Kinetic State Tracking for a Class of Singularly Perturbed Systems

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The trajectory-following control problem for a general class of nonlinear multi-input, multi-output two-time scale system is revisited. While most earlier works used singular perturbation theory and assumed that an isolated real root exists for the nonlinear set of algebraic equations that constitute the slow subsystem, here two-time scale systems are analyzed in the context of integral manifolds. This accounts for the existence of multiple roots. It is shown that the slow subsystem has a center manifold, and for small values of the slow state, an approximate solution of the nonlinear set of transcendental equations can be computed. Geometric singular perturbation theory is used as the model reduction technique, and dynamic inversion is used to formulate stabilizing control laws for slow state tracking. The control laws are independent of the scalar perturbation parameter, and an upper bound for it is determined such that boundedness of all the closed-loop signals is guaranteed. The methodology is validated through numerical simulation of a generic two-degree of freedom kinetic model and a nonlinear, coupled, six degree-of-freedom model of the F/A-18A Hornet. Results presented in the paper show that this methodology permits close tracking of the reference trajectory while maintaining all the control signals well within their specified bounds.

Nomenclature

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<thead>
<tr>
<th>Symbol</th>
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<tr>
<td>e</td>
<td>Tracking error</td>
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<tr>
<td>M</td>
<td>Mach number</td>
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<tr>
<td>p, q, r</td>
<td>Aircraft roll, pitch and yaw rates</td>
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<td>t</td>
<td>Slow time scale</td>
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<td>t(_0)</td>
<td>Initial time</td>
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<td>u</td>
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<td>x, z</td>
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<td>y</td>
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\(\alpha, \beta, \delta_e, \delta_a, \delta_r\) angle-of-attack, sideslip angle, elevator, aileron and rudder

\(\epsilon\) Small scalar parameter

\(\eta\) Throttle input

\(\gamma\) Flight path angle

\(\phi\) Roll attitude angle

\(\psi\) Heading angle

\(\theta\) Pitch attitude angle

\(\tau\) Fast time scale

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I. Introduction

Mathematical modeling of many physical systems calls for high-order dynamic equations. The presence of certain parameters such as spring constant, mass, moments of inertia and Reynold’s number to name a few, are the cause of stiffness and increased order of these equations. It is highly difficult to arrive at exact analytical solutions of these nonlinear governing equations with known and sometimes unknown variable coefficients and therefore an approximate solution is computed. Singular perturbation theory is an scheme used to simplify systems which inherently possess both fast and slow dynamics. Such systems are characterised by a small parameter multiplying the highest derivative. Suppression of this small parameter reduces the order of the system, and thus the label ‘singularly perturbed’. The commencement of singular perturbation theory dates back to the 1904 work of Prandtl on fluid boundary layers. But it was not until the 19th century that applications of perturbation methods were explored for control design.

The main contribution of perturbation methods is at the level of modeling where it has been used as a model-reduction technique as well as a means of removing the numerical stiffness in the original system. In particular, the method of matched asymptotic expansions reduces study of the full-order system of equations to the study of two other degenerate models. The first captures the dominant phenomena and the neglected phenomena is handled in the next. For the full-order system,

\[ \dot{x} = f(x, z, \epsilon); \]
\[ \epsilon \dot{z} = g(x, z, \epsilon), \]

the lower order models are developed to be of the form,

\[ \dot{x} = f(x, z, \epsilon); \]
\[ 0 = g(x, z, \epsilon), \]
\[ x' = 0; \]
\[ z' = g(x, z, \epsilon), \]

where \( \epsilon \) represents the scalar perturbation parameter. It has been shown that the behaviour of the system of equations (Eqs.1) is constrained within \( O(\epsilon) \) bound of Eqs.2 (‘slow dynamics’) provided the dynamics of Eqs.3 (‘fast dynamics’) is stabilizing.\(^4\) One problem evident with the reduced model, (Eq.2) is the solution of the transcendental or the algebraic set of equations for the fast states \( z \). It is known that there maybe many solutions satisfying the equation. To get around this problem, in the control literature it has been assumed that in the domain of interest, Eq.2 has isolated real roots.

Tracking properties of general singularly perturbed systems were first studied by Gruje \(^5\) in 1982. This work laid down the foundations of tracking theory in a Lyapunov sense. Later in 1988 this work was extended for nonlinear time-varying singularly perturbed systems.\(^5\) However, it is assumed that separate controls are available for both; the slow and the fast subsystems and the algebraic set of equations have a trivial solution. Jayasuriya \(^7\) designed a controller structure for a linear singularly perturbed model consisting of a pre-compensator to achieve the tracking objective and a stabilizing compensator in conjunction with a Luenberger observer to estimate the slow subsystem dynamics. Asymptotic tracking was guaranteed as long as the closed loop system possessed two clusters of eigenvalues with their ratio less than a predetermined upper bound.
Christofides et al. developed robust controller design for systems with stabilizable fast subsystem, and; input/output linearizable slow subsystem with input-to-state stable inverse dynamics. This work considered a general class of nonlinear time-varying singularly perturbed systems that have dynamics linear in the fast states.

Another approach to tracking was presented in an article by Heck in 1991. This paper addressed the design of sliding-mode controllers for a class of linear time-invariant systems where tracking of slow variables is desired. For both the slow and the fast subsystems, a sliding mode controller is designed and a composite of these controls is then implemented on the full-order system. Concept of composite control, that is designing separate controller for each of the subsystems and then implementing their cumulative to the full higher order system was initiated by Suzuki and Miura in 1976 and since then this concept has been extensively used by researchers for robust stabilization of systems with time-scale properties.

In robotics, the flexible modes constitute the fast subsystem. Vandegrift et al. developed a nonlinear tracking controller for the link-tip positions of a multi-link flexible robot arm. The concept of composite control was implemented with feedback linearization used as the control technique. Other work on the same lines is discussed by Beji and Abichou who dealt with tracking control problem of a parallel robot whose motor dynamics make up the fast subsystem. In both of these articles, the dynamics appear linear in the fast state that guarantees existence of a unique solution of the algebraic equation.

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Noticeable from the above study of literature are certain facts. Although all the systems studied fall under the category of Eqs. different design methodologies have been developed for varied physical systems, for which, further, several control techniques have been employed. Moreover, the control laws developed for general form of physical systems, assume existence of unique solution of the transcendental equation. For general dynamical system models, the existence of isolated roots for the fast states is not guaranteed. Study of singularly perturbed systems with such behaviour has been the focus of what is called the ‘Geometric Singular Perturbation Theory’, foundation of which, was laid down by Fenichel in 1979. Since then, this theory has been employed for transforming multiple-time scale systems into standard form (Eq.1). and to develop reduced-order models. Geometric methods study the reduced-order models (Eqs.2,3) through perspective of integral manifolds. Work by Sharkey and O’Reilly designed stabilizing control laws for a special class of singularly perturbed systems wherein the control appears only in fast dynamics. The global nature of the above stabilization results was proved by Chen later on 1998.

In this paper, concepts from geometric singular perturbation theory are used on a general class of time-varying singularly perturbed systems. Specifically, systems that are nonlinear both in slow and fast states which have not been addressed in literature to date. Three major contributions are made. First, our work is no longer restricted to systems that have a unique solution for the nonlinear algebraic set of equations of the slow subsystem. We acknowledge the presence of multiple roots by proving that a center manifold exists for the slow subsystem. This allows for the incorporation of results from the center manifold theory that will be helpful in obtaining approximate roots of the transcendental equations. We then design tracking control law for the slow and fast subsystems to track the desired reference and computed approximation respectively. Second, each of the above control laws developed do not require knowledge of the scalar perturbation parameter. Using Lyapunov theory, boundedness of all closed loop signals is guaranteed. We derive an upper bound \( \varepsilon^* \) for the singular perturbation parameter for which the error remains bounded. Lastly, we demonstrate the performance of the developed control algorithm for two challenging nonlinear problems, a generic two-degree of freedom kinetic model and a nonlinear, coupled, six degree-of-freedom model of the F/A-18A Hornet.

The paper is organized as follows. Section II describes the models considered in this work and formulates the control problem. Section III presents the necessary concepts of geometric singular perturbation theory and motivation of this work. Section IV develops the reduced-order models and formulates the control laws required to achieve tracking. The proof of stability is also presented in this section. Numerical simulations
are presented in Section V. Finally conclusions are discussed in Section VI.

II. Problem Formulation

The dynamic system considered is the nonlinear affine in control singularly perturbed system mathematically written as,

\[
\dot{x} = f(x, z) + g(x, z)u; \quad x(t_0) = x(0) \tag{4}
\]

\[
\epsilon \dot{z} = l(x, z) + k(x, z)u; \quad z(t_0) = z(0) \tag{5}
\]

\[
y = x \tag{6}
\]

where \(x \in \mathbb{R}^m\) is the set of slow variables of the system and \(z \in \mathbb{R}^p\) is the vector of the fast variables. \(\epsilon\) is a small scalar positive parameter that characterizes the multiple time-scale behaviour of the system. The output is denoted as \(y \in \mathbb{R}^m\). The input vector \(u \in \mathbb{R}^m\) consists at least as many controls as the number of outputs. The dot over the variables indicates derivative with respect to \(t\), referred to as the slow time-scale of the system.

- The functions \(g(x, z)\) and \(k(x, z)\) represent the control-influence terms, while all other terms such as inertial coupling, gravitational forces are all contained in \(f(x, z)\) and \(l(x, z)\).
- For a rigid body one may notice that \(x\) are the translational velocities, while \(z\) represents the angular velocities. The rotational dynamics written out for a rigid body contains the nonlinear inertial coupling terms. The function \(l(x, z)\) captures this nonlinearity in the fast states.
- The output vector consists of the slow kinetic states. It is desired that the output \(y\) tracks a desired smooth trajectory specified by \(y_r\) and its derivative \(\dot{y}_r\).
- **Assumption 1**: The number of control inputs required in this study are at least equal to the number of outputs.

III. Background: Geometric Singular Perturbation Theory

Singular Perturbation Theory is a tool used by control engineers to obtain reduced-order approximations of the full-order stiff equations of motion which are difficult to analyze. The theory is valid so long as the parameter \(\epsilon\) remains sufficiently small and the time-scale behaviour is preserved. The Method of Matched Asymptotic Expansions and its variation, the Method of Composite Expansions have been the foremost methods employed to develop these reduced-order models. The alternative geometric approach describes the motion of the full-order system using the concept of invariant manifolds. Both the above mentioned approaches narrow down to the exact same reduced-order models, but with different assumptions about the system. Asymptotic methods assume that the dynamical system possesses isolated roots, while the geometric approach is more general and takes into consideration multiple non-isolated roots of nonlinear systems. This very fact was the motivation to carry out analysis in this work using the geometric approach. Consider an open-loop nonlinear system,

\[
\dot{x} = f(x, z); \tag{7}
\]

\[
\epsilon \dot{z} = l(x, z). \tag{8}
\]

The above equations can be rewritten in the fast time scale \(\tau = \frac{(t-t_0)}{\epsilon}\),

\[
x' = \epsilon f(x, z); \tag{9}
\]

\[
z' = l(x, z). \tag{10}
\]

In the above equations the prime denotes differentiation with respect to the fast time scale. The independent variables \(t\) and \(\tau\) are referred to as the slow and fast time scales respectively, and Eqs.7-8 and Eqs.9-10 (referred as slow and fast systems respectively) are equivalent whenever \(\epsilon \neq 0\).
At first, the system is studied for $\epsilon = 0$. The fast system reduces to $p$ dimensions with variables $\mathbf{x}$ as constant parameters:

\begin{align}
\mathbf{x}' &= 0; \\
\mathbf{z}' &= \mathbf{l}(\mathbf{x}, \mathbf{z}).
\end{align}

On the other hand, the order of the slow system reduces to dimension $m$. The resulting set of differential-algebraic equations from here on will be referred as the reduced slow system.

\begin{align}
\mathbf{x} &= \mathbf{f}(\mathbf{x}, \mathbf{z}); \\
0 &= \mathbf{l}(\mathbf{x}, \mathbf{z}).
\end{align}

The reduced slow system appears to be locally flattened vector space of the complete slow system. Thus, the set of points $(\mathbf{x}, \mathbf{z}) \in \mathbb{R}^m \times \mathbb{R}^p$ is expected to have a $C^r$ smooth manifold $\mathcal{M}_0$ of dimension $m$ inside the zero set of function $\mathbf{l}(\cdot)$, provided the functions $\mathbf{f}(\cdot)$ and $\mathbf{l}(\cdot)$ are assumed to be $C^r$, $r \geq 1$.

**Assumption 2**: The functions $\mathbf{f}(\mathbf{x}, \mathbf{z})$ and $\mathbf{l}(\mathbf{x}, \mathbf{z})$ are sufficiently smooth, that is, $C^r$, with $r \geq 1$.

The above assumption asserts the presence of a smooth diffeomorphism that maps the flow onto a local lower-dimension Euclidean space. The requirement upon the dynamical system to be continuous and at least once differentiable assures smoothness of the manifold $\mathcal{M}_0$.

Further, Eqs.13-14 suggest that $\mathbf{x}$ evolve according to the reduced slow system, with $\mathbf{z} = \mathbf{Z}_0(\mathbf{x})$:

\begin{align}
\mathbf{x} &= \mathbf{f}(\mathbf{x}, \mathbf{Z}_0(\mathbf{x}));
\end{align}

where $\mathbf{Z}_0(\mathbf{x})$ is the solution of the algebraic part, Eq.14 and defines the manifold,

\begin{align}
\mathcal{M}_0 : \mathbf{z} = \mathbf{Z}_0(\mathbf{x}); & \quad \mathbf{x} \in \mathbb{R}^m, \mathbf{z} \in \mathbb{R}^p
\end{align}

When viewed from the perspective of the reduced fast system (Eqs.11-12), manifold $\mathcal{M}_0$ is the set of fixed points $(\mathbf{x}, \mathbf{Z}_0(\mathbf{x}))$. Thus, $\mathcal{M}_0$ is trivially invariant. Moreover, if every point $(\mathbf{x}, \mathbf{Z}_0(\mathbf{x}))$ of the $\mathcal{M}_0$ is a hyperbolic fixed point of Eq.12, then the manifold said to be *normally hyperbolic*. Loosely, this implies that the flow in the directions normal to the manifold $\mathcal{M}_0$ is faster than of those tangential to it. Furthermore, the normally hyperbolic invariant manifold has local, $C^r$ smooth, stable and unstable manifolds: $\mathcal{W}^S_{loc}(\mathcal{M}_0)$ and $\mathcal{W}^U_{loc}(\mathcal{M}_0)$. These manifolds are unions over all $(\mathbf{x})$ in $\mathcal{M}_0$ of the local stable and unstable manifolds of the reduced fast system’s hyperbolic fixed points, $(\mathbf{x}, \mathbf{Z}_0(\mathbf{x}))$. These ideas are described graphically in Figure 1-2. For convenience of display, consider the following system

\begin{align}
\dot{x}_1 &= -x_1; \\
\dot{x}_2 &= -x_2; \\
\epsilon \dot{z} &= -z.
\end{align}

The invariant manifold is given by $\mathcal{M}_0 : z = 0$. Thus the complete $x_1 - x_2$ plane is the invariant manifold. Origin is the stable hyperbolic equilibrium of the reduced slow system, therefore any trajectory starting on the manifold, approaches origin in time (Figure 2). The set of points $(x_1, x_2, z)$ approach the manifold at an exponential rate forward in time and therefore the complete space is the stable manifold $\mathcal{W}^S(\mathcal{M}_0)$.

Studying the reduced fast system suggests that for any point with non-zero $z(0)$, the flow approaches normal to the manifold. These normal lines (with constant $x_1$ and $x_2$) are called the fast fibers, denoted as $\mathcal{F}^S_{(x_1, x_2)}$, with the superscript to signify its stability characteristic, $S$ for stable and $U$ for unstable. The union of all these fibers for all $(x_1, x_2)$ constitute the stable manifold. Intuitively one may conclude that for initial conditions not on the manifold, the reduced fast system describes the transition to the manifold, following which the system evolves according to reduced slow system.

For the full-order system, similar inferences can be made. The presence of $\epsilon$ in Eq.8 indicates that the fast variables grow relatively faster than the other states of the system, and if their open-loop system is stabilizing, these states quickly settle down to their equilibrium. While the other variables continue to evolve in time, with the fast variables fixed by an equilibrium hyper surface. Mathematically, $\exists t^* : t^* > t_0$, after which the solutions $\mathbf{x}(t, \epsilon)$ and $\mathbf{z}(t, \epsilon)$ lie on a distinct $m$ dimensional-invariant manifold $\mathcal{M}_\epsilon$,

\begin{align}
\mathcal{M}_\epsilon : \mathbf{z} = \mathbf{Z}(\mathbf{x}, \epsilon); & \quad \mathbf{x} \in \mathbb{R}^m, \mathbf{z} \in \mathbb{R}^p
\end{align}
Figure 1. Geometric sketch of system of equations Eqs.17-19 for $\epsilon = 0$

Figure 2. The flow of the system of equations Eqs.17-19 for $\epsilon = 0$ projected onto the $x_1$-$x_2$ plane.
For the example considered, Eqs.17-19, the invariant manifold continues to be the $x_1 - x_2$ plane. In addition, the family of lines parallel to the $z$-axes still constitute the family of fast fibers. But these fast fibers are now invariant relative to the system, only as a family, not individually as was the case earlier. Consider Figure.3 to study this behaviour. To generate this figure, $\epsilon$ was chosen to be 0.2. For an initial condition starting out a particular fiber, evolves now in two parts: one component along the manifold $M_\epsilon$, which is governed by the reduced slow system, and the other component in the normal direction, whose flow is governed by the exponential rate of convergence of the fibers. For the points that are already on the manifold, are seen to evolve similar to the flow sketched in Figure.2. Thus, the reduced-order models provide good insight of the behaviour of the full-order system.

The above geometric constructs are more formally, statements of Fenichel’s persistence theory. First, the following assumptions about the open loop system Eqs.9-10 are made:

**Assumption 3:** There exists a set $\mathcal{M}_0$ that is contained in $\{ (x,z) : l(x,z) = 0 \}$ such that $\mathcal{M}_0$ is a compact boundary less manifold.

**Assumption 4:** $\mathcal{M}_0$ is normally hyperbolic relative to Eq.12 and in particular, it is required that for all points $z \in \mathcal{M}_0$, there are $k$(respectively $l$) eigenvalues of $D_{x}l(0,0,z)$ with positive (respectively negative) real parts are bounded away from zero, where $k + l = p$.

**Theorem 1:** (Fenichel Theorem for compact boundary less manifolds). Let Eq.9-10 satisfy the assumptions (2),(3) and (4). If $\epsilon > 0$ is sufficiently small, then there exists a manifold $\mathcal{M}_\epsilon$ that is $C^{r-1}$ smooth locally invariant under Eqs.9-10 and $C^{r-1} ~ O(\epsilon)$ close to $\mathcal{M}_0$. In addition, there exist perturbed local stable and unstable manifolds of $\mathcal{M}_\epsilon$. They are the unions of invariant families of fast stable and unstable fibers of dimensions $l$ and $k$ respectively, and they are $C^{r} ~ O(\epsilon)$ close, for all $r < \infty$, to their unperturbed counterparts.

Besides, the above assertion, from study of system Eqs.17-19, it is apparent that if the fast fibers were unstable then a point not on the manifold, would be blown farther away in time. Moreover, for the system considered the manifolds $\mathcal{M}_0$ and $\mathcal{M}_\epsilon$ were obtained to be identically equal. This is not the case with non-linear systems. The invariant manifold $\mathcal{M}_\epsilon$ is locally tangential to $\mathcal{M}_0$. In fact, any manifold that is locally tangential to $\mathcal{M}_0$, is a candidate manifold $\mathcal{M}_\epsilon$. This conclusion stems from the center manifold theory. To
study the behaviour of this manifold, rewrite the fast system as (using the technique called suspension)\(^{26}\)

\[
\begin{align*}
x' &= \epsilon f(x, z); \\
\epsilon' &= 0 \\
z' &= l(x, z);
\end{align*}
\]

(21)\(^{26}\)

The perturbed system obtained by linearizing the above equations about the point \((\epsilon = 0, x, Z(x, \epsilon))\) is written as, in compact form:

\[
\begin{align*}
\Delta w' &= Fw + F_1 z \\
\Delta z' &= Lz + L_1 w
\end{align*}
\]

(24)\(^{26}\)

(25)\(^{26}\)

where \(w = [x, \epsilon]^T\) and \(F, F_1, L, L_1\) are constant matrices of appropriate size obtained by using Taylor series expansion of Eqs.21-23. If all the eigenvalues of \(F\) have zero real parts while all the eigenvalues of \(L\) have negative real parts the manifold for Eqs.9-10, then \(M_\epsilon\), is precisely the center manifold, and it belongs to span of the generalized eigenvectors associated with eigenvalues with zero real parts. The requirement on eigenvalues of \(F\) supports the existence of time-scales in the system, for if the eigenvalues were non-zero, then all the states would be fast variables and the system is not singularly perturbed. Thus, this suggests that the eigenvalue restriction on \(F\) is always satisfied by systems with multiple time scale property. The other requirement of negative eigenvalues of \(L\) is to make sure that the trajectories not on the manifold approach it in forward time.

A. Computing the invariant manifold

With this connection between Fenichel’s work and center manifold theory, one can go ahead to solve for the invariant manifold \(M_\epsilon\) defined in Eq.20. Invariance of the manifold, implies that it satisfies, the manifold condition:

\[
\epsilon \left[ \frac{\partial Z}{\partial x} f(x, Z(x, \epsilon)) \right] = g(x, Z(x, \epsilon))
\]

(26)\(^{26}\)

Then one plugs in the perturbation expansion for \(Z(x, \epsilon) = Z_0(x) + \epsilon Z_1(x) + O(\epsilon^2)\) into Eq.26 to solve order by order for \(Z(x, \epsilon)\). At order \(\epsilon^0\), one finds the equation: \(0 = g(x, Z_0(x))\) which defines \(Z_0(x)\). For nonlinear system analysis, the domain of interest is known, and therefore one may use the implicit function theorem to solve for \(Z_0(x)\) since matrix \(L\) is invertible. It is inverse problem that a control engineer is posed with, that is to find \(Z_0(x)\) as a smooth function of its arguments.

With the earlier analysis, it is known that manifold \(M_\epsilon\) is a center manifold. At this point, theorems from center manifold theory are invoked to approximate \(Z_0(x)\). If origin is the fixed point of the linearized system, then the next theorem asserts that one can find \(Z(x)\) to any degree of accuracy.

**Theorem 2**\(^{27}\): Let \(\phi : \mathbb{R}^m \times \mathbb{R} \to \mathbb{R}^p\) satisfy \(\phi(0, 0, 0) = 0\) and \(|(M\phi)(x, \epsilon)| = O(|\epsilon|^q, |x|^n)\) for \(|x| \to 0\) and \(\epsilon \to 0\) where \(q > 1\) and \(n > 1\) and

\[
(M\phi)(x, \epsilon) = \epsilon \left[ \frac{\partial Z}{\partial x} f(x, Z(x, \epsilon)) \right] - g(x, Z(x, \epsilon)).
\]

(27)\(^{27}\)

Then as \(x \to 0\) and \(\epsilon \to 0\),

\[
|Z(x, \epsilon) - \phi(x, \epsilon)| = O(|\epsilon|^q, |x|^n).
\]

(28)\(^{27}\)

IV. Control Law Development

Consider the system,

\[
\begin{align*}
x &= f(x, z) + g(x, z)u; \quad x(t_0) = x(0) \\
\epsilon z &= l(x, z) + k(x, z)u; \quad z(t_0) = z(0) \\
y &= x
\end{align*}
\]

(29)\(^{27}\)

(30)\(^{27}\)

(31)\(^{27}\)

where the vector fields \(f(\cdot), g(\cdot), l(\cdot)\) and \(k(\cdot)\) are assumed to be sufficiently smooth. It is desired to steer the slow-state variables to a desired reference. Equivalently from an earlier discussion, it can be concluded
that the desired equilibrium must lie on the manifold, and the fast subsystem must be designed such that the initial conditions not on the manifold are directed to quickly fall onto it. Further, in order to compute the equilibrium of solutions upon which the fast states must lie in order to accomplish control objective, the local approximation result for center manifolds is invoked. Composite control technique is then adopted to first, set the desired reference as the equilibrium of the slow subsystem, and second to assign the computed manifold as the hyperbolic fixed point of the fast subsystem.

Apply a change of coordinates with the tracking error defined as \( e = x - y_r \), to transform the equilibrium of the slow dynamics to the origin.

\[
\begin{align*}
\dot{e} &= f^*(e, z) + g^*(e, z)u; \quad (32) \\
\epsilon \dot{z} &= l^*(e, z) + k^*(e, z)u; \quad (33)
\end{align*}
\]

where \( f(e + y_r, z) - y_r = f^*(..) \), \( g(e + y_r, z) = g^*(..) \) and so on. Further, the slow subsystem is constructed as laid down in the earlier section as,

\[
\begin{align*}
\dot{e} &= f^*(e, z) + g^*(e, z)u_s; \quad (34) \\
0 &= l^*(e, z) + k^*(e, z)u_s; \quad (35)
\end{align*}
\]

From earlier discussion it is known that there exists a smooth manifold of dimension \( n \), and it is a center manifold and label the local approximation to the manifold as \( \Phi_0(e, u_s) \). Therefore, the slow subsystem is obtained to be,

\[
\dot{e} = f^*(e, \Phi_0(e, u_s)) + g^*(e, \Phi_0(e, u_s))u_s. \quad (36)
\]

Then, design \( u_s \), such that

\[
\dot{e} = -K_e e, \quad (37)
\]

where \( K_e \) is positive definite symmetric matrix. Since \( u_s \) is a function of the error, the manifold can be written then as \( \Phi_0(e) \).

In order to design the controller for the fast subsystem, rewrite the set of equations Eqs.32-33, in the fast time scale and with new set of coordinates, \( z = \zeta + \Phi_0(e) \).

\[
\begin{align*}
\epsilon' &= \epsilon f^*(e, \zeta + \Phi_0) + g^*(e, \zeta + \Phi_0)u; \quad (38) \\
\zeta' &= l^*(e, \zeta + \Phi_0) + k^*(e, \zeta + \Phi_0)u - \epsilon \Phi_0 \quad (39)
\end{align*}
\]

Thus, for \( \epsilon = 0 \), the fast subsystem is obtained as:

\[
\begin{align*}
\epsilon' &= 0; \quad (40) \\
\zeta' &= l^*(e, \zeta + \Phi_0) + k^*(e, \zeta + \Phi_0)(u_s + u_f) \quad (41)
\end{align*}
\]

Now, with the knowledge of \( u_s(e) \), design \( u_f(e, \zeta) \) such that

\[
\zeta' = -K_\zeta \zeta \quad (42)
\]

where \( K_\zeta \) is a positive definite symmetric matrix.

Further, it remains to show that with control computed as \( u = u_s(e) + u_f(e, \zeta) \), maintains closed-loop stability. First, rewrite the closed-loop system as,

\[
\begin{align*}
\dot{e} &= f^*(e, \Phi_0(e, u_s)) + g^*(e, \Phi_0(e, u_s))u_s + f^*(e, \zeta + \Phi_0) - f^*(e, \Phi_0(e, u_s)) \\
&\quad + g^*(e, \zeta + \Phi_0)u - g^*(e, \Phi_0(e, u_s))u_s; \quad (43) \\
\epsilon \dot{\zeta} &= l^*(e, \zeta + \Phi_0) + k^*(e, \zeta + \Phi_0)(u_s + u_f) - \epsilon \Phi_0 \quad (44)
\end{align*}
\]

From, Eqs.36,37,42

\[
\begin{align*}
\dot{e} &= -K_e e + f^*(e, \zeta + \Phi_0) - f^*(e, \Phi_0(e, u_s)) \\
&\quad + g^*(e, \zeta + \Phi_0)u - g^*(e, \Phi_0(e, u_s))u_s; \quad (45) \\
\epsilon \dot{\zeta} &= -K_\zeta \zeta - \epsilon \Phi_0 \quad (46)
\end{align*}
\]
Choose a Lyapunov function $\nu(e, \zeta) = V(e) + W(\zeta)$, where $V(e) = e^T e$ and $V(\zeta) = \zeta^T \zeta$ to study the stability properties of the closed-loop system. Then along the trajectory of Eqs. 45-46,

$$
\dot{\nu} = 2e^T \dot{e} + 2\zeta^T \dot{\zeta} \\
= 2e^T [-K_e e + f^*(e, \zeta + \Phi_0) - f^*(e, \Phi_0(e, u_s)) + g^*(e, \zeta + \Phi_0)u - g^*(e, \Phi_0(e, u_s))u_s] \\
+ \frac{2}{\epsilon} \zeta^T \left[-K_\zeta \zeta - \epsilon \dot{\Phi}_0\right]
$$  

(47)

Further, let $K_1 = \|2K_e\|$ and $K_2 = \|2K_\zeta\|$, and

$$
2e^T\|f^*(e, \zeta + \Phi_0) - f^*(e, \Phi_0(e, u_s)) + g^*(e, \zeta + \Phi_0)u - g^*(e, \Phi_0(e, u_s))u_s\| \leq \beta_1 \|e\| \|\zeta\|  
$$  

(49)

Then,

$$
\dot{\nu} \leq -K_1 \|e\|^2 - \frac{1}{\epsilon} K_2 \|\zeta\|^2 + \beta_1 \|e\| \|\zeta\| - \|\zeta\| \|2\Phi_0\| 
$$  

(50)

$$
\dot{\nu} \leq - \left[ \begin{array}{c} \|e\| \\ \|\zeta\| \end{array} \right]^T \left[ \begin{array}{cc} K_1 & -\frac{K_2}{2} \\ -\frac{\beta_1}{2} & \frac{1}{\epsilon} K_2 \end{array} \right] \left[ \begin{array}{c} \|e\| \\ \|\zeta\| \end{array} \right] - \|\zeta\| \|2\Phi_0\| 
$$  

(51)

Thus, if $\epsilon < \epsilon^*$, where,

$$
\epsilon^* = \frac{4K_1K_2}{\beta_1^2} 
$$  

(52)

$$
\dot{\nu} \leq -K \left[ \begin{array}{c} \|e\| \\ \|\zeta\| \end{array} \right]^T \left[ \begin{array}{c} \|e\| \\ \|\zeta\| \end{array} \right] - \xi 
$$  

(53)

Thus, from uniform boundedness theorem, one can conclude that the errors $e$ and $\zeta$ are bounded.

This result is similar to the conclusions made by Fenichel. Consider the closed-loop system Eqs. 45-46. For $\epsilon = 0$,

$$
\dot{e} = -K_e e + f^*(e, \zeta + \Phi_0) - f^*(e, \Phi_0(e, u_s)) + g^*(e, \zeta + \Phi_0)u - g^*(e, \Phi_0(e, u_s))u_s; \\
0 = -K_\zeta \zeta 
$$  

(54)

(55)

$\zeta = 0$, becomes the solution of the algebraic equation, therefore,

$$
\dot{e} = -K_e e 
$$  

(56)

$e = 0$ is the asymptotically stable equilibrium of the reduced slow-system. Similarly, the reduced fast-system has $\zeta = 0$ as its asymptotically equilibrium. Therefore, from the Fenichel’s theorem one can conclude existence of $\mathcal{M}_e O(\epsilon)$ close to $\zeta = 0$.

One other point to be noted is the extra term $\|\Phi_0\|$ in the derivative of the Lyapunov function. Since, in the approach followed in this work, $\Phi_0$ is an approximation of solution of the algebraic equation and thus not invariant, leads to a non-zero derivative term to appear. It is easy to notice when $\Phi_0 = 0$, errors of the closed-loop system can be concluded to asymptotically approach origin. This is the case for systems which are linear in fast states, where an exact manifold solution can be computed. From discussion presented in the earlier section, this exact manifold becomes invariant and its derivative is trivially equal to zero.

V. Numerical Simulations

A. Purpose and Scope

Validation of theoretical developments presented above is demonstrated in this section through simulation. The first example is a generic planar nonlinear system. This planar example enables the study of the geometric constructs that are generally difficult to visualize in higher dimension problems. Starting from the analysis of the open-loop system, step-by-step procedure of control development is laid out for the system to follow a desired slow kinetic state. The comparison between the manifold approximation and the attained
actual fast state is made. The tracking results are studied for different time-varying trajectories. Further
the controller performance is studied by varying the feedback gains. Moreover, the behaviour of the system
in the absence of the nonlinearity in the fast dynamics is also analyzed.

The next simulation develops control laws for the highly nonlinear F/A-18A Hornet. The motive of
this example is to test the performance of the controller for a highly nonlinear two-time scale system. It is
required to perform a turning maneuver while maintaining zero sideslip, and specified angle-of-attack profile.
Dynamic inversion is the control technique applied to accomplish the desired performance.

B. A Two-Degree of Freedom Generic Kinetic Nonlinear Model

The fast dynamics are modified to include an arbitrarily chosen quadratic nonlinearity in the fast state.
Further ‘pseudo’ control term with unit effectiveness is introduced in the system dynamics. In the system
described below, \( x \in \mathbb{R} \) and \( z \in \mathbb{R} \) represent the slow and fast states respectively. The control \( u \in \mathbb{R} \), is
developed to follow a desired smooth trajectory \( x_r \in \mathbb{R} \).

\[
\begin{align*}
\dot{x} &= -x + (x + \kappa - \lambda)z + u; \\
\epsilon \dot{z} &= x - (x + \kappa)z + z^2 + u
\end{align*}
\]  

(57) 

(58)

\( \kappa \) and \( \lambda \) are bounded, positive constants design variables that represented scaled, nondimensionalized concentrations in the original model. The value, \( \epsilon = 0.2 \) is retained in the modified model.

1. Open-Loop System Analysis

Before proceeding on to control design, it is first demonstrated that in fact the concept of approximating the
unperturbed manifold \( M_0 \) for the open-loop system resembles closely the exact manifold solution. Consider
the open-loop system,

\[
\begin{align*}
\dot{x} &= -x + (x + \kappa - \lambda)z; \\
\epsilon \dot{z} &= x - (x + \kappa)z + z^2
\end{align*}
\]  

(59) 

(60)

Following the procedure laid out in Section III, the reduced slow and fast subsystems are:

\[
\begin{align*}
\dot{x} &= -x + (x + \kappa - \lambda)z; \\
0 &= x - (x + \kappa)z + z^2 + u
\end{align*}
\]  

(61) 

(62)

\[
\begin{align*}
x' &= 0 \\
z' &= x - (x + \kappa)z + z^2
\end{align*}
\]  

(63) 

(64)

For Eq.62, the manifold approximation is

\[
\Phi_0(x) = \frac{x}{x + \kappa}
\]  

(65)

and equivalently flow on the manifold follows,

\[
\dot{x} = -\frac{\lambda x}{x + \kappa}
\]  

(66)

which has the origin as it asymptotically stable equilibrium for \( \{\forall x \in \mathbb{R} | x \neq -\kappa \} \). The linearized system
has an eigenvalue of \( \frac{-\lambda}{\kappa} \) at the origin. But because of the local nature of the manifold approximation, one
can only claim origin to be locally stable equilibrium of the system.

In order to study the behaviour of the fast subsystem about the manifold, use a change of coordinates
\( \zeta = z - \Phi_0 \). Then,

\[
\begin{align*}
x' &= 0; \\
\zeta' &= -(x + \kappa)\zeta + (\zeta + \frac{x}{x + \kappa})^2 + \frac{\partial \Phi_0}{\partial x} x'
\end{align*}
\]  

(67) 

(68)
and for any general system the last term in the reduced fast subsystem is identically zero. Further, one may conclude that for small $|x|$, then $\zeta = 0$, or rather $z = \Phi_0$, is the equilibrium solution of the system and $z = \Phi_0$ is an invariant manifold. Also, for $x = -1(-\kappa)$, the approximation reaches a singularity. Thus conclusions can be made only for small $|x| < 1$. The detailed proof of these conclusions is well-discussed by Carr.\textsuperscript{27} The flow of the open-loop system (Eqs. 59-60) is presented in Figure 4. Note that only for points $x < 1$, the flow asymptotically approaches origin. For large values of the slow state, the equilibrium is shifted to $z = 1$. Thus, the simulation validates the local nature of stability results. Further, as mentioned before, the flow on the manifold is governed by reduced slow subsystem and for points away from the manifold follow the reduced fast subsystem until the flow reaches the invariant manifold.

![Figure 4. Phase Plane of the open-loop generic kinetic nonlinear model with $\epsilon = 0.2$, $\kappa = 1$ and $\lambda = 0.5$.](image)

2. Controller Design

For the closed-loop system, equivalent to Eqs. 34-35,

\[
\dot{e} = -(e + x_r) + (e + x_r + \kappa - \lambda)z - \dot{x}_r + u_s \quad (69)
\]

\[
0 = (e + x_r) - (e + x_r + \kappa)z + z^2 + u_s. \quad (70)
\]

With a manifold approximation,

\[
\Phi_0 = \frac{e + x_r + u_s}{e + x_r + \kappa} \quad (71)
\]

the slow control is designed to be,

\[
u_s(e, x_r) = \frac{\dot{x}_r - K_e(e + x_r + \kappa) + \lambda(e + x_r)}{2e + 2x_r + 2\kappa - \lambda} \quad (72)
\]

Similar procedure is adopted to design the fast controller,

\[
u_f(e, \zeta, x_r) = -K_\zeta \zeta + (e + x_r + \kappa)\zeta - (\zeta + \Phi_0)^2. \quad (73)
\]

Then, $u = u_s + u_f$, the closed-loop becomes,

\[
\dot{e} = -K_e e - (K_\zeta - 2\kappa + \lambda)\zeta + 2e\zeta - (\zeta + \Phi_0)^2 \quad (74)
\]

\[
\dot{\zeta} = -K_\zeta \zeta - \epsilon \frac{\partial \Phi_0}{\partial e} \dot{e} \quad (75)
\]
Thus, for small $|\epsilon|$, one may conclude only boundedness of the closed-loop system about origin. Specifically, the asymptotic convergence cannot be proved due to the presence of the quadratic nonlinearity.

### 3. Results and Discussion

**Case 1(a)** The first specified reference is a ramp of slope 0.5 starting from zero to the value 5. The gains chosen are $K_e = 5$ and $K_\zeta = 1$. The initial conditions chosen are $x(0) = 0.1$ and $z(0) = 0.5$. Figure 5 compares closed-loop state to specified reference. Notice that all along, the system maintains the same trend as the reference and remains within $|0.2|$ error throughout. The final value attained is $x(t = 30) = 4.821$. Such close trajectory following is possible only if the fast state remains close to its approximation. This fact is displayed in Figure 6. The fast state remains within $|0.0489|$ of the desired approximate solution of the manifold. At all times the trend is followed. For perfect tracking of constant $x_r = 5$, from Eq. 72, $u_s = 0.2174$ and correspondingly $\Phi_0 = 0.8696$. This results in a value of $u_f = -0.7561$ and therefore $u = -0.5388$. But since in the results, error is not zero, $\Phi_0(t = 30) = 0.9457$ while the fast state actually reached is $z(t = 30) = 0.9456$. The final value of control computed is $u(t = 30) = -0.2111$. Figure 7 presents the computed control input. Note the small magnitude of control input throughout the simulation. The drop at $t = 10$, corresponds to the change in the reference from ramp to the steady-state. The overshoots at initial time due to initial condition error can be avoided by appropriate choice of feedback gains.

![Figure 5](image)

**Figure 5. Case 1(a) Kinetic slow state compared to specified reference ($\kappa = 1$, $\lambda = 0.5$, $\epsilon = 0.2$, $K_e = 5$, $K_\zeta = 1$)**

**Case 1(b)** In order to study the effect of feedback gains, values of $K_e$ and $K_\zeta$ were drastically changed to 1 and 0.1. Lowering the value of $K_\zeta$, the rate at which points not on the manifold drive toward the manifold is brought down. Thus it takes longer for the fast states to follow closely the manifold approximation. While lowering $K_e$, directly effects the error dynamics. With the rate at which error converges to the trajectory reduced, the slow-state does not follow the trend of the trajectory well. Moreover, the control does not have large overshoots as the changes now are made at a slower rate. These exact features are seen in simulation (Figures. 8-10). These characteristics of dynamic-inversion are well-known in control literature and this simulation emphasizes the fact that these characteristics are preserved as long as $\epsilon < \epsilon^*$. 

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Figure 6. Case 1(a) Kinetic fast state compared to manifold approximation $\Phi_0$. ($\kappa = 1$, $\lambda = 0.5$, $\epsilon = 0.2$, $K_e = 5$, $K_\zeta = 1$)

Figure 7. Case 1(a) Computed control ($\kappa = 1$, $\lambda = 0.5$, $\epsilon = 0.2$, $K_e = 5$, $K_\zeta = 1$)
Figure 8. Case 1(b) Kinetic slow state compared to specified reference ($\kappa = 1$, $\lambda = 0.5$, $\epsilon = 0.2$, $K_e = 1$, $K_\zeta = 0.1$).

Figure 9. Case 1(b) Kinetic fast state compared to manifold approximation $\Phi_0$. ($\kappa = 1$, $\lambda = 0.5$, $\epsilon = 0.2$, $K_e = 1$, $K_\zeta = 0.1$).
Figure 10. Case 1(b) Computed control ($\kappa = 1$, $\lambda = 0.5$, $\epsilon = 0.2$, $K_e = 1$, $K_\zeta = 0.1$)

Figure 11. Case 1(c) Kinetic slow state compared to specified reference (absence of nonlinear term in fast dynamics) ($\kappa = 1$, $\lambda = 0.5$, $\epsilon = 0.2$, $K_e = 5$, $K_\zeta = 1$)
Figure 12. Case 1(c) Kinetic fast state compared to manifold approximation $\Phi_0$ (absence of nonlinear term in fast dynamics). $(\kappa = 1, \lambda = 0.5, \epsilon = 0.2, K_e = 5, K_\zeta = 1)$

Figure 13. Case 1(c) Computed Control (absence of nonlinear term in fast dynamics) $(\kappa = 1, \lambda = 0.5, \epsilon = 0.2, K_e = 5, K_\zeta = 1)$
Figure 14. Case 1(d) Kinetic slow state compared to specified sine-wave reference ($\kappa = 1$, $\lambda = 0.5$, $\epsilon = 0.2$, $K_e = 5$, $K_\zeta = 1$)

Figure 15. Case 1(d) Kinetic fast state compared to manifold approximation (following sine-wave) $\Phi_0$. ($\kappa = 1$, $\lambda = 0.5$, $\epsilon = 0.2$, $K_e = 5$, $K_\zeta = 1$)
Case 1(c) Next, in the absence of quadratic nonlinearity, clearly for the closed-loop model (Eqs.74-75), the origin becomes the globally asymptotically stable. These results are presented in Figures.11-13. The final values at $t = 30$ seconds are $\Phi_0 = 0.8695$ and $u = 0.2177$. These are the exact values computed to maintain perfect tracking. Notice in the absence of quadratic nonlinearity, $u_f = 0$ and thus $u = u_s$ at final time.

Case 1(d) The above examples demonstrate that the controller is able to guide the system to follow the trend of the desired reference. To test the controller performance for a continuously time-varying reference, the next simulation is set up such that the a sine-wave $0.2\sin(0.2t)$ is traced. The results are presented in Figures.14-16. The initial conditions chosen are $x(0) = 0.1$ and $z(0) = 0.5$. The feedback gains chosen are $K_e = 5$ and $K_\zeta = 1$. The errors between the actual and reference is within $\pm 0.0106$. For the fast state, the error is within $\pm 0.0037$ of its approximation $\Phi_0$. Again the control is within bounds, though the initial overshoot maybe adjusted further by varying the feedback gains.

C. Turning maneuver maintaining zero sideslip and specific angle-of-attack profile for F/A-18A Hornet

The complete nonlinear dynamic model written in the stability-axes used in the present study is represented by nine states, $(M, \alpha, \beta, p, q, r, \phi, \theta, \psi)$ and four controls $(\eta, \delta_e, \delta_a, \delta_r)$. Out of nine states, it is well-known that angular rates form the set of fast-states. Thus, writing in the notation used in this work, $x = [M, \alpha, \beta, \phi, \theta, \psi]^T$ and $z = [p, q, r]^T$. In this study the aerodynamic database for the symmetric F/A-18 high angle-of-attack research vehicle (HARV)\textsuperscript{28} is used. The aerodynamic coefficients are given as analytical functions of the sideslip, angle-of-attack, angular rates, elevator, aileron and the rudder. Considering the number of controls available, only three out of the six slow states can be controlled. Throttle is maintained constant ($\eta = 0.523$) and not used as a control for this study. This is a result of one of the drawbacks of dynamic inversion. Since the control influence matrix is inverted, abnormally huge values of throttle are obtained. Thus, it is best to control throttle separately if needed. It is desired that a 45 degree turn be made at zero sideslip and specified angle-of-attack profile. Therefore, pitch-attitude angle $\theta$ and bank angle $\phi$ are left uncontrolled.
Since, the aircraft equations of motion are highly coupled, the first step in design of the control law is to transform them such that explicit presence of slow and fast states appears. Let \( \mathbf{x}_s = [\alpha, \beta, \psi]^T \) represent the subset of slow states being followed and \( \mathbf{u} = \delta_e, \delta_a, \delta_r \) represent the control variables, then

\[
\begin{align*}
\dot{\mathbf{x}}_s &= \mathbf{S}(\mathbf{x}_s, \theta, \phi) + \mathbf{F}(\mathbf{x}_s, \theta, \phi)\mathbf{z} + \mathbf{E}(\mathbf{x}_s)\mathbf{u} \\
\dot{\mathbf{z}} &= \mathbf{Q}(\mathbf{z}) + \mathbf{J}(\mathbf{x}_s) + \mathbf{G}(\mathbf{x}_s)\mathbf{z} + \mathbf{U}(\mathbf{x}_s)\mathbf{u}
\end{align*}
\]  

(76)  

(77)  

The above set of equations introduces set of nonlinear functions that are described briefly below. The exact form of these functions is derived in the Appendix.

- In the translational equations of motion, all the functions such as gravitational forces and aerodynamic forces due to angle-of-attack and sideslip are collectively represented as \( \mathbf{S}(\mathbf{x}_s, \theta, \phi) \).
- Terms in the translational equations of motion due cross products between angular rates and the slow states are labeled \( \mathbf{F}(\mathbf{x}_s, \theta, \phi)\mathbf{z} \).
- Thus, the remaining terms in the slow state equations are the control effectiveness terms that are termed as \( \mathbf{E}(\mathbf{x}_s) \).
- \( \mathbf{Q}(\mathbf{z}) \) represent the nonlinearity in the fast dynamics equations. Specifically these are the terms due to cross product between angular rates.
- \( \mathbf{J}(\mathbf{x}_s) \) are aerodynamic moment terms that solely depend on slow state. Similarly, \( \mathbf{G}(\mathbf{x}_s) \) is the matrix of aerodynamic moment terms that depend linearly on angular rates.
- The term \( \mathbf{U}(\mathbf{x}_s) \) are the control effectiveness terms in the angular rate dynamics.

With the above representation of equations, the control design is straightforward. The approximate manifold is computed as,

\[
\Phi_0(\mathbf{x}_s, \mathbf{u}_s) = \mathbf{G}^{-1}(\mathbf{x}_s)\mathbf{J}(\mathbf{x}_s) + \mathbf{U}(\mathbf{x}_s)\mathbf{u}_s.
\]  

(78)  

Then the reduced slow subsystem is obtained as,

\[
\dot{\mathbf{x}}_s = \mathbf{S}(\mathbf{x}_s, \theta, \phi) + \mathbf{F}(\mathbf{x}_s, \theta, \phi)\mathbf{G}^{-1}(\mathbf{x}_s)\mathbf{J}(\mathbf{x}_s) + (\mathbf{G}^{-1}(\mathbf{x}_s)\mathbf{U}(\mathbf{x}_s) + \mathbf{E}(\mathbf{x}_s))\mathbf{u}_s
\]  

(79)  

Therefore, the slow-control is designed to be,

\[
\mathbf{u}_s = (\mathbf{G}^{-1}(\mathbf{x}_s)\mathbf{U}(\mathbf{x}_s) + \mathbf{E}(\mathbf{x}_s))^{-1} [\dot{\mathbf{x}}_s - \mathbf{K}_e(\mathbf{x}_s - \mathbf{x}_r) - \mathbf{S}(\mathbf{x}_s, \theta, \phi) - \mathbf{F}(\mathbf{x}_s, \theta, \phi)\mathbf{G}^{-1}(\mathbf{x}_s)\mathbf{J}(\mathbf{x}_s)]
\]  

(80)  

In the above expression, the existence of the inverse needs to be discussed. The aerodynamic moments of the aircraft are known to be function of the angular rates and this fact guarantees non-singularity of matrix \( \mathbf{G}(\mathbf{x}_s) \). The control effectiveness terms \( \mathbf{U}(\mathbf{x}_s) \) and \( \mathbf{E}(\mathbf{x}_s) \) represent the aerodynamic force and moment coefficients due to the control surfaces, which again are known to be non-zero.

Further, with the above design of the slow control, the reduced fast subsystem becomes,

\[
\mathbf{z}' = \mathbf{Q}(\mathbf{z}) + \mathbf{J}(\mathbf{x}_s) + \mathbf{G}(\mathbf{x}_s)\mathbf{z} + \mathbf{U}(\mathbf{x}_s)\mathbf{u}_s + \mathbf{U}(\mathbf{x}_s)\mathbf{u}_f
\]  

(81)  

Then, the fast control is designed to be,

\[
\mathbf{u}_f = \mathbf{U}^{-1}(\mathbf{x}_s) [-\mathbf{K}_e(\mathbf{z} - \Phi_0) - \mathbf{Q}(\mathbf{z}) - \mathbf{J}(\mathbf{x}_s) + \mathbf{G}(\mathbf{x}_s)\mathbf{z} - \mathbf{U}(\mathbf{x}_s)\mathbf{u}_s]
\]  

(82)  

2. Results and Discussion

Case 2 The maneuver specified is a smooth 45 degree turn, with a step input profile for the angle-of-attack maintaining zero sideslip. The feedback gain matrices are chosen to be,

\[
\mathbf{K}_e = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}
\]  

(83)
Figure 17. Case 2. Mach number, angle-of-attack and sideslip for F/A-18

Figure 18. Case 2. Kinematic angles for F/A-18
Figure 19. Case 2. Angular rates for F/A-18

Figure 20. Case 2. Control for F/A-18
and

\[
K_\zeta = \begin{bmatrix}
5 & 0 & 0 \\
0 & 5 & 0 \\
0 & 0 & 5
\end{bmatrix}.
\] (84)

The initial conditions set are \(\alpha(0) = 2\text{deg}, M(0) = 0.3\text{ with all other states equal to zero. Figures 17-20 present simulation results. All the slow followed states follow reference closely. The angle-of-attack error is within \pm0.016\text{deg}, while all other errors are identically zero. Notice, the change in bank angle as the airplane starts to turn. The roll-rates are smooth and within \pm20\text{deg/sec. Also, the roll rates are seen to closely follow the desired approximate manifolds. With increase in angle-of-attack, it is expected for velocity to fall. Moreover, for pitch-up, the elevator is pulled-up, following the convention positive elevator for downward deflection. With the pitch-up maneuver, the mach is seen to drop as expected. Aileron and rudder inputs are applied to negate non-zero roll and yaw rates. The controls are well-within bounds. Even though \(\theta\) and \(\phi\) were left uncontrolled, it is seen that their magnitudes remain constant. Simulating the behaviour of the aircraft for 250seconds, it is seen that the pitch-rate and accordingly pitch-attitude angle settle down to zero.

VI. Conclusions

In this work control formulation for tracking the slow states of a general class of nonlinear singularly perturbed systems based on the study of its geometric constructs has been developed. For a given set of nonlinear algebraic equations, an approximate analytical form of the reduced system manifold was computed. Developments of control laws for each of the subsystems and boundedness of closed-loop signals was demonstrated through composite Lyapunov function approach. Controller performance was demonstrated through simulation for two challenging nonlinear examples.

The simulation results for the open-loop nonlinear planar model confirm that the approximation of the invariant manifold closely follows the exact manifold solution. It was demonstrated that the tracking error was maintained between \([0.2]\) at all times. It is seen that dynamic-inversion control characteristics were preserved as long as the perturbation parameter was within computed bounds. In the absence of nonlinear terms in the fast dynamics, perfect tracking was demonstrated. Additionally, while following a sinusoidal wave the tracking error is maintained between \(\pm0.0106\). Simulations of the F/A-18A Hornet demonstrate that the controller is capable of handling highly nonlinear systems. Perfect tracking for heading and sideslip was achieved with an error of \(\pm0.016\text{deg}\) in angle-of-attack. The angular rates are within bounds and seen to follow the desired manifold approximation well. Even though mach number and Euler-angles \(\phi\) and \(\theta\) were left uncontrolled their magnitudes is seen to remain bounded and vary as expected.

Appendix

The mathematical model of the aircraft, used in the present study is represented by the following dynamic and kinematic equations

\[
M = \frac{1}{mv_s} \left[ T_m \eta \cos \alpha \cos \beta - \frac{1}{2} C_D(\alpha, q, \delta e) \rho v_s^2 M^2 S - mg \sin \gamma \right] \] (85)

\[
\dot{\alpha} = q - \frac{1}{\cos \beta} \left\{ (pcos \alpha + rsin \alpha) \sin \beta \right\} \] (86)

\[
\dot{\beta} = \frac{1}{mv_s M} \left[ -T_m \eta \cos \alpha \sin \beta + \frac{1}{2} C_L(\alpha, q, \delta e) \rho v_s^2 M^2 S + mg \cos \mu \cos \gamma \right] \] (87)

\[
\dot{p} = \frac{I_y - I_z}{I_z} qr + \frac{1}{2I_z} \rho v_s^2 M^2 SbC_l(\beta, p, r, \delta e, \delta a, \delta r) \] (88)

\[
\dot{q} = \frac{I_z - I_x}{I_y} pr + \frac{1}{2I_y} \rho v_s^2 M^2 ScC_m(\alpha, q, \delta e) \] (89)
\[ \dot{r} = \frac{I_x - I_y}{I_z} pq + \frac{1}{2I_z} \rho v_s^2 M^2 SbC_n(\beta, p, r, \delta c, \delta a, \delta r) \] (90)

\[ \dot{\phi} = p + q \sin \phi \tan \theta + r \cos \phi \tan \theta \] (91)

\[ \dot{\theta} = q \cos \phi - r \sin \phi \] (92)

\[ \dot{\psi} = (q \sin \phi + r \cos \phi) \sec \theta \] (93)

Wind axes orientation angles, \( \mu \) and \( \gamma \), are defined as follows:

\[ \sin \gamma = \cos \alpha \cos \beta \sin \theta - \sin \beta \sin \phi \cos \theta - \sin \alpha \cos \beta \cos \phi \cos \theta \] (94)

\[ \sin \mu \cos \gamma = \sin \theta \cos \alpha \sin \beta + \sin \phi \cos \theta \cos \beta - \sin \alpha \sin \beta \cos \phi \cos \theta \] (95)

\[ \cos \mu \cos \gamma = \sin \theta \sin \alpha + \cos \alpha \cos \phi \cos \theta \] (96)

In order to write the equations in the form Eqs.76-77,

\[ S(x_s, \theta, \phi) = \begin{bmatrix} -\frac{1}{mv_s \beta \cos \beta} \left[ \frac{1}{2} C_L(\alpha) \rho v_s^2 M^2 S - mg \cos \mu \cos \gamma \right] \\ \frac{1}{mv_s \beta M} \left[ \frac{1}{2} C_Y(\beta) \rho v_s^2 M^2 S + mg \sin \mu \cos \gamma \right] \\ 0 \end{bmatrix} \] (97)

\[ F(x_s, \theta, \phi) = \begin{bmatrix} -\cos \tan \beta & 1 & -\sin \tan \beta \\ \sin \alpha & 0 & -\cos \alpha \\ 0 & \sec \theta \sin \phi & \cos \phi \sec \theta \end{bmatrix} \] (98)

\[ F(x_s) = \begin{bmatrix} -\frac{1}{2 \rho C_{L,1}} \rho v_s^2 M^2 SbC_{1,1} \\ 0 \\ 0 \end{bmatrix} \] (99)

\[ Q(z) = \begin{bmatrix} l_x - l_z \rho v_s^2 M^2 SbC_{1,1} \\ 0 \\ l_x - l_z \rho v_s^2 M^2 SbC_{1,1} \\ 0 \\ l_x - l_z \rho v_s^2 M^2 SbC_{1,1} \end{bmatrix} \] (100)

\[ J(x_s) = \begin{bmatrix} \frac{1}{2 \rho} \rho v_s^2 M^2 SbC_{1,1} \\ \frac{1}{2 \rho} \rho v_s^2 M^2 SbC_{1,1} \end{bmatrix} \] (101)

\[ G(x_s) = \begin{bmatrix} \frac{1}{2 \rho} \rho v_s^2 M^2 SbC_{1,1} \\ \frac{1}{2 \rho} \rho v_s^2 M^2 SbC_{1,1} \end{bmatrix} \] (102)

\[ U(x_s) = \begin{bmatrix} \frac{1}{2 \rho} \rho v_s^2 M^2 SbC_{1,1} \\ \frac{1}{2 \rho} \rho v_s^2 M^2 SbC_{1,1} \end{bmatrix} \] (103)

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