Leader-following Connectivity Preservation Rendezvous of Multi-agent Systems Based Only Position Measurements

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Abstract—Recently, we investigated the problem of leader-following rendezvous with connectivity preservation for a double integrator multi-agent system subject to external disturbances by a full information feedback control. In this paper, we further study the same problem via the position feedback control only. Additionally, the formulation of the problem in this paper is more general than the previous one in that the disturbances to different followers can be different and these disturbance signals will not be used by the control law. Thus the result of this paper is more practical than that of the previous paper.

I. INTRODUCTION

Consider a collection of double integrator systems of the following form

\[ \dot{q}_i = u_i + d_i, \quad i = 1, \ldots, N. \] (1)

where \( q_i \in \mathbb{R}^n, u_i \in \mathbb{R}^n, d_i \in \mathbb{R}^n \) are the position, input, and the external disturbance of the subsystem \( i \) of (1). It is assumed that, for \( i = 1, \ldots, N \), \( d_i \) is generated by an exosystem as follows

\[ \dot{w}_i = S_i w_i, \quad \dot{d}_i = D_i w_i \] (2)

where \( w_i \in \mathbb{R}^{s_i}, S_i \in \mathbb{R}^{s_i \times s_i}, \) and \( D_i \in \mathbb{R}^{n \times s_i} \) are constant matrices. Without loss of generality, we assume the pair \( (D_i, S_i) \) is detectable.

Also, let \( q_0 \in \mathbb{R}^n \) be a reference trajectory generated by a system as follows

\[ \dot{q}_0 = S_{01} q_0 + S_{02} \dot{q}_0 \] (3)

where \( S_{01}, S_{02} \in \mathbb{R}^{n \times n} \) are arbitrary constant matrices.

It is noted that, when \( D_i = 0_{n \times s_i} \), (1) is a double integrator system, and when \( S_{01} = S_{02} = 0_{n \times n} \), (3) is also a double integrator system. However, in this paper, we don’t require (3) to be a double integrator system and it can be seen that system (3) contains double integrator system and harmonic system as special cases.

Like [2], we view the system composed of (1) and (3) as a multi-agent system of \((N + 1)\) agents with (3) as the leader and the \( N \) subsystems of (1) as \( N \) followers. With respect to the system composed of (1) and (3), we can define a digraph

\[ \mathcal{G}(t) = (\mathcal{V}, \mathcal{E}(t)) \] where \( \mathcal{V} = \{0, 1, \ldots, N\} \) with 0 associated with the leader system and \( i = 1, \ldots, N, \) associated with the \( i \)th subsystem of (1), and \( \mathcal{E}(t) \subseteq \mathcal{V} \times \mathcal{V} \). The set \( \mathcal{V} \) is called the node set of \( \mathcal{G}(t) \) and the set \( \mathcal{E}(t) \) is called the edge set of \( \mathcal{G}(t) \).

The rendezvous problem with connectivity preservation of the double integrator multi-agent system was studied recently in [1], [2], [9] and [10]. In particular, the problem was studied in [1], [10] via full state feedback control assuming the leader system was also a double integrator and the follower was not subject to external disturbances. The problem was further studied in [9] via position feedback control only. Recently, the problem in [10] is generalized in [2] to the case where the leader system can be a linear autonomous system described in (3) and all follower subsystems are allowed to be subject to a disturbance generated by (3). In this paper, we will further generalize the result of [2] in two aspects. First, we allow the disturbances to various followers to be different. In particular, they can be different from the leader signal. Second, we will solve the problem by a position feedback control law as described in (4). This control law depends neither on the velocity of the system nor on the external disturbances, is thus more practical and economic than the one in [2]. It is noted that since the closed-loop system is nonlinear, the validity of the output feedback control law cannot be directly established by the result of the state feedback control and linear observer theory. We have to derive our result using a rigorous Lyapunov-like analysis.

The rest of this paper is organized as follows. In Section II, we will formulate our problem precisely. In Section III, we will present our main result, which will be illustrated by an example in Section IV. Finally, we close this paper in Section V with some concluding remarks.

The following notation will be used throughout this paper: given the column vectors \( a_i, i = 1, \ldots, s \), we denote \( \text{col}(a_1, \ldots, a_s) = [a_1^T, \ldots, a_s^T]^T \).

II. PROBLEM FORMULATION

Let us first characterize the edge set \( \mathcal{E}(t) \) introduced in [2] as follows.

Given any \( r > 0 \) and \( \epsilon \in (0, r) \), for any \( t \geq 0 \), \( \mathcal{E}(t) = \{(i, j) \mid i, j \in \mathcal{V}\} \) is defined such that

1) \( \mathcal{E}(0) = \{(i, j) \mid \|q_i(0) - q_j(0)\| < (r - \epsilon), i, j = 1, \ldots, N\} \cup \{(0, j) \mid \|q_0(0) - q_j(0)\| < (r - \epsilon), j = 1, \ldots, N\} \)

\[ 1 \text{See Appendix for a summary of digraph.} \]
1, \ldots, N};
2) if \( \| q_i(t) - q_j(t) \| \geq r \), then \((i, j) \notin \mathcal{E}(t)\);
3) \((i, 0) \notin \mathcal{E}(t)\), for \( i = 0, 1, \ldots, N \);
4) for \( i = 0, 1, \ldots, N, j = 1, \ldots, N \), if \((i, j) \notin \mathcal{E}(t^-)\)
and \( \| q_i(t) - q_j(t) \| < (r - \epsilon) \), then \((i, j) \in \mathcal{E}(t)\).
5) for \( i = 0, 1, \ldots, N, j = 1, \ldots, N \), if \((i, j) \in \mathcal{E}(t^-)\)
and \( \| q_i(t) - q_j(t) \| < r \), then \((i, j) \in \mathcal{E}(t)\).

As pointed out in [2], the definition of edge is somewhat different from that in literature mainly in that the node 0
associated with the leader as well as the edges adjacent to the node 0 is part of the graph. Since the leader does not have
a control, there is no edge from a follower to the leader. If \( \epsilon = 0 \), then the above definition is similar to that given in
[6], [7]. Thus the physical interpretation of \( r \) is the sensing radius of the distance sensor of each follower. The number \( \epsilon \)
is introduced to the effect of hysteresis.

To describe our control law, we use the notation \( \mathcal{N}_i(t) \) to denote the neighbor set of the node \( i \) for \( i = 0, 1, \ldots, N \).
Define a subgraph \( \mathcal{G}(t) = (\mathcal{V}, \mathcal{E}(t)) \) of \( \mathcal{G}(t) \), where \( \mathcal{V} = \{0, 1, \ldots, N\} \) \( \mathcal{E}(t) \subseteq \mathcal{V} \times \mathcal{V} \) is obtained from \( \mathcal{E}(t) \) by removing all edges between the node 0 and the nodes in \( \mathcal{V} \). Clearly, \( \mathcal{G}(t) \) is an undirected graph. For \( i = 1, \ldots, N \), let \( \mathcal{N}_i(t) = \mathcal{N}_i(t) \cap \mathcal{V} \). It can be seen that, for \( i = 1, \ldots, N \), \( \mathcal{N}_i(t) \) is the neighbor set of the node \( i \) with respect to \( \mathcal{V} \).

Our control law takes the following form:

\[
\begin{align*}
  u_i &= h_i(q_i - q_j, \zeta_i, \zeta_j, j \in \mathcal{N}_i(t)), \quad i = 1, \ldots, N \\
  \dot{\zeta}_i &= \dot{q}_i(q_i, \zeta_i, \zeta_i, q_i, j \in \mathcal{N}_i(t)) \quad (4)
\end{align*}
\]

where \( h_i, g_i \) are sufficiently smooth functions to be specified later, and \( \zeta_i \in \mathbb{R}^{2n+3} \)
is to estimate \( \text{col}(q_i - \hat{q}_i, q_i - \hat{q}_i) \).

In contrast with the control law in [2], the control law (4) only depends on the position information of the neighboring subsystems. Thus it is called the distributed position feedback control law.

The leader-following rendezvous problem with connectivity preservation is described as follows.

**Definition 2.1:** Given the multi-agent system composed of (1), (2) and (3), \( r > 0 \) and \( \epsilon \in (0, r) \), and arbitrary positive real numbers \( P_i, \kappa_i, i = 1, \ldots, N \), find a distributed control law of the form (4) such that, for all initial conditions \( q_0(0), \hat{q}_0(0), w_i(0), q_i(0), \hat{q}_i(0), \zeta_i(0), i = 1, \ldots, N \), that make \( \mathcal{G}(0) \) connected, and satisfy \( \| q_i(0) - \hat{q}_0(0) \| \leq P_i \), and \( \| \zeta_i(0) - \text{col}(q_i(0), \hat{q}_i(0), w_i(0), q_i(0), \hat{q}_i(0)) \| \leq \kappa_i \), the closed-loop system has the following properties:

1) \( \mathcal{G}(t) \) is connected for all \( t \geq 0 \);
2) \( \lim_{t \to -\infty} (q_i - q_0) = 0 \) and \( \lim_{t \to -\infty} (\dot{q}_i - \dot{q}_0) = 0 \), \( i = 1, \ldots, N \).

**Remark 2.1:** If, for \( i = 1, \ldots, N \), \( D_i = D \) for some matrix \( D \), \( w_i = \text{col}(q_i, \hat{q}_i) \) and \( y_i = \text{col}(q_i, \hat{q}_i) \), then the above problem is reduced to the problems studied in [2]. What makes our current problem challenging and thus interesting is that our control law will be independent of not only \( \dot{\hat{q}}_i \) but also \( w_i \).

## III. Main Result

We will make use of some techniques in output regulation problem to deal with our problem. For this purpose, we can convert our system in state space form as follows

\[
\begin{align*}
  \dot{x}_i &= Ax_i + Bu_i + E_i w_i \\
  y_i &= C_m x_i \\
  e_i &= x_i - x_0, \quad i = 1, \ldots, N
\end{align*}
\]

where, for \( i = 1, \ldots, N \), \( x_i = \begin{bmatrix} q_i \\ p_i \end{bmatrix} \) with \( p_i = \hat{q}_i, y_i \in \mathbb{R}^n \)
\( e_i \) are the state, measurement output, and regulated output of agent \( i \), respectively. Also, let \( x_0 = \begin{bmatrix} \hat{q}_0 \\ p_0 \end{bmatrix} \in \mathbb{R}^{2n} \)
with \( p_0 = \hat{q}_0 \). Then,

\[
\dot{x}_0 = S_0 x_0
\]

where \( S_0 = \begin{bmatrix} 0_{n \times n} & I_n \\ S_{01} & S_{02} \end{bmatrix} \). Various matrices in (5) are as follows:

\( A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \otimes I_n, B = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \otimes I_n, E_i = \begin{bmatrix} 0_{n \times s_i} \\ D_i \end{bmatrix}, C_m = \begin{bmatrix} 1 & 0 \end{bmatrix} \otimes I_n \).

**Remark 3.1:** Let \( \hat{A}_i = \begin{bmatrix} A & E_i \\ S_{01} & S_{i1} \end{bmatrix} \) and \( \hat{C}_{mi} = \begin{bmatrix} C_m & 0_{n \times s_i} \end{bmatrix} \). Then, noting that the pair \((D_i, S_i)\) can always be assumed to be detectable, it can be verified that the pair \((\hat{C}_{mi}, \hat{A}_i)\) is also detectable. Thus, there exists \( L_i = \begin{bmatrix} L_{i1} \\ L_{i2} \end{bmatrix} \) with \( L_{i1} \in \mathbb{R}^{n \times n} \) and \( L_{i2} \in \mathbb{R}^{s \times n} \) such that \( \hat{A}_i + L_i \hat{C}_{mi} \) is Hurwitz.

One of the main objectives of this paper is to deal with the external disturbance \( E_i w_i \). For this purpose, let

\[
X_i = \begin{bmatrix} I_{2n} & 0_{2n \times s_i} \end{bmatrix}, \quad U_i = \begin{bmatrix} S_{01} & S_{02} \end{bmatrix} - D_i
\]

Then it can be verified that performing on (5) the following coordinate transformation

\[
\begin{bmatrix} \tilde{x}_i \\ \tilde{u}_i \\ \tilde{v}_i \end{bmatrix} = \begin{bmatrix} \tilde{q}_i \\ \tilde{p}_i \\ \tilde{v}_i \end{bmatrix}, \quad i = 1, \ldots, N
\]

with \( \tilde{v}_i = \begin{bmatrix} x_i \\ w_i \end{bmatrix}, \quad i = 1, \ldots, N \), converts system (5) to the following double-integrator system without disturbance

\[
\begin{bmatrix} \dot{\tilde{q}}_i \\ \dot{\tilde{p}}_i \end{bmatrix} = \begin{bmatrix} \tilde{p}_i \\ \hat{q}_i \end{bmatrix}, \quad i = 1, \ldots, N
\]

**Remark 3.2:** The transformation (8) is inspired by the output regulation theory [3]. In fact, associated with (5) are the following linear matrix equations

\[
X_i S_i = AX_i + BU_i + E_i \\
0 = X_i + \begin{bmatrix} -I_{2n} & 0_{2n \times s_i} \end{bmatrix}
\]

with \( S_i = \begin{bmatrix} S_0 & 0_{2n \times s_i} \\ S_{i1} \end{bmatrix}, E_i = \begin{bmatrix} 0_{2n \times 2n} & E_i \end{bmatrix} \).

(10) is called regulator equations associated with the ith follower [5]. It can be verified that (7) is a solution pair
of (10). The transformation (8) is a standard technique for converting an output regulation problem to a stabilization problem.

As in [2], our control law will utilize the bounded potential function \( \psi(.) \) introduced in [10] as follows.

\[
\psi(s) = \frac{s^2}{r - s + \varepsilon^2}, \quad 0 \leq s \leq r
\]  

(11)

where \( Q \) is some positive number. The function is nonnegative and bounded over \([0, r]\), and its derivative \( \frac{d\psi(s)}{ds} = -\frac{2s - 2r + \varepsilon^2}{(r - s)^{3/2}} \) is positive for all \( s \in (0, r) \). Moreover, the function has the property that, for any \( \alpha > 0, \beta \geq 0, \) and any \( \varepsilon \in (0, r) \), there exists some \( Q > 0 \) such that

\[
\psi(r) \geq \alpha \psi(r - \varepsilon) + \beta
\]  

(12)

In fact, as pointed out in [2], (12) holds whenever \( \varepsilon > \frac{(\alpha - \varepsilon)^2}{\beta} \).

Now we can propose the dynamic distributed position feedback control law as follows:

\[
u_i = -\sum_{j \in N_i(t)} \nabla_{\hat{q}_i} \psi(\|\hat{q}_i - \hat{q}_j\|) - \sum_{j \in N_i(t)} a_{ij}(t)(\xi_{2i} - \xi_{2j})
- a_{i0}(\xi_{2i} - p_0) + U_i \text{col}(\tilde{\eta}_i, \tilde{w}_i)
\]

\[
\dot{\xi}_i = A_i \xi_i + B_i u_i + E_i \tilde{w}_i + L_{i1}(C_m \xi_i - y_i)
\]

\[
\dot{\tilde{w}}_i = S_i \tilde{w}_i + L_{i2}(C_m \xi_i - y_i)
\]

\[
\tilde{\eta}_i = S_0 \tilde{\eta}_i + \gamma(\sum_{j=1}^{N} a_{ij}(t)(\tilde{\eta}_j - \tilde{\eta}_i) - a_{i0}(t)(\tilde{\eta}_i - x_0))
\]  

(13)

where, for \( i = 1, \ldots, N, \) \( j = 0, \ldots, N, \)

\[
a_{ij}(t) = \begin{cases} 1, & (j, i) \in \mathcal{E}(t) \\ 0, & \text{otherwise} \end{cases}
\]  

(14)

\[
\xi_1 = \left[ \begin{array}{c} \xi_{1i} \\ \xi_{2i} \end{array} \right] \quad \text{with} \quad \xi_{1i} \in \mathbb{R}^n \quad \text{and} \quad \xi_{2i} \in \mathbb{R}^n, \quad \gamma \quad \text{is a sufficiently large positive number, and} \quad L_i \quad \text{is as described in Remark 3.1.}
\]

Since \( A_i + L_i C_m \) is Hurwitz, there exist positive definite matrices \( P_i, i = 1, \ldots, N, \) such that \( (A_i + L_i C_m)^T P_i + P_i (A_i + L_i C_m) = -I_{2n_i} + \varepsilon I \). It can be seen that the control law is in the form of (4) with \( \xi_i = (\xi_i, \tilde{w}_i, \tilde{\eta}_i) \).

Let \( \xi_i = \xi_i - x_i, \tilde{w}_i = \tilde{w}_i - w_i \) and \( \tilde{\eta}_i = \eta_i - x_0, \quad i = 1, \ldots, N. \) Then, under the control law (13), the closed-loop system of each agent becomes

\[
\hat{\dot{\xi}}_i = \hat{p}_i, \quad i = 1, \ldots, N
\]

\[
\hat{\dot{\tilde{w}}}_i = -\sum_{j \in N_i(t)} \nabla_{\hat{q}_i} \psi(\|\hat{q}_i - \hat{q}_j\|) - \sum_{j \in N_i(t)} a_{ij}(t)(\hat{p}_i - \hat{p}_j)
- \sum_{j \in N_i(t)} a_{ij}(t)(\xi_{2i} - \xi_{2j}) + a_{i0}(\xi_{2i} + D_i \tilde{w}_i)
\]

\[
+ \left[ \begin{array}{c} S_{01} \\ S_{02} \end{array} \right] \tilde{\eta}_i
\]

\[
\hat{\xi}_i = S_0 \tilde{\eta}_i + \gamma(\sum_{j=1}^{N} a_{ij}(t)(\tilde{\eta}_j - \tilde{\eta}_i) - a_{i0}(t)(\tilde{\eta}_i))
\]  

(15)

Before establishing our main result, we note that, associated with the graph \( \mathcal{G}(t), t \geq 0 \), we can define matrices

\[
H(t) = \begin{bmatrix} a_1(t) & -a_{12}(t) & \cdots & -a_{1N}(t) \\ -a_{21}(t) & a_2(t) & \cdots & -a_{2N}(t) \\ \vdots & \vdots & \ddots & \vdots \\ -a_{N1}(t) & -a_{N2}(t) & \cdots & a_N(t) \end{bmatrix}
\]  

(16)

where \( a_i(t) = \sum_{j=0, j \neq i}^{N} a_{ij}(t), i = 1, \ldots, N, \)

\[
\begin{bmatrix} P_0(t) \end{bmatrix} = \begin{bmatrix} H(t) \otimes I_n \quad \Lambda(t) \end{bmatrix} \begin{bmatrix} \frac{\Lambda(t)}{2} \\ \theta I \end{bmatrix}
\]  

(17)

where \( \theta \) is some real number, \( \lambda = 2Nn + s_1 + \cdots + s_N, \Lambda(t) = \begin{bmatrix} 0_{N n \times N n} \quad H(t) \otimes I_n \quad D \end{bmatrix} \text{ with } D = \text{diag}(D_1, \ldots, D_N), \) and

\[
P(t) = \begin{bmatrix} \frac{H(t) \otimes I_n \quad \Lambda(t)}{2} \quad \theta I \quad 0_{2N \times N} \quad Y(t) \end{bmatrix}
\]  

(18)

where

\[
Y(t) = \gamma H(t) \otimes I_{2n} - I_N \otimes \frac{S_0 + S_0^T}{2}
\]

with \( \gamma \) some real number. We have the following lemma.

**Lemma 3.1:** Assume the graph \( \mathcal{G}(t) \) is connected for all \( t \geq 0 \). Then

1. there exists positive number \( \theta \) such that \( P_0(t) \) is positive definite for all \( t \geq 0 \);
2. there exists positive number \( \gamma \) such that \( P(t) \) is positive definite for all \( t \geq 0 \).

**Proof:** Part 1. Note that, for any \( t \geq 0, H(t) = -M + \Delta \) where \( M \) is a Metzler matrix and \( \Delta = \text{diag}[a_{10}(t), \ldots, a_{N0}(t)] \). By Remark 1.1, \( H(t) \) is positive definite since the graph \( \mathcal{G}(t) \) is connected. By Lemma 3.1 in [2], if there exists finite number \( \theta > 0 \) such that, for all \( t \geq 0, \)

\[
\theta > \lambda_M(\frac{H(t) \otimes I_n \Lambda(t)}{2})
\]

where \( \lambda_M(A) \) denotes the largest eigenvalue of a square matrix \( A \), then \( P_0(t) \) is positive definite for all \( t \geq 0 \).

It is noted that

\[
\begin{bmatrix} \Lambda(t) \end{bmatrix} \begin{bmatrix} H(t) \otimes I_n \Lambda(t) \end{bmatrix} = \begin{bmatrix} 0_{N n \times N n} \quad H(t) \otimes I_n \quad D \end{bmatrix} \times
\begin{bmatrix} 0_{N n \times N n} \quad H(t) \otimes I_n \quad D \end{bmatrix}
= \begin{bmatrix} \Gamma_1 \quad \Gamma_2 \quad \Gamma_3 \end{bmatrix}
\]  

(19)

with \( s = s_1 + \cdots + s_N. \) Thus, \( P_0(t) \) is positive definite for all \( t \geq 0 \) if, for all \( t \geq 0, \)

\[
\theta > \lambda_M\left(\frac{1}{4} \begin{bmatrix} \frac{H(t) \otimes I_n \quad D \quad D^T(H(t) \otimes I_n)D} \end{bmatrix} \right)
\]  

(20)
Since $H(t)$ is uniquely determined by $\tilde{G}(t)$, and there are only finitely many different connected graphs with $N + 1$ nodes, such a finite number $\theta$ always exists.

Part 2). Let

$$Z = \left[ \begin{array}{c} -\frac{1}{2}I_N \otimes \left[ S_{\theta_1} \quad S_{\theta_2} \right] \\ 0_{n \times 2Nn} \end{array} \right]$$

$$P(t) = \left[ \begin{array}{cc} P_0(t) & Z \\ Z^T & Y(t) \end{array} \right]$$

Then by Lemma 3.1 in [2], if there exists finite real number $\gamma$ such that, for all $t \geq 0$,

$$\gamma > \frac{\lambda_M(I_N \otimes S_{\theta_0} + Z^T P_0^{-1}(t) Z)}{\lambda_m(H(t))}$$

(21)

with $\lambda_m(A)$ denotes the smallest eigenvalue of a square matrix $A$, then $P(t)$ is positive definite for all $t \geq 0$.

Since $H(t)$ is uniquely determined by $\tilde{G}(t)$, and there are only finitely many different connected graphs with $N + 1$ nodes, such a finite constant always exists.

We can now state our main result as follows.

**Theorem 3.1**: The leader-following connectivity preservation rendezvous problem for system composed of (1), (2) and (3) is solvable by the control law (13) where $\gamma$ is a sufficiently large positive constant.

**Proof**: Let

$$\bar{\eta} = \text{col}(\bar{\eta}_1, \bar{\eta}_2, \cdots, \bar{\eta}_N), \quad \bar{q} = \text{col}(\bar{q}_1, \bar{q}_2, \cdots, \bar{q}_N)$$

$$\bar{p} = \text{col}(\bar{p}_1, \bar{p}_2, \cdots, \bar{p}_N), \quad \bar{x} = \text{col}(\bar{x}_1, \bar{x}_2, \cdots, \bar{x}_N)$$

$$\mu_i = \left[ \begin{array}{c} \xi_i \\ \bar{w}_i \end{array} \right], \quad i = 1, \cdots, N$$

$$\mu = \text{col}(\mu_1, \mu_2, \cdots, \mu_N)$$

Let $\bar{\mu} = \text{col}(\bar{\xi}_1, \cdots, \bar{\xi}_N, \bar{\xi}_2, \cdots, \bar{\xi}_N, \bar{\bar{w}}_1, \cdots, \bar{\bar{w}}_2) = T_\mu$

with $T = $

$$\left[ \begin{array}{cccccccccccc} I_n & 0_{n \times n} & 0_{n \times n} & \cdots & 0_{n \times n} & 0_{n \times n} & 0_{n \times n} & 0_{n \times n} \\
0_{n \times n} & I_n & 0_{n \times n} & \cdots & 0_{n \times n} & 0_{n \times n} & 0_{n \times n} & 0_{n \times n} \\
0_{n \times n} & 0_{n \times n} & I_n & \cdots & 0_{n \times n} & 0_{n \times n} & 0_{n \times n} & 0_{n \times n} \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\
0_{n \times n} & 0_{n \times n} & 0_{n \times n} & \cdots & I_n & 0_{n \times n} & 0_{n \times n} & 0_{n \times n} \\
0_{n \times n} & 0_{n \times n} & 0_{n \times n} & \cdots & 0_{n \times n} & I_n & 0_{n \times n} & 0_{n \times n} \\
0_{n \times n} & 0_{n \times n} & 0_{n \times n} & \cdots & 0_{n \times n} & 0_{n \times n} & I_n & 0_{n \times n} \\
0_{n \times n} & 0_{n \times n} & 0_{n \times n} & \cdots & 0_{n \times n} & 0_{n \times n} & 0_{n \times n} & I_n \end{array} \right]$$

It is noted that $T^{-T} T^{-1} = I_n$.

Given $r > 0$, $\epsilon \in (0, r)$, and arbitrary positive real numbers $P_i, \kappa_i, i = 1, \cdots, N$, the control law is determined by two design parameters $Q$ and $\gamma$. Let us first determine $\gamma$.

By Lemma 3.1, there are $\theta > 0$ and $\gamma > 0$ such that $P(t)$ is positive definite for all possible connected $\tilde{G}(t)$ with $N + 1$ nodes. Fix $\theta$ and $\gamma$.

To determine $Q$, we introduce the following energy function for system (15).

$$V(q, p, u, \eta, t) = \frac{1}{2} \sum_{i=1}^{N} \left( \sum_{j \in \mathcal{N}(i)} \psi(||\eta_i - \eta_j||) + 2a_{i0} \psi(||\eta_i||) \right)$$

$$+ \bar{p}_i^T \bar{p}_i + 2\theta \mu_i^T P_i \mu_i + \bar{\eta}_i^T \bar{\eta}_i \right)$$

$$= \frac{1}{2} \sum_{i=1}^{N} \left( \psi(||\xi_i - \eta_j||) + 2a_{i0} \psi(||\xi_i||) \right)$$

$$+ \bar{p}_i^T \bar{p}_i + \bar{\eta}_i^T \bar{\eta}_i + \theta \mu_i^T T^{-T} P_i T^{-1} \mu_i$$

(22)

with $P = \text{diag}(P_1, \cdots, P_N)$. Let

$$Q_{max} = \frac{\alpha(r - \epsilon)^2}{\epsilon} + \beta$$

(23)

where $\alpha = \frac{N(N-1)}{2} + N$ and

$$\beta = \frac{1}{2} \sum_{i=1}^{N} (P_i^2 + \delta_i \kappa_i^2)$$

(24)

where $\delta_i = \max \{1, 2\theta \lambda_{s, i}(\bar{P}_i)\}$. Then pick any $Q > Q_{max}$.

Now, we will show that the above control law is such that the graph $\tilde{G}(t)$ is connected for all $t \geq 0$. Let the energy function be given by (22). Then it can be seen that for all initial conditions $x_0(0), u_0(0), q_i(0), p_i(0), \xi_i(0), \bar{w}_i(0), \eta_i(0)$ that make $\tilde{G}(0)$ connected and satisfy $||p_i(0) - p_i(0)|| \leq P_i$, $||\text{col}(\xi_i(0), \bar{w}_i(0), \eta_i(0)) - \text{col}(\xi_i(0), \bar{w}_i(0), x_0(0))|| \leq \kappa_i$ our choice of $Q$ is such that

$$V(0) = V(q(0), p(0), u(0), \eta(0), 0) \leq Q_{max}$$

(25)

It can be verified that the time derivative of the function (22) along the closed-loop system (15) satisfies

$$\dot{V} = \sum_{i=1}^{N} \sum_{j \in \mathcal{N}(i)} \psi(||\eta_i - \eta_j||) + \sum_{i=1}^{N} \bar{p}_i^T \bar{p}_i$$

$$+ \sum_{i=1}^{N} \theta(\mu_i^T P_i \mu_i + \mu_i^T P_i \mu_i + \bar{\eta}_i^T \bar{\eta}_i)$$

(26)

By using the same argument as what is used in the proof of Theorem 3.1 in [2], we can conclude that there exists a finite integer $k > 0$ such that

$$\tilde{G}(t) = \tilde{G}(0), \quad t \in [0, t_1)$$

$$\tilde{G}(t) = \tilde{G}(t_1) \supset \tilde{G}(t_{i-1}), \quad t \in [t_i, t_{i+1}), \quad i = 1, \cdots, k - 1$$

$$\tilde{G}(t) = \tilde{G}(t_k) \supset \tilde{G}(t_{k-1}), \quad t \in [t_k, \infty)$$

Thus, for all $t \geq t_k$, along any trajectory of the closed-loop system, we have

$$\dot{V}(t) = - \left[ \begin{array}{c} \bar{p} \\ \bar{\eta} \end{array} \right] T \left[ \begin{array}{c} \bar{p} \\ \bar{\eta} \end{array} \right], \quad t \geq t_k.$$
with $P(t_k)$ positive definite. Thus
\[ V(t) \leq V(0) \leq Q_{\max} < Q, \quad \text{for } t \geq t_k \] (28)
Therefore, for all $t \geq t_k$, the graph $\mathcal{G}(t)$ is connected.

Moreover, by using the same argument as what is used in the proof of Theorem 3.1 in [2], we can conclude also that, $\bar{p}$, $\bar{q}$, $\mu$ and, $\bar{\eta}$ are bounded for all $t \geq 0$, and for $i = 1, \ldots, N$,
\[ \lim_{t \to \infty} (p_i - p_0) = 0 \]
\[ \lim_{t \to \infty} (q_i - q_0) = 0 \]

IV. EXAMPLE

Consider the following double integrator systems with disturbance with $N = 4$ and $n = 2$
\[ \ddot{q}_i = u_i + d_i, \quad i = 1, 2, 3 \] (29)
The leader system is
\[ \dot{x}_0 = S_0x_0 \] (30)
where $S_0 = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \otimes I_2$.
All the followers are subject to different external disturbances generated by
\[ \dot{\hat{w}}_i = S_1w_i, \quad i = 1, 2, 3 \]
\[ d_i = D_iw_i \] (31)
with $S_1 = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$, $S_2 = 0$, $S_3 = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$, $D_1 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, $D_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$ and $D_3 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$.
The values for $L_i$, $i = 1, 2, 3$ can be calculated as follows:
\[ L_1 = \begin{bmatrix} -11.1058 & -1.5488 \\ 0.2136 & -12.8942 \\ -30.0300 & -14.7458 \\ 7.6435 & -47.6888 \\ 7.1842 & -35.6389 \\ 33.1849 & -4.725 \end{bmatrix} \]
\[ L_3 = \begin{bmatrix} 4.6733 & -5.488 \\ -12.8942 & 1.5488 \\ -30.0300 & -14.7458 \\ 7.6435 & -47.6888 \\ 7.1842 & -35.6389 \\ 33.1849 & -4.725 \end{bmatrix} \]
Thus, $\lambda_M(\bar{P}_1) = 4.6733$, $\lambda_M(\bar{P}_2) = 2.5605$ and $\lambda_M(\bar{P}_3) = 8.3629$.
Assume the sensing range is $r = 8$ and $\epsilon = 0.5$. For $i = 1, 2, 3$, let
$\lambda = 14$, $\kappa_i = 20$. (32)
By Eq. (20), we can obtain $\theta > 2.4016$, then take $\theta = 2.5$.
Also using (12) with $\alpha = \frac{N(N-1)}{2} + N = 6$ and $\beta = 19010$
gives $(\alpha(r-\epsilon)^2 + \beta) = 19685$. Then taking $Q = 20000$ makes (12) satisfied. Thus the potential function is
\[ \psi(s) = \frac{s^2}{8 - s + \frac{64}{20000}}, \quad 0 \leq s \leq r \] (33)
By Eq. (21), we obtain $\gamma > 154.7491$, and let $\gamma = 155$ such that $P(t)$ is positive definite.
For the purpose of simulation, let the initial values of various variables be
$\begin{align*}
x_0(0) &= \begin{bmatrix} 1 & 1 & 1 & 1 \end{bmatrix}^T \\
x_1(0) &= \begin{bmatrix} 1 & 1 & 1 & 1 \end{bmatrix}^T \\
x_2(0) &= \begin{bmatrix} 1 & 1 & 1 & 1 \end{bmatrix}^T \\
x_3(0) &= \begin{bmatrix} 1 & 1 & 1 & 1 \end{bmatrix}^T \\
\end{align*}$
$\begin{align*}
w_0(0) &= \begin{bmatrix} 1 & 1 & 1 & 1 \end{bmatrix}^T \\
w_1(0) &= \begin{bmatrix} 1 & 1 & 1 & 1 \end{bmatrix}^T \\
w_2(0) &= \begin{bmatrix} 1 & 1 & 1 & 1 \end{bmatrix}^T \\
w_3(0) &= \begin{bmatrix} 1 & 1 & 1 & 1 \end{bmatrix}^T \\
\end{align*}$
$\begin{align*}
\tilde{x}_1(0) &= \begin{bmatrix} 1 & 1 & 1 & 1 \end{bmatrix}^T \\
\tilde{x}_2(0) &= \begin{bmatrix} 1 & 1 & 1 & 1 \end{bmatrix}^T \\
\tilde{x}_3(0) &= \begin{bmatrix} 1 & 1 & 1 & 1 \end{bmatrix}^T \\
\tilde{x}_4(0) &= \begin{bmatrix} 1 & 1 & 1 & 1 \end{bmatrix}^T \\
\end{align*}$
\[ \tilde{\xi}_1(0) = \begin{bmatrix} 1 & 1 & 1 & 1 \end{bmatrix}^T \]
\[ \tilde{\xi}_2(0) = \begin{bmatrix} 1 & 1 & 1 & 1 \end{bmatrix}^T \]
\[ \tilde{\xi}_3(0) = \begin{bmatrix} 1 & 1 & 1 & 1 \end{bmatrix}^T \]
\[ \tilde{\xi}_4(0) = \begin{bmatrix} 1 & 1 & 1 & 1 \end{bmatrix}^T \]
It can be verified that these initial values are such that
\[ ||p_i(0) - p_0(0)|| \leq P_i \]
\[ ||\zeta_i(0) - \text{col}(x_i(0), \eta_i(0), w_i(0))|| \leq \kappa_i \]
and $\tilde{E}(0) = \{(0,1), (1,2), (2,3), (1,3)\}$ which forms a connected graph.
With these parameters, we can simulate the performance of the control law (13), and some of the simulation results are shown in Figures 1 to 3. Figure 1 shows the distances of the edges $\{(0,1), (1,2), (2,3), (1,3)\}$ which constitute the initial edge set. It can be seen that, for all $t \geq 0$, these distances are smaller than the sensing range $r = 8$. Thus, the connectivity of the network is maintained. Figures 2 and 3 further show that both the position and the velocity of all the followers asymptotically approach the position and the velocity of the leader, respectively.

![Fig. 1. Distances between initially connected agents](image-url)


**APPENDIX**

We first introduce some graph notation which can be found in [4]. A digraph $G = (\mathcal{V}, \mathcal{E})$ consists of a finite set of nodes $\mathcal{V} = \{1, \ldots, N\}$ and an edge set $\mathcal{E} = \{(i, j) : i, j \in \mathcal{V}, i \neq j\}$. A node $i$ is called a neighbor of a node $j$ if the edge $(i, j) \in \mathcal{E}$. $\mathcal{N}_i$ denotes the subset of $\mathcal{V}$ that consists of all the neighbors of the node $i$. If the graph $G$ contains a sequence of edges of the form $(i_1, i_2), (i_2, i_3), \ldots, (i_k, i_{k+1})$, then the set $\{(i_1, i_2), (i_2, i_3), \ldots, (i_k, i_{k+1})\}$ is called a path of $G$ from $i_1$ to $i_{k+1}$, and node $i_{k+1}$ is said to be reachable from node $i_1$. The edge $(i, j)$ is called undirected if $(i, j) \in \mathcal{E}$ implies $(j, i) \in \mathcal{E}$. The graph is called undirected if every edge in $\mathcal{E}$ is undirected. A graph is called connected if there exists a node $i$ such that any other nodes are reachable from node $i$. A digraph $G_s = (\mathcal{V}_s, \mathcal{E}_s)$ is a subgraph of $G = (\mathcal{V}, \mathcal{E})$ if $\mathcal{V}_s \subseteq \mathcal{V}$ and $\mathcal{E}_s \subseteq \mathcal{E} \cap (\mathcal{V}_s \times \mathcal{V}_s)$.

A matrix $\mathcal{L} = [l_{ij}] \in \mathbb{R}^{N \times N}$ with zero row sum is said to be a Laplacian matrix of a graph $G$ if, for $i, j = 1, \ldots, N, i \neq j$, $l_{ij} < 0$ if $(j, i) \in \mathcal{E}$, and $l_{ij} = l_{ji}$ if $(j, i)$ is a undirected edge of $\mathcal{E}$. Clearly, $\mathcal{L}1_N = 0$.

Given any matrix $M = [m_{ij}] \in \mathbb{R}^{N \times N}$ with nonnegative off-diagonal elements and zero row sums, $M$ is called Metzler matrix. If $L$ is the Laplacian of some graph $G$, then $-L$ is a Metzler matrix. A graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ is said to be the graph of $M$ if $\mathcal{V} = \{1, \ldots, N\}$ and $\mathcal{E} = \{(i, j) \mid m_{ji} > 0, i \neq j, i, j = 1, \ldots, N\}$. Clearly, if $L$ is any Laplacian matrix of a graph $G$, then $\Gamma(L) = \mathcal{G}$.

A matrix $M = [m_{ij}]_{N \times N}$ with nonnegative off-diagonal elements and zero row sums is called Metzler matrix. If $L$ is the Laplacian of some graph $G$, then $-L$ is a Metzler matrix. A graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ is said to be the graph of $M$ if $\mathcal{V} = \{1, \ldots, N\}$ and $\mathcal{E} = \{(i, j) \mid m_{ij} > 0, i \neq j, i, j = 1, \ldots, N\}$.

**Remark 1.1:** It is shown in [8] that a Metzler matrix has at least one zero eigenvalue and all the nonzero eigenvalues have negative real parts. Furthermore, a Metzler matrix has exactly one zero eigenvalue and its null space is span{1} if and only if the associated graph is connected. A symmetric Metzler matrix is negative semi-definite. Let $M$ be a symmetric Metzler matrix whose graph is connected. Let $\Delta = \text{diag}\{a_1, \ldots, a_N\}$ where $a_i \geq 0$ for $i = 1, \ldots, N$. Then for any nonzero, nonnegative diagonal matrix $\Delta$, $-M + \Delta$ is positive definite.

**REFERENCES**


