E} \times \vec{\partial}\phi)^2 - 2(\vec{E}_0 \times \vec{\partial}\eta)(\delta\vec{E} \times \vec{\partial}\phi)$$

$$- 2(\vec{E}_0\delta\vec{E}) + 2(\vec{\partial}\phi\vec{\partial}\eta)$$
(4)

⁴ This problem was also considered in [6] where the e.m. field is integrated out to produce an effective lagrangian for the scalar field only. The other essential difference with us is that we consider *P*-wave modes of the scalar field which describe physical transverse fluctuations of the F-string and not the *S*-wave modes which do not correspond to physical excitations of the string.

Note that one should keep the last two linear terms because they produce additional quadratic terms after taking the square root. These terms involve the longitudinal polarization of the e.m. field and cancel out at quadratic order. The resulting quadratic Lagrangian is

$$2L_q = \delta \vec{E}^2 (1 + (\vec{\partial}\phi)^2) - \delta \vec{B}^2 + (\partial_0\eta)^2 - (\vec{\partial}\eta)^2 (1 - \vec{E}_0^2) + \vec{E}_0^2 (\vec{\partial}\eta \cdot \delta\vec{E}) . \quad (5)$$

Let us introduce the gauge potential for the fluctuation part of the e.m. field as (A_0, \vec{A}) and substitute the values of the background fields from (2)

$$2L_q = (\partial_0\vec{A} - \vec{\partial}A_0)^2 (1 + \frac{1}{r^4}) - (\vec{\nabla} \times \vec{A})^2 + (\partial_0\eta)^2 - (\vec{\partial}\eta)^2 (1 - \frac{1}{r^4}) + \frac{1}{r^4} (\partial_0\vec{A} - \vec{\partial}A_0) \cdot \vec{\partial}\eta . \quad (6)$$

The equations that follow from this lagrangian contain dynamical equations for the vector potential and for the scalar field, and a separate equation which represents a constraint. These equations in the Lorenz gauge $\vec{\partial}\vec{A} = \partial_0A_0$ are

$$- \partial_0^2 \vec{A} (1 + \frac{1}{r^4}) + \Delta \vec{A} + \frac{1}{r^4} \vec{\partial} \partial_0 (A_0 + \eta) = 0 \quad (\alpha)$$

$$- \partial_0^2 A_0 + \Delta A_0 + \vec{\partial} \frac{1}{r^4} \vec{\partial} (A_0 + \eta) - \vec{\partial} \frac{1}{r^4} \partial_0 \vec{A} = 0 \quad (\beta)$$

$$- \partial_0^2 \eta + \Delta \eta - \vec{\partial} \frac{1}{r^4} \vec{\partial} (A_0 + \eta) + \vec{\partial} \frac{1}{r^4} \partial_0 \vec{A} = 0 \quad (\gamma)$$

Equation (β) is a constraint: the time derivative of the *lhs* is zero, as can be shown using the equation of motion (α) .

Let us choose $A_0 = -\eta$. This condition can be viewed as (an attempt to) preserve the BPS relation which holds for the background: $\vec{E} = \vec{\partial}\phi$. Another point of view is that this fixes the general coordinate invariance which is inherent in the Born-Infeld lagrangian in such a way as to make the given perturbation to be normal to the surface. Of course transversality is insured automatically but this choice makes it explicit. The general treatment of this subject can be found in [7].

With this condition the equations (β) and (γ) become the same, and the first equation is also simplified:

$$- \partial_0^2 \vec{A} (1 + \frac{1}{r^4}) + \Delta \vec{A} = 0 , \quad (7)$$

$$- \partial_0^2 \eta + \Delta \eta + \vec{\partial} \frac{1}{r^4} \partial_0 \vec{A} = 0 . \quad (8)$$

This should be understood to imply that once we obtain a solution, A_0 is determined from η , but in addition we are now obliged to respect the gauge condition which goes over to $\vec{\partial}\vec{A} = -\partial_0\eta$.

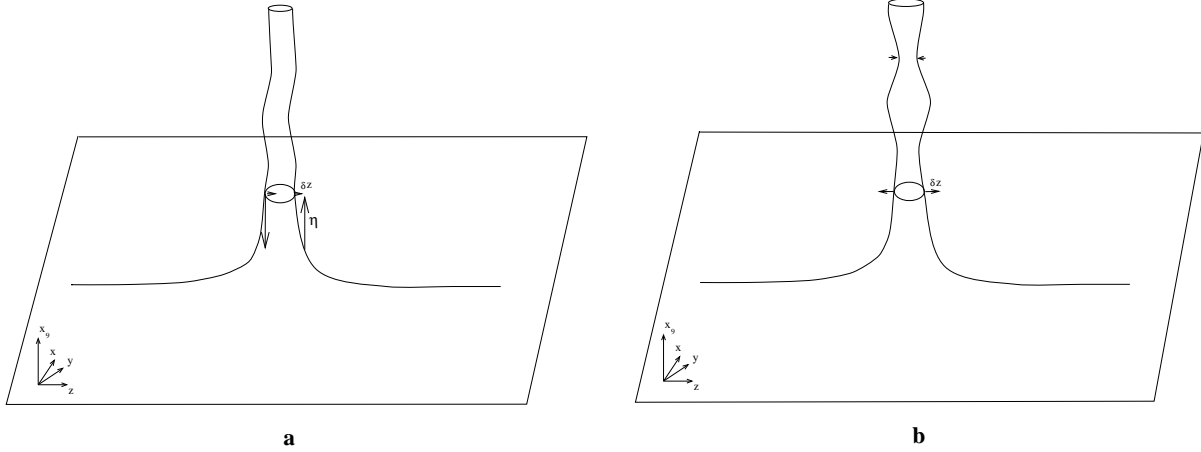


Figure 1: In order that all points on the S_2 section of the tube (which is schematically shown here as a circle) move all in the same direction \hat{z} by an equal distance δz , the field η has to take on different values at, say, opposite points of the S_2 . In effect, $\eta = \delta \frac{-1}{r} = \frac{1}{r^2} \hat{z} \delta z$. If, on the other hand, it is taken to be an S -wave (as in the paper [6]) that would correspond to Fig 1b, which at best is a problematic ‘internal’ degree of freedom of the tube.

3 Neumann boundary conditions and dipole radiation

We will seek for a solution with definite energy (frequency w) in the following form: \vec{A} should have only one component A_z , and η be an $l = 1$ spherical P -wave

$$A_z = \zeta(r) e^{-i\omega t} \quad , \quad \eta = \frac{z}{r} \psi(r) e^{-i\omega t}$$

The geometrical meaning of such a choice for η is explained in Fig 1. With this ansatz the equations become

$$\left(1 + \frac{1}{r^4}\right) \omega^2 \zeta + \frac{1}{r^2} \partial_r (r^2 \partial_r \zeta) = 0 \quad (9)$$

$$\frac{z}{r} \omega^2 \psi + \frac{z}{r} \frac{1}{r^2} \partial_r (r^2 \partial_r \psi) + \frac{z}{r} \frac{2}{r^2} \psi - i\omega \partial_z \left(\frac{\zeta}{r^4}\right) = 0 \quad , \quad (10)$$

with the gauge condition becoming $\partial_r \zeta = i\omega \psi$. It can be seen again, that the second equation follows from the first by differentiation. This is because the former coincides with the constraint in our ansatz.

Therefore the problem is reduced to finding the solution of a single scalar equation, and determining the other fields through subsidiary conditions. The equation itself surprisingly turned out to be the one familiar from [1] for the transverse fluctuations.

There it was solved by going over to a coordinate ξ which measures the distance radially along the surface of the brane

$$\xi(r) = \omega \int_1^r du \sqrt{1 + \frac{1}{u^4}} ,$$

and a new wavefunction

$$\tilde{\zeta} = \zeta (1 + r^4)^{1/4} .$$

This coordinate behaves as $\xi \sim r$ in the outer region ($r \rightarrow \infty$) and $\xi \sim -1/r$ on the string ($r \rightarrow 0$). The exact symmetry of the equation $r \leftrightarrow 1/r$ goes over to $\xi \leftrightarrow -\xi$. The equation, when written in this coordinate becomes just the free wave equation, plus a narrow symmetric potential at $\xi \sim 0$

$$-\frac{d^2}{d\xi^2} \tilde{\zeta} + \frac{5/\omega^2}{(r^2 + 1/r^2)^3} \tilde{\zeta} = 0 . \quad (11)$$

The asymptotic wave functions can be constructed as plain waves in ξ ,

$$\zeta(r) = (1 + r^4)^{-1/4} e^{\pm i\xi(r)} ,$$

or in the various limits:

$$\begin{aligned} r \rightarrow 0 & \quad \zeta \sim e^{\pm i\xi(r)} , \\ r \rightarrow \infty & \quad \zeta \sim \frac{1}{r} e^{\pm i\xi(r)} . \end{aligned}$$

These formulae give us the asymptotic wave function in the regions $\xi \rightarrow \pm\infty$, while around $\xi = 0$ ($r = 1$) there is a symmetric repulsive potential which drops very fast $\sim 1/\xi^6$ on either side of the origin. The scattering is described by a single dimensionless parameter $\omega\sqrt{c}$, and in the limit of small ω and/or coupling $c = \pi g_s$ the potential becomes narrow and high, and can be replaced by a δ -function with an equivalent area $\sim \frac{1}{\omega\sqrt{g_s}}$ under the curve. Therefore the scattering matrix becomes almost independent of the exact form of the potential. The end result is that most of the amplitude is reflected back with a phase shift close to π , thus dynamically realizing the Dirichlet boundary condition in the low energy limit.

In order to obtain ψ (and η) we need to differentiate ζ with respect to r :

$$i\omega\psi = \frac{-1}{4} \frac{4r^3}{(1+r^4)^{5/4}} e^{\pm i\xi(r)} \pm \frac{i\omega}{(1+r^4)^{1/4}} \left(1 + \frac{1}{r^4}\right)^{1/2} e^{\pm i\xi(r)} . \quad (12)$$

Again it is easy to obtain the simplified limiting form:

$$\begin{aligned} r \rightarrow 0 & \quad i\omega\psi \sim \left(-r^3 \pm \frac{i\omega}{r^2}\right) e^{\pm i\xi(r)} \\ r \rightarrow \infty & \quad i\omega\psi \sim \left(\frac{-1}{r^2} \pm \frac{i\omega}{r}\right) e^{\pm i\xi(r)} \end{aligned}$$

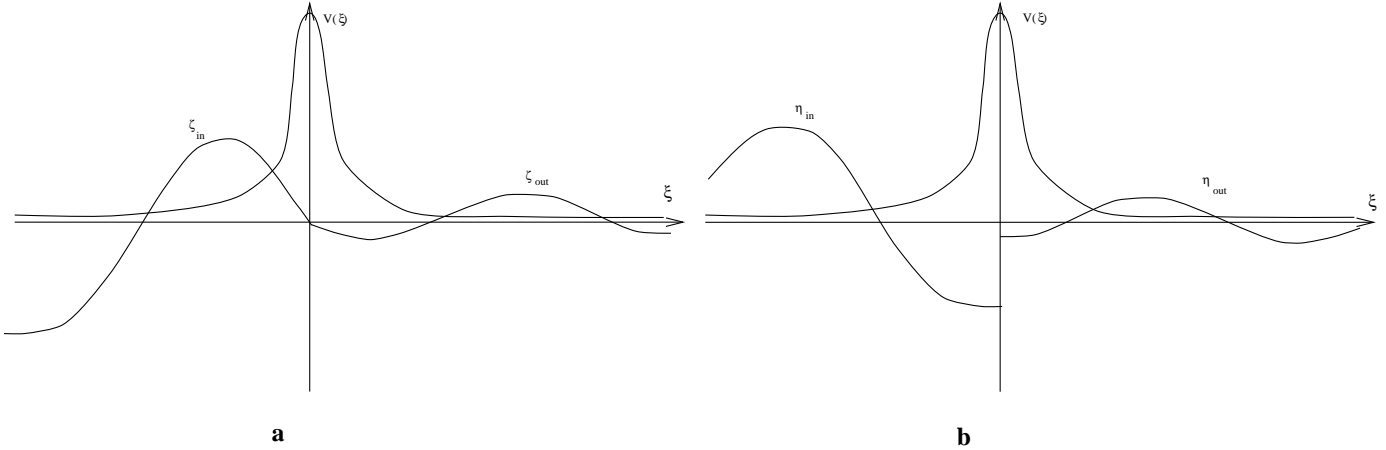


Figure 2: The figure 2a depicts the scattering of the ζ wave. Note the discontinuity in the derivative which is proportional to $1/\omega$. Figure 2b shows the scattering of the η wave. Being the derivative of ζ , it results in a discontinuity of the function itself, making it into a cosine, which means free (Neumann) boundary condition at $\xi = 0$.

This brings about several consequences for ψ . Firstly, it causes ψ to grow as $\sim 1/r^2$ as $r \rightarrow 0$. This is the correct behaviour because when converted to displacement in the z direction, it means constant amplitude. Secondly, the i that enters causes the superposition of the incoming and reflected waves to become a cosine from a sine, as is the case for ζ waves. This corresponds to a 0 phase shift and implies Neumann boundary condition for the η wave (Fig 2).

Because of the ω factor in the gauge condition we need to be careful about normalizations, thus we shall choose to fix the amplitude of the η wave to be independent of ω . Then the magnitude of the e.m. field in the inner region becomes independent of ω as well. Combined with the transmission factor, proportional to $\omega\sqrt{c}$, this gives the correct dependence of the total power emitted by an oscillating charge $\sim \omega^4 g_s^2$.

In conclusion, we need to analyze the outgoing scalar wave. This wave has both real and imaginary parts, the former is from differentiating the phase, while the latter is from the prefactor. The imaginary part is $\sim 1/r^2$ which drops faster than radiation. The real part does contribute to the radiation at spatial infinity, as can be shown from the integral of the energy density $\int (\partial_r \eta)^2 dr \sim \int \omega^4 / r^2 \cdot 4\pi r^2 dr$. This is not altogether surprising, as we are dealing with a supersymmetric theory where the different fields are tied together. Thus the observer at spatial infinity will see both an electromagnetic dipole radiation field and a scalar wave.

The problem of longitudinal fluctuations was treated in [6], though not in a completely satisfactory way. There the scalar field was taken to be an S -wave, which as should be apparent from Fig 1b, does not correspond to the string oscillating as a

whole. In addition the electromagnetic field was effectively integrated out, thus one cannot obtain the dipole radiation at spatial infinity.

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