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TWELVE LIMIT CYCLES IN A CUBIC ORDER PLANAR SYSTEM WITH Z_2-SYMMETRY

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Abstract. In this paper, we report the existence of twelve small limit cycles in a planar system with 3rd-degree polynomial functions. The system has Z_2-symmetry, with a saddle point, or a node, or a focus point at the origin, and two focus points which are symmetric about the origin. It is shown that such a Z_2-equivariant vector field can have twelve small limit cycles. Fourteen or sixteen small limit cycles, as expected before, cannot not exist.

1. **Introduction.** One of the well-known mathematical problems is the second part of the 16th Hilbert's problem, which considers the maximal number of limit cycles for a general planar polynomial system. A recent survey article [5] (and more references therein) has comprehensively discussed this problem and reported the recent progress. To start discussion of the 16th Hilbert's problem, assume that the system under consideration is described by the following equations:

$$\dot{x} = P_n(x, y), \quad \dot{y} = Q_n(x, y), \tag{1.1}$$

where the dot denotes the differentiation with respect to time, and $P_n(x, y)$ and $Q_n(x, y)$ represent n^{th}-degree polynomials of x and y. The problem is to find the maximum number of the limit cycles that system (1.1) may exhibit. In general, this is a very difficult question, in particular, for finding large (global) limit cycles. So far the best result reported in literature for cubic order systems is eleven limit cycles [5, 14].

If we restrict the problem to the neighborhood of isolated fixed points, then the question is reduced to studying degenerate Hopf bifurcations. In the past 50 years, many researchers have considered the local problem and many results have been obtained (e.g., see [1, 4, 6, 7, 8]). Recently, the following cubic order system:

$$\begin{aligned} \dot{x} &= a\,x + b\,y + a_{30}\,x^3 + a_{21}\,x^2\,y + a_{12}\,x\,y^2 + a_{03}\,y^3, \\ \dot{y} &= b\,x + a\,y + b_{30}\,x^3 + b_{21}\,x^2\,y + b_{12}\,x\,y^2 + b_{03}\,y^3, \end{aligned} \tag{1.2}$$

has been investigated for the existence of limit cycles [2, 3, 10, 11, 12, 13]. Here, the Jacobian of the system evaluated at the equilibrium $(x,\, y) = (0,\, 0)$ has eigenvalues $a \pm |b|$. Thus, the origin is either a saddle point when $|b| > |a|$, or a node if $|a| > |b|$. Further, let

$$a_{03} = -\,b, \quad b_{03} = -\,a \quad \text{and} \quad a_{12} = a, \tag{1.3}$$

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then system (1.2) is symmetric with the origin and has two Hopf-type focus points at $(x,\, y) = (0,\, \pm 1)$. Under the above conditions, ten small limit cycles have been found from system (1.2) when the origin is a saddle point [2]. Recently, it has been shown that twelve small limit cycles exist for a special case of system (1.2) [13]. These results have been reported in several international conferences [3, 10, 11, 12].

Note that the third condition, $a_{12} = a$ given in Eq. (1.3), makes the eigenvalues of the system evaluated at the two focus points be a purely imaginary pair. Under the first two conditions of Eq. (1.3), the eigenvalues of the system evaluated at $(0,\, \pm 1)$ are given by $\lambda_{\pm} = \frac{1}{2}\left[(a_{12}-a) \pm \sqrt{(a_{12}-a)^2 - 8\,(b^2-a^2+b\,b_{12}-a\,a_{12})}\,\right]$, from which the real part of the eigenvalues is obtained as $v_0 = \frac{1}{2}\,(a_{12} - a)$ if $(a_{12}-a)^2 < 8\,(b^2-a^2+b\,b_{12}-a\,a_{12})$.

The perturbation method based on multiple scales and Maple program developed in [9] was used to compute the focus values (equivalent to finding normal forms of differential equations), which was proved computationally efficient. Based on the focus values, parameter perturbation was carried out to show the existence of the exact small limit cycles [2, 13]. In this paper, we first summarize the results for the special case of twelve limit cycles, and then outline our study on the limit cycles of a general Z_2-equivariant vector field. We shall show that the maximal number of small limit cycles of such a system is twelve. Fourteen or sixteen small limit cycles, as expected before, cannot exist.

Now we briefly outline the result of twelve limit cycle for a special case [3, 10, 11, 12, 13]. Consider system (1.2) and assume $|b| > |a|$, which implies that the origin is a saddle point. In addition to the conditions given in Eq. (1.3), further suppose that the following condition:

$$b_{12} = \frac{1 + 4\,a^2}{2\,b} - b, \tag{1.4}$$

holds, then system (1.2) is symmetric with the origin and has two Hopf-type focus points at $(0, \pm 1)$, with the eigenvalues $\pm\, i$. Then, we have *six* free parameters: $a,\, b,\, a_{21},\, b_{21},\, a_{30}$ and b_{30}, which will be involved in the expressions of the focus values.

Since the system is symmetric with the two focus points $(0, \pm 1)$, we only need to consider one of them, say, $(0, 1)$. Thus, introduce the transformation: $x = 2\,b\,u,\ y = 1 + 2\,a\,u - v$ into system (1.2) to obtain

$$\begin{aligned}
\dot{u} \;=&\; v + 2\,(a_{21}\,b - a^2)\,u^2 + 4\,a\,u\,v - \frac{3}{2}\,v^2 \\
&+ 4\,b\,(a_{30}\,b + a_{21}\,a)u^3 - 2\,(a_{21}b - a^2)u^2 v - 2\,a\,u\,v^2 + \frac{1}{2}\,v^3, \\
\dot{v} \;=&\; -u - 4\,(b_{21}\,b^2 - a_{21}\,a\,b - 2a\,b^2 + 2\,a^3 + a)\,u^2 + 2\,(2\,a^2 - 2\,b^2 + 1)\,u\,v \\
&- 8\,(b_{30}\,b^3 - a_{30}\,a\,b^2 - a_{21}\,a^2\,b + b_{21}\,a\,b^2 - a^2\,b^2 + a^4 + \frac{1}{2}\,a^2)\,u^3 \\
&+ 4\,(b_{21}\,b^2 - a_{21}\,a\,b - 2a\,b^2 + 2\,a^3 + a)\,u^2 v - (2\,a^2 - b^2 + 1)\,u\,v^2,
\end{aligned} \tag{1.5}$$

whose Jacobian evaluated at the origin is in the real Jordan canonical form.

Applying the Maple program [9] to system (1.5) yields the focus values, v_i, expressed in terms of the parameters $a,\, b,\, a_{21},\, b_{21},\, a_{30}$ and b_{30}. Indeed, we have obtained the explicit symbolic expressions for the six focus values: $v_i(a,\, b,\, a_{21},\, b_{21},\, a_{30},\, b_{30})$, $i = 1, 2, \ldots, 6$. For example, v_1 is given by

$$v_1 = \frac{1}{2}\, b\, \big[\, 2ab\,(4a^2-4b^2+3)+3ba_{30}+b_{21}b\,(4b^2-4a_{21}b-1) \\ + 2a_{21}a\,(2b^2+2ba_{21}-4a^2-1)\,\big]\,. \tag{1.6}$$

Since $b \neq 0$, we can solve a_{30} from the above equation to obtain $a_{30} = a_{30}(b,\, a_{21},\, a,\, b_{21})$. With this a_{30}, executing the Maple program results in v_2, given by

$$v_2 = \frac{1}{9}\, b^2\,(b_N + 6\, b\, b_D\, b_{30}), \tag{1.7}$$

where b_N and b_D are expressed in terms of a, b, a_{21}, b_{21}. There are two cases in solving the equation $v_2 = 0$. Since $b \neq 0$, so if $b_D \neq 0$ one can uniquely determine b_{30} as $b_{30} = -\frac{b_N}{6\,b\,b_D}$. This is generic case to be considered in the next section. When $b_D = 0$, which is referred as a special case, b_{30} is not used at this order, but b_{21} can be determined, and then simplify v_2. It should be noted that since one more parameter is needed to solve the equation $v_2 = 0$, thus the remaining parameters b_{30}, a_{21}, a and b must be used to solve the four equations $v_i = 0$, $i = 2, 3, 4, 5$. Hence, the special case gives finite possible solutions, while the generic case can have infinite possible solutions for solving the same number of equations.

Now we can use the simplified v_2 to determine a, and then further simplify the expressions v_3, v_4 and v_5 to obtain

$$\begin{aligned} v_3 &= \frac{2\,a\,b^2}{625\,(2\,b^2 - 2\,b\,a_{21} - 1)^4}\;\, F\,F_1, \\ v_4 &= \frac{-\,2\,a\,b^2}{3515625\,(2\,b^2 - 2\,b\,a_{21} - 1)^7}\;\, F\,F_2, \\ v_5 &= \frac{a\,b^2}{31640625000\,(2\,b^2 - 2\,b\,a_{21} - 1)^{10}}\;\, F\,F_3, \end{aligned} \tag{1.8}$$

where F_i $(i = 1, 2, 3)$ are i^{th}-degree polynomials of b_{30}. Solving b_{30} from $F_1 = 0$ and then eliminating a_{21} from equations $F_2 = 0$ and $F_3 = 0$ results in a polynomial equation of b^2:

$$F_4 = 10195528\, b^6 - 21299025\, b^4 + 3454965\, b^2 + 259405 = 0, \tag{1.9}$$

which, in turn, gives a total of four solutions for the special case. We consider one of the solutions, given by the critical values:

$$\begin{aligned} b^* &= 1.3798788398, & a_{21}^* &= 1.0897036998, & b_{30}^* &= 0.7093364483, \\ a^* &= 0.0877426100, & b_{21}^* &= 0.1470131077, & a_{30}^* &= -\,0.0284721124, \end{aligned} \tag{1.10}$$

and then $b_{12}^* = -\,1.0063695748$. For these critical values, $v_6 \approx 0.007259$. The six parameters: a, b, a_{21}, b_{21}, a_{30} and b_{30} can then be perturbed from the critical values to find the exact six small amplitude limit cycles around the origin $(u, v) = (0, 0)$. The results are summarized in the following theorem. The proof can be found in [13].

THEOREM 1.1. *Given the cubic system (1.2) which is assumed to have a saddle point at the origin and a pair of symmetric fine focus points at* $(x,\, y) = (0,\, 1)$ *and* $(0,\, -1)$. *Further suppose* $a_{12} = -\,b_{03} = 0.0877426100$, $a_{03} = -\,1.3798788398$, $b_{12} = -\,1.0063695748$. *Then if* b, a_{21}, $b_{30}(b, a_{21})$, $a(b, a_{21})$, $b_{21}(b, a_{21}, a)$ *and*

$a_{30}(b, a_{21}, a, b_{21})$ *are perturbed as*

$$\begin{aligned}
b &= b^* + \epsilon_1, \\
a_{21} &= a_{21}^* + \epsilon_2, \\
b_{30} &= b_{30}(b^* + \epsilon_1,\, a_{21}^* + \epsilon_2) + \epsilon_3, \\
a &= a(b^* + \epsilon_1,\, a_{21}^* + \epsilon_2) - \epsilon_6, \\
b_{21} &= b_{21}(b^* + \epsilon_1,\, a_{21}^* + \epsilon_2,\, a(b^* + \epsilon_1, a_{21}^* + \epsilon_2) - \epsilon_6) - \epsilon_4, \\
a_{30} &= a_{30}(b^* + \epsilon_1,\, a_{21}^* + \epsilon_2,\, a(b^* + \epsilon_1, a_{21}^* + \epsilon_2) - \epsilon_6, \\
&\quad\; b_{21}(b^* + \epsilon_1,\, a_{21}^* + \epsilon_2,\, a(b^* + \epsilon_1,\, a_{21}^* + \epsilon_2) - \epsilon_6) - \epsilon_4) - \epsilon_5,
\end{aligned} \tag{1.11}$$

where $0 < \epsilon_6 \ll \epsilon_5 \ll \epsilon_4 \ll \epsilon_3 \ll (\epsilon_2, \epsilon_1) \ll 1$*, system (1.2) has exact twelve small limit cycles. The notation* (ϵ_2, ϵ_1) *means that* ϵ_2 *and* ϵ_1 *are in the same order, with* $\epsilon_2 = (\delta + \bar{\epsilon})\,\epsilon_1$ *for some* $\delta > 0$ *and some small* $\bar{\epsilon} > 0$.

The normal form of the perturbed system is then given by

$$\dot{r} = r\,(v_0 + v_1\, r^2 + v_2\, r^4 + v_3\, r^6 + v_4\, r^8 + v_5\, r^{10} + v_6\, r^{12}), \tag{1.12}$$

where $0 < v_0 \ll -v_1 \ll v_2 \ll -v_3 \ll v_4 \ll -v_5 \ll v_6$. A numerical example is presented in [11, 12, 13] to illustrate the theoretical result under the following perturbations: $\epsilon_1 = 0.1 \times 10^{-1}$, $\epsilon_2 = 0.10846 \times 10^{-1}$, $\epsilon_3 = 0.1 \times 10^{-7}$, $\epsilon_4 = 0.6 \times 10^{-13}$, $\epsilon_5 = 0.1 \times 10^{-17}$ and $\epsilon_6 = 0.2 \times 10^{-22}$. The six positive roots for r are obtained as: $r_1 = 0.001998424223$, $r_2 = 0.005539772256$, $r_3 = 0.013978579010$, $r_4 = 0.027014927622$, $r_5 = 0.063377651980$ and $r_6 = 0.143875184886$.

The phase portrait of the special case under the above perturbations is shown in Fig. 1, where the two boxes are used to mark the neighborhood of the fine focus points $(0, \pm 1)$, one of which is enlarged, as shown in Fig. 2 to depict the six small limit cycles near $(0, 1)$, three of them are stable and the other three are unstable. It should be pointed out that one cannot use numerical simulation to get the accurate small limit cycles in the neighborhood of a highly singular point, and has to use certain theoretical approach (like the one presented in this paper) to show the existence of the small limit cycles. In fact, the phase portrait of the system near the focus point $(0, 1)$ is obtained from the normal form, rather than numerical simulations.

2. **Twelve Limit Cycles in the Generic Case.** In this section, we turn to the generic case in which $b_D \neq 0$ (see Eq. (1.7)), and thus b_{30} is uniquely determined as $b_{30} = -\frac{b_N}{6\, b\, b_D}$. For the generic case, there are six free coefficients: a, b, a_{21}, b_{21}, a_{30} and b_{30}, which are involved in the expressions of the focus values. Therefore, unlike the special case, it is now possible to have $v_i = 0,\ i = 1, 2, \ldots, 6$, but $v_7 \neq 0$ This implies that it may be possible to have seven small limit cycles around each of the focus points $(0, \pm 1)$. We need to find the expression of v_7 and solve the six polynomial equations. In particular, we have to solve the four very large coupled polynomial equations: $v_i = 0,\ i = 3, 4, 5, 6$. This causes very high complexity in computation. Thus, we first need to simplify Eq. (1.5). To achieve this, let

$$\begin{aligned}
\bar{a}_{21} &= a_{21}\, b - a^2, \\
\bar{a}_{30} &= a_{30}\, b + a_{21}\, a, \\
\bar{b}_{21} &= b_{21}\, b^2 - a_{21}\, a\, b - 2a\, b^2 + 2\, a^3 + a, \\
\bar{b}_{30} &= b_{30}\, b^3 - a_{30}\, a\, b^2 - a_{21}\, a^2\, b + b_{21}\, a\, b^2 - a^2\, b^2 + a^4 + \frac{1}{2}\, a^2,
\end{aligned} \tag{2.1}$$

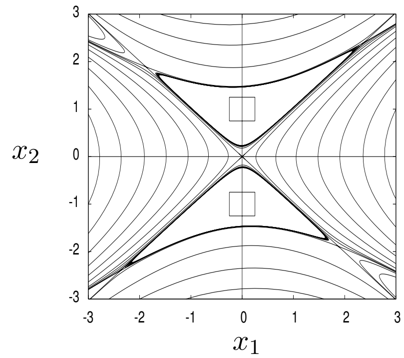

FIGURE 1. The phase portrait of system (1.2) having 12 limit cycles, when the origin is a saddle point.

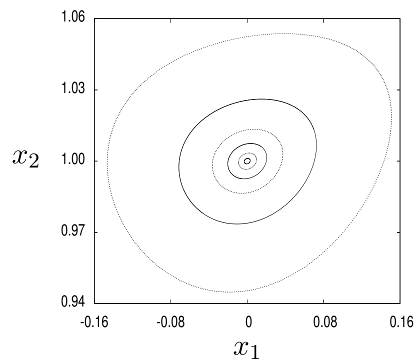

FIGURE 2. The six limit cycles around the focus point $(0, 1)$: ——- stable; $\cdots\cdots$ unstable

then Eq. (1.5) becomes

$$\begin{aligned}\dot{u} &= v + 2\,\bar{a}_{21}\,u^2 + 4\,a\,u\,v - \frac{3}{2}\,v^2 + 4\,b\,\bar{a}_{30}\,u^3 - 2\,\bar{a}_{21}\,u^2 v - 2\,a\,u\,v^2 + \frac{1}{2}\,v^3,\\ \dot{v} &= -\,u - 4\bar{b}_{21}u^2 + 2(2a^2 - 2b^2 + 1)uv - 8\bar{b}_{30}u^3 + \bar{b}_{21}u^2 v - (2a^2 - b^2 + 1)uv^2. \end{aligned} \tag{2.2}$$

Thus, based on Eq. (2.2), we apply the Maple program [9] to find the focus values: $v_i = v_i(a,\, b,\, \bar{a}_{21},\, \bar{b}_{21},\, \bar{a}_{30},\, \bar{b}_{30}),\ \ i = 1, 2, \ldots, 7$. Then, letting $v_1 = 0$ yield the solution for $\bar{a}_{30}$ and setting $v_2 = 0$ results in $\bar{b}_{30} = \bar{b}_{30}(a,\, b,\, \bar{a}_{21},\, \bar{b}_{21}) = -\,\frac{\bar{b}_N}{6\,\bar{b}_D}$.

The remaining focus values are

$$\begin{aligned} v_3 &= \frac{4\,F\ F_1}{9\,\left[a(8\bar{a}_{21}-1) - 10\bar{b}_{21}(a\bar{a}_{21}+2a^2-2b^2+1) + 8a(a^2-b^2)\right]^2},\\ v_4 &= \frac{F\ F_2}{405\,\left[a(8\bar{a}_{21}-1) - 10\bar{b}_{21}(a\bar{a}_{21}+2a^2-2b^2+1) + 8a(a^2-b^2)\right]^3},\\ v_5 &= \frac{F\ F_3}{233280\,\left[a(8\bar{a}_{21}-1) - 10\bar{b}_{21}(a\bar{a}_{21}+2a^2-2b^2+1) + 8a(a^2-b^2)\right]^4},\\ v_6 &= \frac{F\ F_4}{146966400\,\left[a(8\bar{a}_{21}-1) - 10\bar{b}_{21}(a\bar{a}_{21}+2a^2-2b^2+1) + 8a(a^2-b^2)\right]^5}, \end{aligned} \tag{2.3}$$

where $F_i = F_i(a,\, b,\, \bar{a}_{21},\, \bar{b}_{21})$, $i = 1,\, 2,\, 3,\, 4$. Since F_1 is linear in $\bar{a}_{21}$, one can explicitly solve $\bar{a}_{21}$ from $F_1 = 0$ and then obtain the simplified three polynomial equations: $F_i^* = F_i^*(a,\, b,\, \bar{b}_{21}) = 0$, $i = 2,\, 3,\, 4$. These three polynomial equations are coupled and have to be solved simultaneously. Eliminating b from the two equations $F_2^* = 0$ and $F_3^* = 0$ yields the equation:

$$F_5 = 160\, \bar{b}_{21}^3 + 140\, a\, \bar{b}_{21}^2 - 40\, a^2 \bar{b}_{21} + a^3 = 0. \tag{2.4}$$

Similarly, eliminating b from another two equations $F_2^* = 0$ and $F_4^* = 0$ gives the resultant equation: $F_6 = 0$, where the lengthy expression of F_6 is omitted.

Solving $F_5 = F_6 = 0$ for a and $\bar{b}_{21}$, gives possibility of finding solutions such that $v_i = 0$, $i = 1, 2, \ldots, 6$. It should be pointed out that the above variable elimination process do not miss any possible solutions. (It is possible to have extra solutions, and thus one has to verify all solutions using the original equations.) Finally, eliminating $\bar{b}_{21}$ from the two equations: $F_5 = F_6 = 0$ yields

$$\begin{aligned} F_7 = {} & a(11278332390406413903678327394304a^{12} + 94392683579322681115215785548492 80a^{10} \\ & + 1595554914969941771013315201669920 0a^8 + 78067338319585667946062243018556 00a^6 \\ & + 26896701242331499420963123077015 00a^4 + 13179071965242968131829121466706 25a^2 \\ & + 24822508754255240794213996449750 0) = 0. \end{aligned} \tag{2.5}$$

It is obvious that Eq. (2.5) has only one real solution $a = 0$. But if $a = 0$, then $\bar{b}_{21} = 0$ which, in turn, results in $\bar{a}_{21} = \bar{b}_{30} = \bar{a}_{30} = 0$, leading to a center. Thus, a must be non-zero. This implies that one cannot find possible non-zero values of a and $\bar{b}_{21}$ such that $F_5 = F_6 = 0$. Therefore, there is no solution for $F_2 = F_3 = F_4 = 0$, and so no possible non-trivial solutions exist for a, b, $\bar{a}_{21}$, $\bar{b}_{21}$, $\bar{a}_{30}$ and $\bar{b}_{30}$ such that $v_i = 0$, $i = 1, 2, \ldots, 6$. Hence, *fourteen* small limit cycles are *not* possible even for the *generic* case when the origin is a saddle point, and the *maximal* number of the small limit cycles for the generic case is *twelve*.

Next, we shall find all the solutions for the twelve limit cycles of the generic case. It has been shown in Section 1 that there exist four solutions for the special case. It should be noted that for each of the four solutions, there are a family of small limit cycles depending upon perturbations. For the generic case, on the other hand, since there is an extra free coefficient to be chosen, there are infinite solutions for the twelve limit cycles. Finding these solutions only requires $F_2 = F_3 = 0$ ($F_4 \neq 0$), which can be obtained from Eq. (2.4) in a parametric form. For example, numerically solving Eq. (2.4) for $\bar{b}_{21}$ yields three solutions: $\bar{b}_{21} = 0.2033343806\, a,\, -1.1061229255\, a,\, 0.2778854492\, a$. Then, for each of these solutions, one finds two solutions for b^2. But for $\bar{b}_{21} = 0.2778854492\, a$, one solution is negative. Hence, there exist a total of ten solutions for $(\bar{b}_{21}, b)$. However, by checking $F_2 = F_3 = 0$, only four solutions are satisfied: two for the case when the origin is a saddle point and two for the case when the origin is a node. Note that for the special case discussed in the previous section, solutions exist only for the origin being a saddle point.

In the following, we consider one of the solutions when the origin is a node. Let the critical values be:

$$\begin{aligned} b^* &= 0.4765747114\, a, \\ \bar{b}_{21}^* &= 0.2033343806\, a, \\ \bar{a}_{21}^* &= 0.7000000000 + 1.0149654014\, a^2, \end{aligned}$$

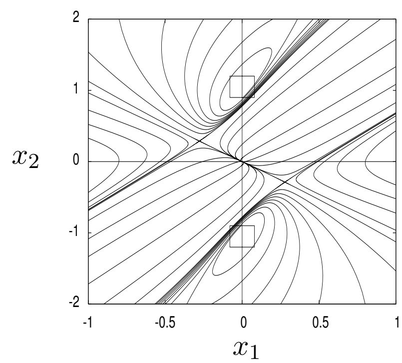

FIGURE 3. The phase portrait of system (1.2) having 12 limit cycles, when the origin is a node.

$$\begin{aligned}\bar{b}_{30}^* &= \frac{a^2(0.0481488581 + 65.9546167690\, a^2 - 9379.2591506305\, a^4)}{0.0008286738 - 0.1076372236\, a^2}, \\ \bar{a}_{30}^* &= -(0.8202076319 + 2.4368685248\, a^2). \end{aligned} \tag{2.6}$$

Then, similarly like Theorem 1.1 of the special case, we can prove the following theorem for the generic case.

THEOREM 2.1. *Given the cubic system (1.2) which is assumed to have a saddle point or a node at the origin and a pair of symmetric fine focus points at* $(x, y) = (0, \pm 1)$. *Further suppose* $a_{12} = -b_{03} = a$, $a_{03} = -0.4765747114\, a$, $b_{12} = \frac{1+4a^2}{2b} - b$. *Then, for an arbitrarily given* $a \neq 0$, *if* b, $\bar{b}_{21}$, $\bar{a}_{21}(b, \bar{b}_{21})$, $\bar{b}_{30}(b, \bar{a}_{21}, \bar{b}_{21})$ *and* $\bar{a}_{30}(b, \bar{a}_{21}, \bar{b}_{21})$ *are perturbed as*

$$\begin{aligned} b &= b^* + \epsilon_1, \\ \bar{b}_{21} &= \bar{b}_{21}^* + \epsilon_2, \\ \bar{a}_{21} &= \bar{a}_{21}(b^* + \epsilon_1,\, \bar{b}_{21}^* + \epsilon_2) + \epsilon_3, \\ \bar{b}_{30} &= \bar{b}_{30}(b^* + \epsilon_1,\, \bar{b}_{21}^* + \epsilon_2,\, \bar{a}_{21}(b^* + \epsilon_1,\, \bar{b}_{21}^* + \epsilon_2) + \epsilon_3) + \epsilon_4, \\ \bar{a}_{30} &= \bar{a}_{30}(b^* + \epsilon_1,\, \bar{b}_{21}^* + \epsilon_2,\, \bar{a}_{21}(b^* + \epsilon_1,\, \bar{b}_{21}^* + \epsilon_2) + \epsilon_3) + \epsilon_5, \\ a_{12} &= a + \epsilon_6, \end{aligned} \tag{2.7}$$

where $0 < |\epsilon_6| \ll |\epsilon_5| \ll |\epsilon_4| \ll |\epsilon_3| \ll (|\epsilon_2|, |\epsilon_1|) \ll 1$, *system (1.2) has exact twelve small limit cycles. The notation* $(|\epsilon_2|, |\epsilon_1|)|$ *means that* ϵ_2 *and* ϵ_1 *are in the same order, with* $\epsilon_2 = (\delta + \bar{\epsilon})\, \epsilon_1$ *for some* $\delta > 0$ *and some small* $\bar{\epsilon} > 0$. *(Note that here* ϵ_i *can be either positive or negative.)*

To end this section, we present a numerical example for the case when the origin is a node. We choose $a = 0.8$ and take the following perturbations: $\epsilon_1 = -0.2 \times 10^{-7}$, $\epsilon_2 = 0.5 \times 10^{-7}$, $\epsilon_3 = -0.5 \times 10^{-7}$, $\epsilon_4 = 0.2 \times 10^{-11}$, $\epsilon_5 = 0.1 \times 10^{-14}$, $\epsilon_6 = -0.2 \times 10^{-19}$, under which the six positive roots for r are: $r_1 = 0.0048210550$, $r_2 = 0.010756682296$, $r_3 = 0.014129671666$, $r_4 = 0.045686004658$, $r_5 = 0.196053309332$ and $r_6 = 0.355629707766$. The phase portrait of this generic case when the origin is a node, under the above perturbations, is shown in Fig. 3, where the marked box near $(0, 1)$ or $(0, -1)$ contains six small limit cycles, like those shown in Fig. 2. But the stabilities of these limit cycles are in the reverse order of those given in Fig. 2.

3. **Other Possible Cases for the Z_2-equivariant Vector Field.** In the previous two sections, we have discussed both special and generic cases which exhibit twelve limit cycles. It has been noted that for the special case, the origin can only be a saddle point, while for the generic case, the origin can be either a saddle point or a node. Further, we have shown that the maximal number of small limit cycles is twelve. Existence of fourteen small limit cycles is not possible. Recall Eq. (1.4) that b_{12} was chosen so that the frequency of the limit cycles is 1. If we let b_{12} be free, that is, the frequency can be chosen arbitrarily, then we have seven coefficients a, b, a_{21}, b_{21}, a_{30}, b_{30} and b_{12}, free to be chosen in the expressions of the focus values. This seems it might be possible to set $v_i = 0$, $i = 1, 2, \ldots, 7$, but $v_8 \neq 0$, implying that eight (in total sixteen for the whole system) limit cycles may exist. Also we want to ask: for any Z_2-equivariant vector field, what is the maximal number of small limit cycles? In this section, we first show that for an arbitrary Z_2-equivariant vector field, the origin can be a focus point, or a saddle point or a node. The first case has not been considered in the previous two sections. In the following, we shall show that for any of these cases, the maximal number of small limit cycles is twelve.

First, we take the standard complex form of the Z_2-equivariant vector field given by [5]:

$$\dot{z} = F_2(z, \bar{z}), \qquad \dot{\bar{z}} = \bar{F}_2(z, \bar{z}), \tag{3.1}$$

where $F_2(z, \bar{z}) = P(w_1, w_2) + i\,Q(w_1, w_2)$, $w_1 = \frac{1}{2}\,(z + \bar{z})$, $w_2 = \frac{1}{2\,i}\,(z - \bar{z})$, and

$$\begin{aligned} F_2(z, \bar{z}) \;=\; & (A_0 + A_1|z|^2 + A_2|z|^4)\,z + (A_3 + A_4|z|^2 + A_5|z|^4)\,\bar{z} \\ & + (A_6 + A_7|z|^2)\,z^3 + (A_8 + A_9|z|^2)\,\bar{z}^3, \end{aligned}$$

up to 5^{th} order. Here, $A_j = a_j + i\,b_j$ are complex. For a consistent comparison with system (1.2), we transform Eq (3.1) into the real form and truncated to the 3rd order:

$$\begin{aligned} \dot{w}_1 \;=\; & (a_0 + a_3)\,w_1 - (b_0 - b_3)\,w_2 + (a_1 + a_4 + a_6 + a_8)\,w_1^3 \\ & - (b_1 - b_4 + 3\,b_6 - 3\,b_8)\,w_1^2\,w_2 + (a_1 + a_4 - 3\,a_6 - 3\,a_8)\,w_1\,w_2^2 \\ & - (b_1 - b_4 - b_6 + b_8)\,w_2^3, \\ \dot{w}_2 \;=\; & (b_0 + b_3)\,w_1 + (a_0 - a_3)\,w_2 + (b_1 + b_4 + b_6 + b_8)\,w_1^3 \\ & + (a_1 - a_4 + 3\,a_6 - 3\,a_8)\,x^2\,w_2 + (b_1 + b_4 - 3\,b_6 - 3\,b_8)\,w_1\,w_2^2 \\ & + (a_1 - a_4 - a_6 + a_8)\,w_2^3. \end{aligned} \tag{3.2}$$

The two eigenvalues of the Jacobian of Eq. (3.2) evaluated at $(w_1, w_2) = (0, 0)$ are $a_0 \pm \sqrt{a_3^2 + b_3^2 - b_0^2}$. Thus, the origin $(0, 0)$ is a saddle point or a node when $a_3^2 + b_3^2 - b_0^2 \geq 0$, while a focus point or a center if $a_3^2 + b_3^2 - b_0^2 < 0$. When $a_3^2 + b_3^2 - b_0^2 = 0$, the origin is either a node or a double zero singular point.

First we consider $a_3^2 + b_3^2 - b_0^2 \geq 0$ for which the origin is a saddle point or a node. For this case, we introduce the following linear transformations:

$$\begin{bmatrix} w_1 \\ w_2 \end{bmatrix} = \begin{bmatrix} 0 & b_0 - b_3 \\ \mp\sqrt{a_3^2 + b_3^2 - b_0^2} & a_3 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}, \tag{3.3}$$

into Eq. (3.2), and then let $a = a_0$, $b = \pm\sqrt{a_3^2 + b_3^2 - b_0^2}$, and rename the coefficients of the third order terms in the resulting equations as a_{30}, a_{21}, a_{12}, a_{03}, b_{30}, b_{21}, b_{12} and b_{03}, which finally yields Eq. (1.2). It is easy to see that system (1.2) is invariant under the rotation about the origin with angle π. To consider small limit cycles, one needs fine focus point to generate degenerate Hopf bifurcation. Due to the Z_2-symmetry, it is naturally to best have two symmetric focus points about the origin. Then, if one focus point has N limit cycles, the whole system would have $2N$ limit cycles, though requiring the two symmetric focus points be Hopf type gives some constraints on the coefficients. If consider one Hopf type focus point at the origin, it may have couple of more free coefficients to choose, but the total number of the limit cycles would be less than that of the case of the two symmetric focus points. This will be discussed next for the case: $a_3^2 + b_3^2 - b_0^2 < 0$.

In addition, noticing that the two eigenvectors of the saddle point (the origin) are $(1,1)$ and $(-1,1)$, so the two focus points can be assumed on the y-axis (or x-axis). Further, by a proper scaling, we can assume that the two focus points are located at $(0, \pm 1)$, which leads to the conditions $a_{03} = -b$ and $b_{03} = -a$, and making the two focus point be Hopf type requires $a_{12} = a$. These are exact three conditions given by Eq (1.3).

Before considering the case $a_3^2 + b_3^2 - b_0^2 < 0$, we want to show that the condition given in Eq. (1.4): $b_{12} = \frac{1+4a^2}{2b} - b$, which normalizes the frequency of limit cycles to be 1, is not necessary. In other words, if we assume that the frequency can be selected arbitrarily, then b_{12} is free to choose. However, it can been shown that this extra free coefficient does not increase possibility of extra limit cycles. That is, one of the coefficients b_{12}, a and b (not necessarily b_{12}), can be chosen arbitrarily, which does not affect the solutions of limit cycles.

To prove the above statement, let ω be an arbitrary frequency of the limit cycles, then introducing the following parameter scaling and the time scaling:

$$\begin{aligned}&a \to \omega\, a, \quad b \to \omega\, b, \quad a_{30} \to \omega\, a_{30}, \quad a_{21} \to \omega\, a_{21}, \quad b_{30} \to \omega\, b_{30}, \quad b_{21} \to \omega\, b_{21},\\ &\tau = \omega\, t,\end{aligned} \tag{3.4}$$

into system (1.2) results in the following equations:

$$\begin{aligned}\frac{du}{d\tau} &= v + 2\,\bar{a}_{21}\,u^2 + 4\,a\,u\,v - \frac{3}{2}\,v^2 + 4\,b\,\bar{a}_{30}\,u^3 - 2\,\bar{a}_{21}\,u^2 v - 2\,a\,u\,v^2 + \frac{1}{2}\,v^3,\\ \frac{dv}{d\tau} &= -u - 4\bar{b}_{21}u^2 + 2(2a^2 - 2b^2 + 1)uv - 8\bar{b}_{30}u^3 + 4\bar{b}_{21}u^2 v - (2a^2 - b^2 + 1)uv^2,\end{aligned} \tag{3.5}$$

in which a shifting of the focus point $(0,1)$ to $(0,0)$ has been used. The above equation is exactly in the same form of Eq. (2.2), except that the time now is τ. The coefficients $\bar{a}_{21}$, $\bar{b}_{21}$, $\bar{a}_{30}$ and $\bar{b}_{30}$ are given by Eq. (2.1).

Therefore, based on the above discussion and the results obtained in the previous two sections, we have shown that for the case $a_3^2 + b_3^2 - b_0^2 \geq 0$, the maximal number of small limit cycles that system (1.2) can have is twelve.

Next, we turn to the case when $a_3^2 + b_3^2 - b_0^2 < 0$. For this case, the origin is either a focus point ($a_0 \neq 0$) or a center ($a_0 = 0$). Similarly, let $a = a_0$ and $b = \sqrt{-(a_3^2 + b_3^2 - b_0^2)} > 0$, then the eigenvalues of the Jacobian of the system evaluated at the origin are $a \pm b\,i$. Applying a linear transformation into Eq. (3.2) and renaming the coefficients of the third order terms in the resulting equations as a_{30}, a_{21}, a_{12}, b_{30}, b_{21}, b_{12} and b_{30} produces

$$\begin{aligned}\dot{x} &= a\,x + b\,y + a_{30}\,x^3 + a_{21}\,x^2\,y + a_{12}\,x\,y^2 + a_{03}\,y^3,\\ \dot{y} &= -b\,x + a\,y + b_{30}\,x^3 + b_{21}\,x^2\,y + b_{12}\,x\,y^2 + b_{03}\,y^3.\end{aligned} \tag{3.6}$$

Then via the scalings given in Eq. (3.4), one can shift the point $(0,1)$ to $(0,0)$ to obtain

$$
\begin{aligned}
\frac{du}{d\tau} &= v + 2\,\bar{a}_{21}\,u^2 + 4\,a\,u\,v - \frac{3}{2}\,v^2 + 4\,b\,\bar{a}_{30}\,u^3 - 2\,\bar{a}_{21}\,u^2 v - 2\,a\,u\,v^2 + \frac{1}{2}\,v^3, \\
\frac{dv}{d\tau} &= -u - 4\bar{b}_{21}u^2 + 2(2a^2+2b^2+1)uv - 8\bar{b}_{30}u^3 + 4\bar{b}_{21}u^2 v - (2a^2+b^2+1)uv^2. \qquad (3.7)
\end{aligned}
$$

where $\bar{a}_{21}$ and $\bar{a}_{30}$ are given in Eq. (2.1), while $\bar{b}_{21}$ and $\bar{b}_{30}$ become

$$
\begin{aligned}
\bar{b}_{21} &= b_{21}\,b^2 - a_{21}\,a\,b + 2a\,b^2 + 2\,a^3 + a, \\
\bar{b}_{30} &= b_{30}\,b^3 - a_{30}\,a\,b^2 - a_{21}\,a^2\,b + b_{21}\,a\,b^2 + a^2\,b^2 + a^4 + \frac{1}{2}\,a^2. \qquad (3.8)
\end{aligned}
$$

Based on Eq. (3.7), we can follow the same procedures, described in Sections 1 (for the special case) and 2 (for the generic case), to consider the case when the origin is a focus point. In fact, all the formulas for this case are similar to the case when the origin is a saddle point or a node. Therefore, for this case, we can conclude that the maximal number of small limit cycles is also twelve and they indeed exist.

One may think that in addition to the two symmetric fine focus points, making the origin be a center may increase the number of small limit cycles. This requires $a = 0$, for which one can easily show that all the three fixed points (the origin and the two symmetric points) are centers. Thus, this does not increase the number of limit cycles.

Finally, let us consider the origin being a focus point when $a = 0$ (i.e., $a_0 = 0$), and investigate the maximal number of small limit cycles which can exist around the origin. Under the condition $a = 0$, Eq. (3.6) has eight free coefficients a_{ij}'s and b_{ij}'s, while $b = \omega > 0$ can be chosen arbitrarily, which does not affect the number of limit cycles. In general, eight free coefficients may yield nine small limit cycles. But a careful consideration (following a similar procedure as that given in the previous sections) shows that the maximal number of small limit cycles around the origin is five, much less than twelve for the case of two symmetric focus points. The case $a = b = 0$, giving rise to a double zero singular point, obviously cannot have twelve small limit cycles.

Summarizing the results obtained in this section, we have the following theorem.

Theorem 3.1. *For the cubic system (1.2), suppose it has property of Z_2-symmetry, then the maximal number of small limit cycles that the system can exhibit is twelve, i.e., $H_2(3) = 12$.*

Here, as usual, $H(n)$ denotes the Hilbert number of a planar system with nth-degree polynomials. The subscript 2 indicates the Z_2-equivariant vector field.

Finally, a numerical example is given to show the twelve small limit cycles around the two symmetrical focus points when the origin is a focus point. Taking again $a = 0.8$, and using the following perturbations: $\epsilon_1 = 0.4 \times 10^{-4}$, $\epsilon_2 = 0.2 \times 10^{-2}$, $\epsilon_3 = 0.5 \times 10^{-6}$, $\epsilon_4 = 0.2 \times 10^{-10}$, $\epsilon_5 = 0.5 \times 10^{-14}$ and $\epsilon_6 = 0.4 \times 10^{-19}$, yields the six positive roots for r: $r_1 = 0.002181905326$, $r_2 = 0.005739599555$, $r_3 = 0.009556305571$, $r_4 = 0.031425453967$, $r_5 = 0.234338604978$ and $r_6 = 0.350317333017$. The phase portrait of this case, when the origin is a focus point, is shown in Fig. 4, where the areas near the points $(0, \pm 1)$ marked by the two boxes, contain the twelve small limit cycles.

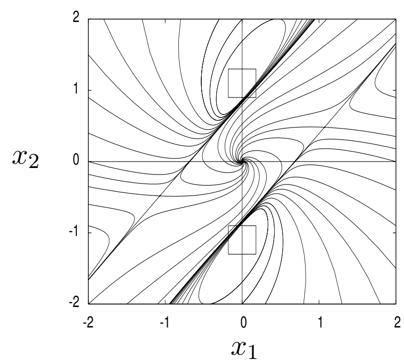

FIGURE 4. The phase portrait of system (1.2) having 12 limit cycles, when the origin is a focus point.

4. **Conclusion.** A cubic order, planar system with Z_2-symmetry is studied for the existence of small limit cycles. With the aid of normal form computation for degenerate Hopf bifurcations, it has been shown that such a system can have maximal twelve small limit cycles. The system has two symmetric fine focus points on the y-axis (or x-axis), while the origin can be a saddle point, or a node, or a focus point. For each case, solutions can be classified as special case or generic case. For the special case, the solutions for twelve limit cycles are finite, while for the generic case, it can have infinite solutions. Furthermore, it has been shown that one coefficient (related to the frequency of limit cycles) can be chosen arbitrarily, which does not affect the number of limit cycles. Fourteen or more small limit cycles are not possible for Z_2-equivariant vector fields.

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