Brief paper

$H_\infty$ control for fast sampling discrete-time singularly perturbed systems

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Abstract

This paper is concerned with the $H_\infty$ control problem via state feedback for fast sampling discrete-time singularly perturbed systems. A new $H_\infty$ controller design method is given in terms of solutions to linear matrix inequalities (LMIs), which eliminates the regularity restrictions attached to the Riccati-based solution. A method for evaluating the upper bound of singular perturbation parameter $\epsilon$ with meeting a prescribed $H_\infty$ performance bound requirement is also given. Furthermore, the results are extended to robust controller design for fast sampling discrete-time singularly perturbed systems with polytopic uncertainties. Numerical examples are given to illustrate the validity of the proposed methods.

Keywords: Discrete-time singularly perturbed systems; Polytopic uncertainty; $H_\infty$ control; Linear matrix inequalities (LMIs); State feedback control

1. Introduction

It is well known that the multiple time-scale systems otherwise known as singularly perturbed systems often raise serious numerical problems in the control engineering field. In order to avoid the difficulties linked with the stiffness of the equations involved in the design, they are usually modeled as singularly perturbed control systems with a small singular perturbation parameter $\epsilon$, being exploited to determine the degree of separation between slow and fast parts of dynamical systems. In the past three decades, singularly perturbed systems have been intensively studied by many researchers, see Kafri and Abed (1996), Li and Li (1995), Li, Chiou, and Kung (1999), Li and Li (1992), Lim and Gajic (2000), Xu and Mizukami (1997), Naidu and Rao (1985) and Singh, Brown, and Naidu (1998) and a survey paper Naidu (2002). In the framework of singularly perturbed systems, a popular approach is the so-called reduction technique, which is a two-step design methodology. Firstly, through the separate stabilization of two lower-dimensional subsystems in two different time scales, a composite stabilizing controller is synthesized from the separate stabilizing controllers of the two subsystems, where the controller could be determined without the knowledge of the small singular perturbation parameter. Based on the reduction technique and Riccati equation approach, the $H_\infty$ control problem for continuous-time singularly perturbed systems has been addressed in Fridman (1996), Lim and Gajic (2000), Pan and Basar (1993, 1994), Shi and Dragan (1999) and Tan, Leung, and Tu (1998). The discrete-time case can be represented by three models: pure singularly perturbed discrete systems which are inherently discrete in nature, the slow sampling rate model, and the fast sampling rate model. The last two models are discretization of the singularly perturbed continuous-time systems. The fast sampling model is more practical in developing the stabilizing optimal control method (Kim, Kim, & Lim, 2002). Based on the two-step methodology and Riccati equation approach, $H_\infty$ controller design methods are given, respectively, in Naidu, Charalambous, Moore, and Abdelrahma (1994) for slow sampling discrete-time singularly perturbed systems and in Datta and RaiChaudhuri (2002) for fast sampling ones. Moreover, a unified approach is provided in Singh, Brown, Naidu, and Heinen (2001) for $H_\infty$ control for both continuous and discrete-time systems.
In general, most of these approaches for control synthesis are independent of $\epsilon$ for avoiding the ill-condition condition. Therefore, it is of great importance to find the $\epsilon$-bound for the stability of the closed-loop systems. By considering critical stability criterion with a bialternate product, Ghosh, Sen, and Datta (1999) and Li et al. (1999) present systematic approaches to determine the exact stability bound of singularly perturbed discrete-time systems. Moreover, an algorithm to find the upper bound of singular perturbation parameter for $D$-stability is given in Hsiao, Pan, and Hwang (2000) and Hsiao, Hwang, and Pan (2003). However, by the authors’ knowledge, the approach for evaluating the upper bound of singular perturbation parameter with meeting $H_\infty$ performance requirement has not been studied. In this paper, such approach will be presented.

In the past decade, linear matrix inequality (LMI) technique has been extensively exploited to solve control problems (Boyd, Ghaoui, Feron, & Balakrishnan, 1994). In contrast to the Riccati approach, the LMIs that arise in system and control theory can be formulated as convex optimization problems that are amenable to compute solution and can be solved effectively (Boyd et al., 1994). Another good feature of the LMI is their ability of adding constraints to the parametrical optimization problem provided they are themselves linear with respect to unknowns (Garcia, Daafouz, & Bernussou, 2002). In particular, for $H_\infty$ synthesis, it has the merit of eliminating the regularity restrictions attached to the Riccati-based solution (Gahinet, Nemirovski, Laub, & Chilali, 1995). Motivated by the merits of the LMI formulations, some significant advances have been achieved for developing LMI-based approaches to the control synthesis for singularly perturbed systems. In particular, based on the LMI technique, state feedback $H_\infty$ controller design methods are given in Fridman (2006) for continuous-time singularly perturbed systems with norm-bounded uncertainties. Garcia and Tarbouriech (2003) addresses the control design problem for linear singularly perturbed systems subject to bounded control. With pole-placement constraints, Lin and Li (2006) presents a sufficient condition for designing robust $H_\infty$ dynamic output feedback controller. Most of the above-mentioned results for control synthesis are for continuous-time singularly perturbed systems. However, for the discrete-time case, there has been no LMI-based formulation for control synthesis.

This paper will be concerned with the $H_\infty$ controller design problem for fast sampling discrete-time singularly perturbed systems. By constructing $\epsilon$-dependent Lyapunov function, an $H_\infty$ controller design method is given in terms of solutions to a set of $\epsilon$-independent LMIs, which can avoid the ill-conditioned numerical problem in LMIs, and eliminates the regularity restrictions attached to the Riccati-based solutions in Vu and Sawan (1993) and Datta and RaiChaudhuri (2002). Moreover, an approach is presented to estimate the upper bound of $\epsilon$ of a singularly perturbed control system subject to $H_\infty$ performance bound constraint. The paper is organized as follows. Section 2 presents system description and problem statement. In Section 3, an $H_\infty$ controller design method is given based on $\epsilon$-independent LMIs, and a method of evaluating the upper bound of singular perturbation parameter $\epsilon$ with meeting a prescribed $H_\infty$ performance bound requirement is also presented. Section 4 extends the results to systems with polytopic uncertainties. The effectiveness of the proposed methods is illustrated by two numerical examples in Section 5. Finally, conclusions are given in Section 6.

Notation: $(\cdot)^T$ is used for the blocks induced by symmetry. The superscript $T$ stands for matrix transposition and the notation $M^{-T}$ denotes the transpose of the inverse matrix of $M$. $\mathcal{L}_2$ is the Lebesgue space consisting of all discrete-time time-varying functions that are square-summable over $[0, \infty)$. $\|z\|_2$ denotes the $L_2$-norm of a vector-valued function $z$.

2. System description and problem statement

Consider the following fast sampling discrete-time singularly perturbed system:

$$
\begin{bmatrix}
    x_{1}(k+1) \\
    x_{2}(k+1)
\end{bmatrix} =
\begin{bmatrix}
    I_{n1} + \epsilon A_{11} & \epsilon A_{12} \\
    A_{21} & A_{22}
\end{bmatrix}
\begin{bmatrix}
    x_{1}(k) \\
    x_{2}(k)
\end{bmatrix} +
\begin{bmatrix}
    \epsilon B_{11} \\
    B_{12}
\end{bmatrix} w(k) +
\begin{bmatrix}
    \epsilon B_{21} \\
    B_{22}
\end{bmatrix} u(k)
$$

(1)

and $x_1(k) \in R^{n_1}$, $x_2(k) \in R^{n_2}$, $w(k) \in R^{m_1}$ is disturbance input, $u(k) \in R^{m_2}$ is the control input, $z(k) \in R^{P}$ is the output to be controlled. The positive singular perturbation parameter is denoted by $\epsilon$.

Remark 1. The model (1) was investigated in Datta and RaiChaudhuri (2002), Li and Li (1995), Li et al. (1999) and Vu and Sawan (1993), respectively.

Denote

$$
\begin{align*}
A_\epsilon &= \begin{bmatrix}
    I_{n1} + \epsilon A_{11} & \epsilon A_{12} \\
    A_{21} & A_{22}
\end{bmatrix}, \\
B_{1\epsilon} &= \begin{bmatrix}
    \epsilon B_{11} \\
    B_{12}
\end{bmatrix}, \\
B_{2\epsilon} &= \begin{bmatrix}
    \epsilon B_{21} \\
    B_{22}
\end{bmatrix}, \\
C_1 &= \begin{bmatrix}
    C_{11} & C_{12}
\end{bmatrix}, \\
x(k) &= \begin{bmatrix}
    x_{1}(k) \\
    x_{2}(k)
\end{bmatrix}
\end{align*}
$$

(2)

then system (1) can be rewritten as follows

$$
\begin{align*}
x(k+1) &= A_\epsilon x(k) + B_{1\epsilon} w(k) + B_{2\epsilon} u(k) \\
z(k) &= C_1 x(k) + D_{11} u(k) + D_{12} u(k).
\end{align*}
$$

(3)

Consider the state feedback control law

$$
u(k) = K x(k)
$$

(4)

then the resulting closed-loop system is given by

$$
\begin{align*}
x(k+1) &= (A_\epsilon + B_{2\epsilon} K) x(k) + B_{1\epsilon} w(k) \\
z(k) &= (C_1 + D_{12} K) x(k) + D_{11} u(k).
\end{align*}
$$

(5)

In this paper, we consider the disturbance attenuation problem characterized by means of the so-called $L_2$ gain of a nonlinear system which is defined as follows:
Definition 2 (Lin & Byrnes, 1996). Given a real number $\gamma > 0$, it is said that the exogenous signals are locally attenuated by $\gamma$ if there exists a neighborhood $U$ of $x = 0$ such that for every positive integer $N$ and for every $w \in l_2[0, N], R^{m_1}$ for which the state trajectory of the closed-loop system (1) or (39) starting $x(0) = 0$ remains in $U$ for all $k \in [0, N]$, the response $z \in l_2([0, N], R^p)$ of (1) or (39) satisfies

$$\sum_{j=1}^{N} \|z_j\|^2 \leq \gamma^2 \sum_{j=1}^{N} \|w_j\|^2, \quad \text{for all } N. \tag{6}$$

The following lemma will be used in the sequel.

Lemma 3. (i). For a scalar $\epsilon$ with $0 < \epsilon < 1$ and a symmetric matrix $Q_{11} > 0$ satisfy

$$-\begin{bmatrix} T_{11} & T_{12} \\ T_{12}^T & T_{22} \end{bmatrix} \succeq \begin{bmatrix} -Q_{11} & Q_{11} \\ Q_{11} & -Q_{11} \end{bmatrix} \tag{7}$$

then

$$-\epsilon \begin{bmatrix} T_{11} & T_{12} \\ T_{12}^T & T_{22} \end{bmatrix} \succeq \begin{bmatrix} -Q_{11} & Q_{11} \\ Q_{11} & -Q_{11} \end{bmatrix} \tag{8}$$

(ii). For a positive scalar $\epsilon^* < 1$, if symmetric matrices $Q_{11} > 0$ and $\Psi_j$ satisfy (7) and

$$\frac{1}{\epsilon^{2}} \begin{bmatrix} -Q & Q_1 \\ Q & -Q \end{bmatrix} + \frac{1}{\epsilon^{2}} \begin{bmatrix} T_{11} & T_{12} \\ T_{12}^T & T_{22} \end{bmatrix} \succeq \begin{bmatrix} -\Psi_1 & \Psi_1 \\ \Psi_1 & -\Psi_1 \end{bmatrix} \tag{9}$$

then

$$\frac{1}{\epsilon^{2}} \begin{bmatrix} -Q & Q_1 \\ Q & -Q \end{bmatrix} + \frac{1}{\epsilon^{2}} \begin{bmatrix} T_{11} & T_{12} \\ T_{12}^T & T_{22} \end{bmatrix} \succeq \begin{bmatrix} -\Psi_1 & \Psi_1 \\ \Psi_1 & -\Psi_1 \end{bmatrix} \tag{10}$$

for $\epsilon \in (0, \epsilon^*)$

Proof. (i). For all nonzero vector $\xi(k) = [\xi_1^T(k) \xi_2^T(k)]^T$, where $\xi_1(k)$ and $\xi_2(k)$ belong to $R^{m_1}$. Pre- and post-multiplying (7) by $\xi^T(k)$ and its transpose, then we have

$$-\begin{bmatrix} \xi_1(k)^T \\ \xi_2(k)^T \end{bmatrix} \begin{bmatrix} T_{11} & T_{12} \\ T_{12}^T & T_{22} \end{bmatrix} \begin{bmatrix} \xi_1(k) \\ \xi_2(k) \end{bmatrix} \succeq \begin{bmatrix} -Q_{11} & Q_{11} \\ Q_{11} & -Q_{11} \end{bmatrix} \begin{bmatrix} \xi_1(k) \\ \xi_2(k) \end{bmatrix}. \tag{11}$$

Let $T_{\xi} = \xi^T(k) \begin{bmatrix} T_{11} & T_{12} \\ T_{12}^T & T_{22} \end{bmatrix} \xi(k), Q_{\xi} = \xi^T(k) \begin{bmatrix} -Q_{11} & Q_{11} \\ Q_{11} & -Q_{11} \end{bmatrix} \xi(k)$, then (11) can be rewritten as follows

$$-T_{\xi} \succeq Q_{\xi}. \tag{12}$$

From $Q_{11} > 0$, it follows that

$$Q_{\xi} = -\epsilon \xi_1(k) - \xi_2(k))^T Q_{11} \xi_1(k) - \xi_2(k)) \leq 0. \tag{13}$$

Considering the following two cases

(a): If $T_{\xi} \succeq 0$, then $-\epsilon T_{\xi} \succeq -T_{\xi}$, for $\epsilon \in (0, 1)$. Combining it and (12), then yields $-\epsilon T_{\xi} \succeq Q_{\xi}$, for $\epsilon \in (0, 1)$.

(b): If $T_{\xi} < 0$, then $-\epsilon T_{\xi} > 0$, for $\epsilon \in (0, 1)$. Combining it and (13), we can obtain $-\epsilon T_{\xi} > Q_{\xi}$, for $\epsilon \in (0, 1)$.

Therefore, $-\epsilon T_{\xi} > Q_{\xi}$, for $\epsilon \in (0, 1)$, which implies that (8) holds.

(ii): From $Q_{11} > 0$, it follows that for all nonzero vector $\xi_1(k) = [\xi_1^T(k) \xi_2^T(k)]^T$, satisfying (13). In particular, left side of (13) is equal to zero if and only if $\xi_1(k) = \xi_2(k)$. Denote

$$y \left( \frac{1}{\epsilon} \right) = a_{\xi} \frac{1}{\epsilon^2} + b_{\xi} \frac{1}{\epsilon} + c_{\xi} \tag{14}$$

where $a_{\xi} = \xi^T(k) \begin{bmatrix} -Q_{11} & Q_{11} \\ Q_{11} & -Q_{11} \end{bmatrix} \xi(k), b_{\xi} = \xi^T(k) \begin{bmatrix} 0 \\ Q_{11} \end{bmatrix}, c_{\xi} = \xi^T(k) \begin{bmatrix} 0 \\ -Q_{11} \end{bmatrix} \xi(k)$. From (7), we have

$$a_{\xi} + b_{\xi} \leq 0. \tag{15}$$

From (13), it follows that $a_{\xi} \leq 0$. Pre- and post-multiplying (9) by $\xi^T(k)$ and its transpose, then we have

$$y \left( \frac{1}{\epsilon^*} \right) = a_{\xi} \frac{1}{\epsilon^{2*}} + b_{\xi} \frac{1}{\epsilon^{*}} + c_{\xi} \leq 0. \tag{16}$$

The following two cases are discussed.

(a): If $a_{\xi} < 0$, then for $\lambda \geq 1/2$, we have $dy(\lambda)/d\lambda = 2a_{\xi}\lambda + b_{\xi} \leq a_{\xi} + b_{\xi}$. Combining it and (15), then yields $dy(\lambda)/d\lambda \leq 0$, which implies that $y(\lambda)$ is a monotone decreasing function for $\lambda \geq 1/2$. By (16) and $\frac{1}{\epsilon^*} \geq 1$, it follows that

$$y \left( \frac{1}{\epsilon} \right) \leq 0, \quad \text{for } \epsilon \in \left[ \frac{1}{\epsilon^{*}}, \infty \right), \quad \text{(i.e., } \epsilon \in (0, \epsilon^{*}]). \tag{17}$$

(b): If $a_{\xi} = 0$, which implies that $\xi(k) = [\xi_1^T(k) \xi_2^T(k)]^T$, with $\xi_1(k) = \xi_2(k)$, it follows that $c_{\xi} = -\xi^T(k) \begin{bmatrix} -\Psi_1 & \Psi_1 \\ \Psi_1 & -\Psi_1 \end{bmatrix} \xi(k) = 0$, then (14) and (16) respectively become $y \left( \frac{1}{\xi} \right) = b_{\xi} \frac{1}{\xi}$ and

$$y \left( \frac{1}{\epsilon^*} \right) = b_{\xi} \frac{1}{\epsilon^{*}} \leq 0. \tag{17}$$

From (17), we have $b_{\xi} \leq 0$, which further implies that $y \left( \frac{1}{\xi} \right) = b_{\xi} \frac{1}{\xi} \leq 0$, for $0 < \xi \leq \epsilon^{*} < 1$.

Thus, for all nonzero vector $\xi(k)$, (14) is less than or equal to zero for $\epsilon \in (0, \epsilon^{*})$, which implies that (10) holds for $\epsilon \in (0, \epsilon^{*})$.

3. $H\infty$ control

In this section, an LMI-based $H\infty$ controller design method is presented. Moreover, a sufficient condition is derived for evaluating the upper bound $\epsilon^{*}$ of $\epsilon$ subject to a prescribed $H\infty$ performance bound constraint.

3.1. $H\infty$ control synthesis

The following theorem presents an LMI-based $H\infty$ controller design method for system (1).

**Theorem 4.** If there exist a symmetric matrix $Q = \begin{bmatrix} Q_{11} & Q_{12} \\ Q_{12}^T & Q_{22} \end{bmatrix} > 0$ and matrices $T_{11}, T_{12}, T_{22}, L_1, L_2$, such that
the following LMI holds,
\[
\begin{bmatrix}
-Q_{11} & Q_{11} & T_{12} \\
Q_{11} & -Q_{11} & T_{11} \\
T_{22} & T_{12} & -Q_{12}
\end{bmatrix} < 0
\]  
(18)

where
\[
A_{41} = A_{11}Q_{11} + A_{12}Q_{12}^T + B_2L_1 \\
A_{42} = A_{12}Q_{22} + B_2L_2 \\
A_{51} = A_{21}Q_{11} + A_{22}Q_{12}^T + B_2L_1 \\
A_{52} = A_{22}Q_{22} + B_2L_2 \\
A_{61} = C_1Q_{11} + C_1Q_{12}^T + D_1L_1 \\
A_{62} = C_2Q_{22} + D_2L_2
\]
then there exists a sufficiently small \( \epsilon^* > 0 \) such that for \( \epsilon \in (0, \epsilon^*) \), the state feedback controller
\[
u(k) = K x(k) = [K_1 \ K_2] \begin{bmatrix} x_1(k) \\ x_2(k) \end{bmatrix}
\]  
(20)
renders the closed-loop system (5) asymptotically stable and with an \( H_\infty \) norm less than \( \gamma \), where
\[
K_1 = (L_1 - L_2Q_{22}^{-1}Q_{12}^T)Q_{11}^{-1}, \quad K_2 = L_2Q_{22}^{-1}.
\]  
(21)

Proof. From \( Q_{11} \epsilon Q_{12} \epsilon Q_{12}^T \epsilon Q_{22} > 0 \), it follows that
\[
\epsilon Q_{11} \epsilon Q_{12} \epsilon Q_{12}^T \epsilon Q_{12}^T > 0, \quad \text{for } 0 < \epsilon < 1.
\]
Let \( P_\epsilon = \begin{bmatrix} Q_{11}^\epsilon & Q_{12}^\epsilon \\ Q_{12}^\epsilon & Q_{22}^\epsilon \end{bmatrix}^{-1} \), then \( P_\epsilon > 0 \). Choose Lyapunov function \( V(k) = x^T(k)P_\epsilon x(k) \), then
\[
V(k + 1) - V(k) + \frac{1}{\gamma} z^T(k)z(k) - \gamma w^T(k)w(k)
\]
\[
= \begin{bmatrix} x^T(k) & w^T(k) \end{bmatrix} M_\epsilon \begin{bmatrix} x(k) \\ w(k) \end{bmatrix}
\]  
(22)
where
\[
M_\epsilon = \begin{bmatrix}
-P_\epsilon + A_{12}^T P_\epsilon A_{12} + \frac{1}{\gamma} C_1^T C_1 & * \\
B_{12}^T P_\epsilon A_{12} + \frac{1}{\gamma} D_{11}^T D_{11} - \frac{1}{\gamma} D_{11}^T D_{11} - \gamma I
\end{bmatrix}.
\]  
(23)

On the other hand, pre- and post-multiplying (19) by diag\( [I, \ 1, \ 1, \ \epsilon I, \ 1, \ I] \) and its transpose, it follows that
\[
\begin{bmatrix}
-\epsilon T_{11} & * & * & * & * \\
-\epsilon T_{12} & -Q_{12} & * & * & * \\
0 & 0 & -\gamma I & * & * \\
\epsilon(A_{41} - T_{12}) & \epsilon(Q_{12} + A_{42}) & \epsilon B_{11} & -\epsilon T_{22} & * \\
\epsilon A_{51} & \epsilon A_{52} & B_{12} & -\epsilon Q_{12} & -Q_{22} & * \\
\epsilon A_{61} & \epsilon A_{62} & D_{11} & 0 & 0 & -\gamma I
\end{bmatrix} < 0.
\]  
(24)

Applying Lemma 3(i) to (24), we can obtain
\[
\begin{bmatrix}
-\epsilon Q_{11} & * & * & * & * \\
-\epsilon Q_{12} & -Q_{22} & * & * & * \\
0 & 0 & -\gamma I & * & * \\
\epsilon Q_{11} + \epsilon^2 A_{41} & \epsilon Q_{12} + \epsilon A_{42} & \epsilon B_{11} & -\epsilon Q_{12} & * \\
\epsilon A_{51} & \epsilon A_{52} & B_{12} & -\epsilon Q_{12} & -Q_{22} & * \\
\epsilon A_{61} & \epsilon A_{62} & D_{11} & 0 & 0 & -\gamma I
\end{bmatrix} < 0, \quad \text{for } 0 < \epsilon < 1.
\]  
(25)
From (21), we have
\[
L_1 = K_1 Q_{11} + K_2 Q_{12}^T, \quad L_2 = K_2 Q_{22}.
\]  
(26)
By (25) and (26) can be rewritten as follows
\[
\begin{bmatrix}
-\epsilon Q_{11} & * & * & * & * \\
-\epsilon Q_{12} & -Q_{22} & * & * & * \\
0 & 0 & -\gamma I & * & * \\
\epsilon Q_{11} + \epsilon^2 T_{11} & \epsilon Q_{12} + \epsilon A_{42} & \epsilon B_{11} & -\epsilon Q_{12} & * \\
\epsilon A_{51} & \epsilon A_{52} & B_{12} & -\epsilon Q_{12} & -Q_{22} & * \\
\epsilon A_{61} & \epsilon A_{62} & D_{11} & 0 & 0 & -\gamma I
\end{bmatrix} < 0, \quad \text{for } 0 < \epsilon < 1.
\]  
(27)

Thus, there exists a sufficiently small \( \epsilon^* \) with \( \epsilon^* < 1 \) such that the following inequality holds,
\[
\begin{bmatrix}
-\epsilon Q_{11} & * & * & * & * \\
0 & -\gamma I & * & * & * \\
(A_{11} + B_{21}K_1)Q_{11} + (A_{12} + B_{21}K_2)Q_{12}^T & * & * & * & * \\
(C_1 + D_{12}K_1)\epsilon & D_{11} & 0 & * & * \\
D_{21} & -\epsilon Q_{12} & -Q_{22} & * & *
\end{bmatrix} < 0,
\]  
(28)

for \( \epsilon \in (0, \epsilon^*) \),

where \( Q_{11} = \begin{bmatrix} Q_{11}^\epsilon & Q_{12}^\epsilon \\ Q_{12}^\epsilon & Q_{22}^\epsilon \end{bmatrix} \) and \( A_{11}, \ B_{11}, \ B_{21}, \ C_1 \) are same as in (2). Applying the Schur complement to (28), we have \( M_\epsilon < 0 \), where \( M_\epsilon \) is same as in (23). Therefore, (22) is less than zero, which implied the closed-loop system (5) is asymptotically stable. For all \( N \in \{1, 2, 3, \ldots\} \), summing (22) from \( k = 1 \) to \( k = N \), it follows that
\[
V(N) - V(0) + \frac{1}{\gamma} \sum_{k=1}^{N} z^T(k)z(k) - \gamma \sum_{k=1}^{N} w^T(k)w(k)
\]
\[
< 0, \quad \text{for all } N.
\]  
(29)
By \( V(N) \geq 0 \) and the above inequality, we have
\[
\sum_{k=1}^{N} z^T(k)z(k) < \gamma^2 \sum_{k=1}^{N} w^T(k)w(k), \quad \text{for all } N
\]
which further implies that (6) holds, i.e., the \( H_\infty \) norm of closed-loop system (5) is less than \( \gamma \). Thus, the proof is complete. \( \square \)
Remark 5. (i): Theorem 4 presents an LMI-based sufficient condition for $H_{\infty}$ control synthesis for fast sampling discrete-time singularly perturbed systems, which can be effectively solved via LMI control toolbox (Gahinet et al., 1995). In particular, the design method eliminates the regularity restrictions attached to the Riccati-based solutions in Datta and RalChaudhuri (2002); Vu and Sawan (1993). Moreover, the design method given by Theorem 4 can be extended to robust $H_{\infty}$ controller design for singularly perturbed systems with polytopic uncertainty (see Section 4). The effectiveness of the design methods given by Theorem 4 and Vu and Sawan (1993) will be illustrated via a numerical example in Section 5.

(ii): Notice that the reduction technique for analysis and control synthesis of singularly perturbed systems can avoid large dimension in the computation, which is very important and effective for singularly perturbed systems. Though the proposed LMI-based approach is of full dimensions, it provides a convex alternative method. In general, the LMI problems with reasonable large dimensions can also be effectively solved by using LMI control toolbox (Gahinet et al., 1995).

3.2. Upper bound of $\epsilon$

In this subsection, a sufficient condition is derived for evaluating the upper bound of singular perturbation parameter $\epsilon$ subject to a prescribed $H_{\infty}$ performance constraint. The following lemma will be used in this sequel.

Lemma 6. If there exist symmetric matrices $Q = \begin{bmatrix} Q_{11} & Q_{12} \\ Q_{12}^T & Q_{22} \end{bmatrix} > 0$, $\Psi_1$ and matrices $T_{11}, T_{12}, T_{22}$ satisfying (10) and the following LMIs

\[
-\Psi_1 \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} + \frac{1}{\epsilon} \begin{bmatrix} -\Psi_1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}
\]

\[
\begin{bmatrix} -T_{11} & * & * & * & * \\ -Q_{12}^T & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ \end{bmatrix} + \frac{1}{\epsilon} \begin{bmatrix} -Q_{12} & 0 & 0 & 0 & 0 \\ 0 & -\gamma I & 0 & 0 & 0 \\ 0 & 0 & -\gamma I & 0 & 0 \\ 0 & 0 & 0 & -\gamma I & 0 \\ 0 & 0 & 0 & 0 & -\gamma I \end{bmatrix}
\]

\[
< 0, \text{ for } \epsilon \in (0, \epsilon^*)
\]

where

\[
\begin{align*}
\Xi_1 &= A_{11}Q_{11} + A_{12}Q_{12}^T, \\
\Xi_2 &= A_{12}Q_{22}, \\
\Xi_3 &= A_{21}Q_{11} + A_{22}Q_{12}^T, \\
\Xi_4 &= A_{22}Q_{22}, \\
\Xi_5 &= C_{11}Q_{11} + C_{12}Q_{12}^T, \\
\Xi_6 &= C_{12}Q_{22}
\end{align*}
\]

then for singular perturbation parameter $\epsilon \in (0, \epsilon^*)$, the fast sampling discrete-time singularly perturbed system (1) with $u(k) = 0$ is asymptotically stable and has an $H_{\infty}$ norm less than $\gamma$.

Proof. By (10) and (30), we have

\[
\begin{bmatrix} -Q_{11} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} + \frac{1}{\epsilon} \begin{bmatrix} T_{11} & 0 & 0 & 0 & 0 \\ 0 & T_{12} & 0 & 0 & 0 \\ 0 & 0 & T_{21} & 0 & 0 \\ 0 & 0 & 0 & T_{22} & 0 \end{bmatrix}
\]

\[
+ \frac{1}{\epsilon} \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}
\]

\[
\begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}
\]

\[
\begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}
\]

\[
\begin{bmatrix} -T_{11} & * & * & * & * \\ -Q_{12} & 0 & 0 & 0 & 0 \\ 0 & 0 & -\gamma I & 0 & 0 \\ 0 & 0 & 0 & -\gamma I & 0 \\ 0 & 0 & 0 & 0 & -\gamma I \end{bmatrix}
\]

\[
< 0, \text{ for } \epsilon \in (0, \epsilon^*)
\]

In the following section, we will present the new results by $\epsilon^* = 0.2$.
for $\varepsilon \in (0, \varepsilon^*)$.

Applying the Schur complement to (34), it follows that
\begin{equation}
\begin{bmatrix}
-P + A_T^T P A_e + \frac{1}{\gamma} C_1^T C_1 & A_T^T P B_{1e} + \frac{1}{\gamma} C_1^T D_{11} \\
B_{1e}^T P A_e + \frac{1}{\gamma} D_{11}^T C_1 & B_{1e}^T P B_{1e} + \frac{1}{\gamma} D_{11}^T D_{11} - \gamma I
\end{bmatrix} < 0,
\end{equation}
for $\varepsilon \in (0, \varepsilon^*)$.  
(35)
where $A_e, B_{1e}$ are same as in (2). Pre- and post-multiplying (35) by vector $[x(k) \ w(k)]$ and its transpose, we have
\begin{equation}
(A_e x(k) + B_{1e} w(k))^T P (A_e x(k) + B_{1e} w(k)) - x^T(k) P x(k) + \frac{1}{\gamma} z^T(k) z(k) - \gamma w^T(k) w(k) < 0,
\end{equation}
for $\varepsilon \in (0, \varepsilon^*)$.  
(36)
Because $P = Q^{-1} > 0$, choose Lyapunov function $V = x^T(k) P x(k)$, then (36) can be rewritten as follows:
\begin{equation}
V(k+1) - V(k) + \frac{1}{\gamma} z^T(k) z(k) - \gamma w^T(k) w(k) < 0.
\end{equation}
Consider the above inequality, then the proof is easily obtained and omitted.  
\[\Box\]

Based on Lemma 6, a method of evaluating the upper bound of singular perturbation parameter $\varepsilon$ with meeting a prescribed $H_\infty$ performance bound requirement is given in the following theorem.

**Theorem 7.** For $0 < \varepsilon < 1$, if there exist symmetric matrices $Q = \begin{bmatrix} Q_{11} & Q_{12} \\ Q_{12} & Q_{22} \end{bmatrix} > 0$, $\Psi_{11}$ and matrices $T_{11}$, $T_{12}$, $T_{22}$ satisfy (7) and (9) and the following LMIs,
\begin{equation}
\begin{bmatrix}
-T_{11} & * & * & * & * \\
Q_{11} - T_{12}^T & -Q_{22} & * & * & * \\
0 & 0 & -\gamma I & * & * \\
\Xi_{34} & \Xi_{32} & B_{12} - Q_{12} & -Q_{22} & * \\
\Xi_{56} & 0 & 0 & 0 & -\gamma I
\end{bmatrix} < 0
\end{equation}
(37)
\begin{equation}
\begin{bmatrix}
-\Psi_{11} & 0 & 0 & \Psi_{11} & 0 & 0 & 0 \\
0 & 0 & 0 & Q_{11}^T A_{11}^T & Q_{12}^T A_{12}^T & Q_{12}^T C_{11} \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\Psi_{11} & A_{11} Q_{12} & 0 & -\Psi_{11} & 0 & 0 & 0 \\
0 & A_{21} Q_{12} & 0 & 0 & 0 & 0 & 0 \\
0 & C_{11} Q_{12} & 0 & 0 & 0 & 0 & 0
\end{bmatrix} < 0
\end{equation}
(38)
where $\Xi_{34}, \Xi_{32}, \Xi_{31}, \Xi_{52}, \Xi_{51}, \Xi_{53}$ are same as in (31). Then for singular perturbation parameter $\varepsilon \in (0, \varepsilon^*)$, the discrete-time singularly perturbed system (1) with $u(k) = 0$ is asymptotically stable and has an $H_\infty$ norm less than $\gamma$.

**Proof.** From (37) and (38), then it follows that (30) holds. On the other hand, from conditions (7) and (9) and applying Lemma 3(ii), we have (10) holds. Therefore, conditions (7), (9), (37) and (38) can guarantee that the conditions of Lemma 6 hold. Thus, by virtue of Lemma 6, for singular perturbation parameter $\varepsilon \in (0, \varepsilon^*)$, the fast sampling discrete-time singularly perturbed system (1) with $u(k) = 0$ is asymptotically stable and has an $H_\infty$ norm less than $\gamma$. Thus, the proof is complete.  
\[\Box\]

**Remark 8.** Theorem 7 provides a method of estimating the upper bound of singularly perturbed parameter $\varepsilon$ subject to the closed-loop system to be stable and with meeting an $H_\infty$ performance bound requirement. An upper bound of $\varepsilon$ can be obtained by solving the following LMI-based optimization problem:

maximize $\varepsilon^*$ subject to (7), (9), (37) and (38),
which can be effectively solved using LMI Control Toolbox (Gahinet et al., 1995).

**4. Robust $H_\infty$ control synthesis**

In this section, the results obtained in Section 3 are extended to robust $H_\infty$ control synthesis.

Consider the following uncertain fast sampling discrete-time singularly perturbed system
\begin{equation}
\begin{bmatrix}
\dot{x}_1(k+1) \\
\dot{x}_2(k+1)
\end{bmatrix} = \begin{bmatrix} I_{n_1} + \varepsilon A_{11} & \varepsilon A_{12} \\ \varepsilon A_{21} & I_{n_2} \end{bmatrix} \begin{bmatrix} x_1(k) \\
\varepsilon x_2(k)
\end{bmatrix} + \begin{bmatrix} \varepsilon B_{11} \\
\varepsilon B_{21}\end{bmatrix} w(k) + \begin{bmatrix} \varepsilon B_{12} \\
\varepsilon B_{22}\end{bmatrix} u(k)
\end{equation}
(39)
and $x_1(k) \in R^{n_1}, x_2(k) \in R^{n_2}$ are state vectors, $w(k) \in R^{m_1}$ is disturbance input, $u(k) \in R^{m_2}$ is the control input, $z(k) \in R^p$ is the controlled output. The positive singular perturbation parameter is denoted by $\varepsilon$. The matrices $\mathcal{A} = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}$,
$\mathcal{B}_1 = \begin{bmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{bmatrix}$, $\mathcal{C}_1 = \begin{bmatrix} C_{11} & C_{12} \end{bmatrix}$, $\mathcal{D}_{11}$ and $\mathcal{D}_{12}$ are appropriately dimensioned. They belong to the following uncertainty polytope (Boyd et al., 1994):
\begin{equation}
\Omega = \left\{ \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}, \begin{bmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{bmatrix}, \begin{bmatrix} C_{11} & C_{12} \end{bmatrix}, \begin{bmatrix} \varepsilon B_{11} & \varepsilon B_{12} \\ \varepsilon B_{21} & \varepsilon B_{22} \end{bmatrix} \right\}
\end{equation}
(40)
For the fast sampling singularly perturbed system (39) with polytopic uncertainties, a method for designing $H_\infty$ controller and a sufficient condition for evaluating the upper bound of singular perturbation parameter $\epsilon$ subject to a prescribed $H_\infty$ performance constraint are given in the following two theorems, respectively.

**Theorem 9.** If there exist a symmetric matrix $Q = \begin{bmatrix} Q_{11} & Q_{12} \\ Q_{12} & Q_{22} \end{bmatrix} > 0$, and matrices $T_{11}, T_{12}, T_{22}, L_1, L_2$, satisfy (7) and the following LMI's

$$
\begin{bmatrix}
-T_{11} & * & * & * & * \\
-Q_{12} & * & * & * & * \\
0 & 0 & -yI & * & * \\
-T_{12} + \bar{T}_{41} & Q_{12} + \bar{T}_{42} & B_{11}' & -T_{22} & * \\
\bar{T}_{51} & \bar{T}_{52} & B_{12}' & -Q_{12} & -Q_{22} \\
\bar{T}_{61} & \bar{T}_{62} & D_{11}' & 0 & 0 & -yI \\
\end{bmatrix} < 0,
\quad \text{for } 1 \leq i \leq r,
$$

(41)

where

$$
\begin{align*}
T_{41} &= A_{11}'Q_{11} + A_{12}'Q_{12} + B_{21}'L_1 \\
T_{42} &= A_{12}'Q_{12} + B_{21}'L_2 \\
T_{51} &= A_{21}'Q_{11} + A_{22}'Q_{12} + B_{22}'L_1 \\
T_{52} &= A_{22}'Q_{12} + B_{22}'L_2 \\
T_{61} &= C_{11}'Q_{11} + C_{12}'Q_{12} + D_{12}'L_1 \\
T_{62} &= C_{12}'Q_{12} + D_{12}'L_2
\end{align*}
$$

then there exists a sufficiently small $\epsilon^* > 0$ such that for $\epsilon \in (0, \epsilon^*)$, the state feedback controller

$$
u(k) = Kx(k) = [K_1 \quad K_2]\begin{bmatrix} x_1(k) \\ x_2(k) \end{bmatrix}
$$

renders the uncertain system (39) robustly stable and with an $H_\infty$ norm less than $y$, where

$$
K_1 = (L_1 - L_2Q_{22}^{-1}Q_{12}'Q_{11}^{-1}), \quad K_2 = L_2Q_{22}^{-1}.
$$

**Proof.** Similar to the arguments in the proof of Theorem 4, the proof is easily obtain and omitted. □

**Theorem 10.** If there exist symmetric matrices $Q = \begin{bmatrix} Q_{11} & Q_{12} \\ Q_{12} & Q_{22} \end{bmatrix} > 0$, and matrices $T_{11}, T_{12}, T_{22}$ satisfy (7) and (9) and the following LMI's,

$$
\begin{bmatrix}
-T_{11} & * & * & * & * \\
-Q_{12} & * & * & * & * \\
0 & 0 & -yI & * & * \\
-T_{12} + \bar{T}_{41} & Q_{12} + \bar{T}_{42} & B_{11}' & -T_{22} & * \\
\bar{T}_{51} & \bar{T}_{52} & B_{12}' & -Q_{12} & -Q_{22} \\
\bar{T}_{61} & \bar{T}_{62} & D_{11}' & 0 & 0 & -yI \\
\end{bmatrix} < 0,
\quad \text{for } 1 \leq i \leq r.
$$

(42)

$$
\begin{bmatrix}
-\Psi_{11} & * & * & * & * \\
0 & 0 & * & * & * \\
0 & 0 & 0 & * & * \\
-\Psi_{11} & A_{11}'Q_{12} & 0 & -\Psi_{11} & * \\
0 & A_{21}'Q_{12} & 0 & 0 & * \\
0 & C_{11}'Q_{12} & 0 & 0 & 0
\end{bmatrix} +
\begin{bmatrix}
-T_{11} & * & * & * & * \\
-Q_{12} & * & * & * & * \\
0 & 0 & -yI & * & * \\
-T_{12} + \bar{T}_{41} & Q_{12} + \bar{T}_{42} & B_{11}' & -T_{22} & * \\
\bar{T}_{51} & \bar{T}_{52} & B_{12}' & -Q_{12} & -Q_{22} \\
\bar{T}_{61} & \bar{T}_{62} & D_{11}' & 0 & 0 & -yI \\
\end{bmatrix} < 0,
\quad \text{for } 1 \leq i \leq r
$$

(43)

where

$$
\tilde{T}_{41} = A_{11}'Q_{11} + A_{12}'Q_{12}, \quad \tilde{T}_{42} = A_{12}'Q_{22} \\
\tilde{T}_{51} = A_{21}'Q_{11} + A_{22}'Q_{12}, \quad \tilde{T}_{52} = A_{22}'Q_{22} \\
\tilde{T}_{61} = C_{11}'Q_{11} + C_{12}'Q_{12}, \quad \tilde{T}_{62} = C_{12}'Q_{22}
$$

then for singular perturbation parameter $\epsilon \in (0, \epsilon^*)$, the discrete-time singularly perturbed uncertain system (39) is robustly stable and has an $H_\infty$ norm less than $y$.

**Proof.** From (42) and (43), we can obtain

$$
\begin{bmatrix}
-\tilde{T}_{11} & * & * & * & * \\
-\tilde{Q}_{12} & * & * & * & * \\
0 & 0 & -yI & * & * \\
-\tilde{T}_{12} + \tilde{\bar{T}}_{41} & \tilde{Q}_{12} + \tilde{\bar{T}}_{42} & \tilde{B}_{11}' & -\tilde{T}_{22} & * \\
\tilde{\bar{T}}_{51} & \tilde{\bar{T}}_{52} & \tilde{B}_{12}' & -\tilde{Q}_{12} & -\tilde{Q}_{22} \\
\tilde{\bar{T}}_{61} & \tilde{\bar{T}}_{62} & \tilde{D}_{11}' & 0 & 0 & -yI \\
\end{bmatrix} < 0,
\quad \text{for } 1 \leq i \leq r.
$$

(43)

where

$$
\tilde{T}_{41} = \sum_{i=1}^{r} a_i A_{11}'Q_{11} + A_{12}'Q_{12}, \quad \tilde{T}_{42} = \sum_{i=1}^{r} a_i A_{12}'Q_{22} \\
\tilde{T}_{51} = \sum_{i=1}^{r} a_i A_{21}'Q_{11} + A_{22}'Q_{12}, \quad \tilde{T}_{52} = \sum_{i=1}^{r} a_i A_{22}'Q_{22} \\
\tilde{T}_{61} = \sum_{i=1}^{r} a_i C_{11}'Q_{11} + C_{12}'Q_{12}, \quad \tilde{T}_{62} = \sum_{i=1}^{r} a_i C_{12}'Q_{22}
$$

Then the proof can be completed by using Lemma 3, and the technique of Suplin and Shaked (2005), and the details are omitted here. □

**Remark 11.** Similar to Theorem 7, an upper bound of singular perturbation parameter $\epsilon$ can be obtained by solving the following LMI-based optimization problem:

Maximize $\epsilon^*$ subject to (42) and (43).
Table 1
Controller gain $K$ and $\gamma^*$

<table>
<thead>
<tr>
<th>$K$</th>
<th>$\gamma^*$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Theorem 4</td>
<td>0.2049</td>
</tr>
<tr>
<td>(Vu &amp; Sawan, 1993)</td>
<td>0.4317</td>
</tr>
</tbody>
</table>

Table 2
Upper bounds of $\epsilon$

<table>
<thead>
<tr>
<th>$\gamma$</th>
<th>Theorem 4</th>
<th>(Vu &amp; Sawan, 1993)</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.205</td>
<td>0.18</td>
<td>Infeasible</td>
</tr>
<tr>
<td>0.207</td>
<td>0.49</td>
<td>0.13</td>
</tr>
<tr>
<td>0.209</td>
<td>0.90</td>
<td>0.43</td>
</tr>
<tr>
<td>0.301</td>
<td>0.99</td>
<td>0.99</td>
</tr>
</tbody>
</table>

5. Examples

In this section, two numerical examples are given to illustrate the effectiveness of the proposed methods, where Example 1 presents a comparison between Theorem 4 and the method in Vu and Sawan (1993), and the robust $H_\infty$ controller design using Theorem 9 is illustrated by Example 2.

Example 12. Consider the following nuclear reactor mood (Abdelrahman, Naidu, Charalambous, & Moore, 1998),

$$\begin{align*}
\dot{x}_1 &= -\lambda x_1 + \beta x_2 \\
\dot{x}_2 &= \frac{\beta}{A} x_1 - \frac{\beta}{A} x_2 + \frac{D}{A}
\end{align*}$$

where $x_1$ and $x_2$ are the normalized precursors’ concentration and neutron density, respectively. $\rho$, $\lambda$, $\beta$, and $A$ are the reactivity, precursors’ decay constant, delayed-neutron yield and neutron generation-time, respectively. The parameters are $\lambda = 0.001$, $\beta = 0.0064$ and $A = 0.08$.

In this section, we discretize the model with a sampling period $T = 0.05$ s and a zero-order holder and obtain the following discrete-time singularly perturbed model with $\epsilon = 0.01$, and a disturbance and the controlled output $z(k)$ are considered in the discrete-time model.

$$\begin{align*}
\begin{bmatrix}
x_1(k+1) \\
x_2(k+1)
\end{bmatrix} &= \begin{bmatrix}
1 - 0.3417\epsilon & 0.3417\epsilon \\
0.2733 & 0.7267
\end{bmatrix}\begin{bmatrix}
x_1(k) \\
x_2(k)
\end{bmatrix} + \begin{bmatrix}
9.0021\epsilon \\
42.7983
\end{bmatrix}u(k) + \begin{bmatrix}
0 \\
0.2
\end{bmatrix}w(k)
\end{align*}$$

$$z(k) = \begin{bmatrix}
0 & 1 & 0 & 0
\end{bmatrix}\begin{bmatrix}
x_1(k) \\
x_2(k) \\
w(k)
\end{bmatrix} + \begin{bmatrix}
0 & 0 & 0 & 1
\end{bmatrix}u(k).$$

Applying Theorem 4 and the method in Vu and Sawan (1993) to design $H_\infty$ controllers, the obtained controller gains and the optimal $H_\infty$ performance indices are given in Table 1. For the two controller gains, the upper bounds of singular perturbation parameter $\epsilon$ for meeting different $H_\infty$ performance requirements are obtained by using Theorem 7, and given in Table 2. The computation results show that the design given by Theorem 4 can guarantee a better $H_\infty$ performance for the closed-loop system and larger upper bounds of $\epsilon$.

Example 13. Consider a tunnel diode circuit where the tunnel diode is characterized by the following T–S fuzzy model (Assawinchaichote & Nguang, 2006):

$$\begin{align*}
E_\epsilon \dot{x}(t) &= \sum_{i=1}^{2} \alpha_i(t) \left(A_i^* x(t) + B_i^w(t) + B_i^u(t)\right) \\
Z(t) &= \sum_{i=1}^{2} \alpha_i(t) \left(C_i^* x(t) + D_i^w(t) + D_i^u(t)\right)
\end{align*}$$

which is discretized with a sampling period $T = 0.001$ s and a zero-order holder, then the following discrete-time singularly perturbed model can be obtained, which is a 2-polytopic linear time-varying system in the form of (40) with

$$\begin{align*}
A_1^1 &= \begin{bmatrix}
l_{n1} + \epsilon A_{11}^1 & \epsilon A_{12}^1 \\
A_{21}^1 & A_{22}^1
\end{bmatrix} = \begin{bmatrix}
1 + 0.2129\epsilon & 1.8140\epsilon \\
-0.1814 & 0.8179
\end{bmatrix} \\
B_{1e}^1 &= \begin{bmatrix}
\epsilon B_{11}^1 \\
B_{12}^1
\end{bmatrix} = \begin{bmatrix}
0 \\
0.1
\end{bmatrix}, \\
B_{2e} &= \begin{bmatrix}
\epsilon B_{21}^1 \\
B_{22}^1
\end{bmatrix} = \begin{bmatrix}
0.1874\epsilon \\
0.1812
\end{bmatrix} \\
C_1 &= \begin{bmatrix}
C_{11} & C_{12}
\end{bmatrix} = \begin{bmatrix}
1 & 1
\end{bmatrix}, \\
D_{11} &= 0.1, \\
D_{12} &= 1 \\
A_2^2 &= \begin{bmatrix}
l_{n1} + \epsilon A_{11}^2 & \epsilon A_{12}^2 \\
A_{21}^2 & A_{22}^2
\end{bmatrix} = \begin{bmatrix}
1 + 0.3932\epsilon & 1.8148\epsilon \\
-0.1815 & 0.8179
\end{bmatrix} \\
B_{1e}^2 &= \begin{bmatrix}
\epsilon B_{11}^2 \\
B_{12}^2
\end{bmatrix} = \begin{bmatrix}
0 \\
0.1
\end{bmatrix}, \\
B_{2e} &= \begin{bmatrix}
\epsilon B_{21}^2 \\
B_{22}^2
\end{bmatrix} = \begin{bmatrix}
0.1875\epsilon \\
0.1812
\end{bmatrix} \\
C_2 &= \begin{bmatrix}
C_{11} & C_{12}
\end{bmatrix} = \begin{bmatrix}
1 & 1
\end{bmatrix}, \\
D_{11} &= 0.1, \\
D_{12} &= 1 \\
\epsilon &= 0.005.
\end{align*}$$

Applying Theorem 9 to design robust $H_\infty$ controllers, the obtained controller gain and optimal robust $H_\infty$ performance index are given in Table 3. For the controller, the upper bounds of singular perturbation parameter $\epsilon$ for meeting different $H_\infty$ performance requirements are obtained by using Theorem 10, and given in Table 4.

6. Conclusion

In this paper, the problem of $H_\infty$ control synthesis for fast sampling discrete-time singularly perturbed systems has been studied. The main contribution is as follows. A new design method is given in terms of solutions to a set of LMIs, which eliminates the regularity restrictions attached to the Riccati-based solutions. Moreover, a technique for evaluating the upper bound of singular perturbation parameter with meeting a prescribed $H_\infty$ performance bound requirement
is given. Furthermore, the results are also extended to robust $H_\infty$ control synthesis for singularly perturbed systems with polytopic uncertainties. The numerical examples have shown the effectiveness of the proposed methods.

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